Komplexe Wurzeln: Aufgaben

Berechnen Sie folgende Wurzeln, und geben Sie die Ergebnisse in arithmetischer Form an:

Aufgabe 3: $a)\ \sqrt{-1}\ ,\quad b)\ \sqrt[3]{-1}\ ,\quad c)\ \sqrt[4]{-1}$

Aufgabe 4: $a)\ \sqrt{i}\ ,\quad b)\ \sqrt[3]{i}\ ,\quad c)\ \sqrt[4]{i}\ ,\quad d)\ \sqrt[6]{i}$

Wurzelziehen: Lösung 3

$$-1 = -1 + i\,0\,, \qquad x = -1, \quad y = 0$$

$$(x < 0\,, \qquad y = 0)$$

$$r = |\,z\,| = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + 0^2} = 1$$

$$\sin \varphi_0 = \frac{y}{\sqrt{x^2 + y^2}} = 0$$

$$\varphi = \varphi_0 + 2\,k\,\pi = \pi + 2\,k\,\pi$$

$$-1 = e^{i\,(\pi + 2\,\pi\,k)} = e^{i\,\pi\,(1 + 2\,k)}$$

$$\sqrt{-1} = (-1)^{\frac{1}{2}} = e^{\,i\,\frac{\pi}{2}\,(1\,+\,2\,k)} = e^{\,i\,\left(\frac{\pi}{2}\,+\,\pi\,k\right)} \qquad (k = 0,\ 1)$$

$$k = 0: \qquad W_0 = e^{\,i\,\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$$

$$k = 1: \qquad W_1 = e^{\,i\,\left(\frac{\pi}{2}\,+\,\pi\right)} = \cos\left(\frac{3\,\pi}{2}\right) + i\sin\left(\frac{3\,\pi}{2}\right) = -i$$

Die beiden Lösungen der Gleichung $z^2 = -1$ sind imaginär.

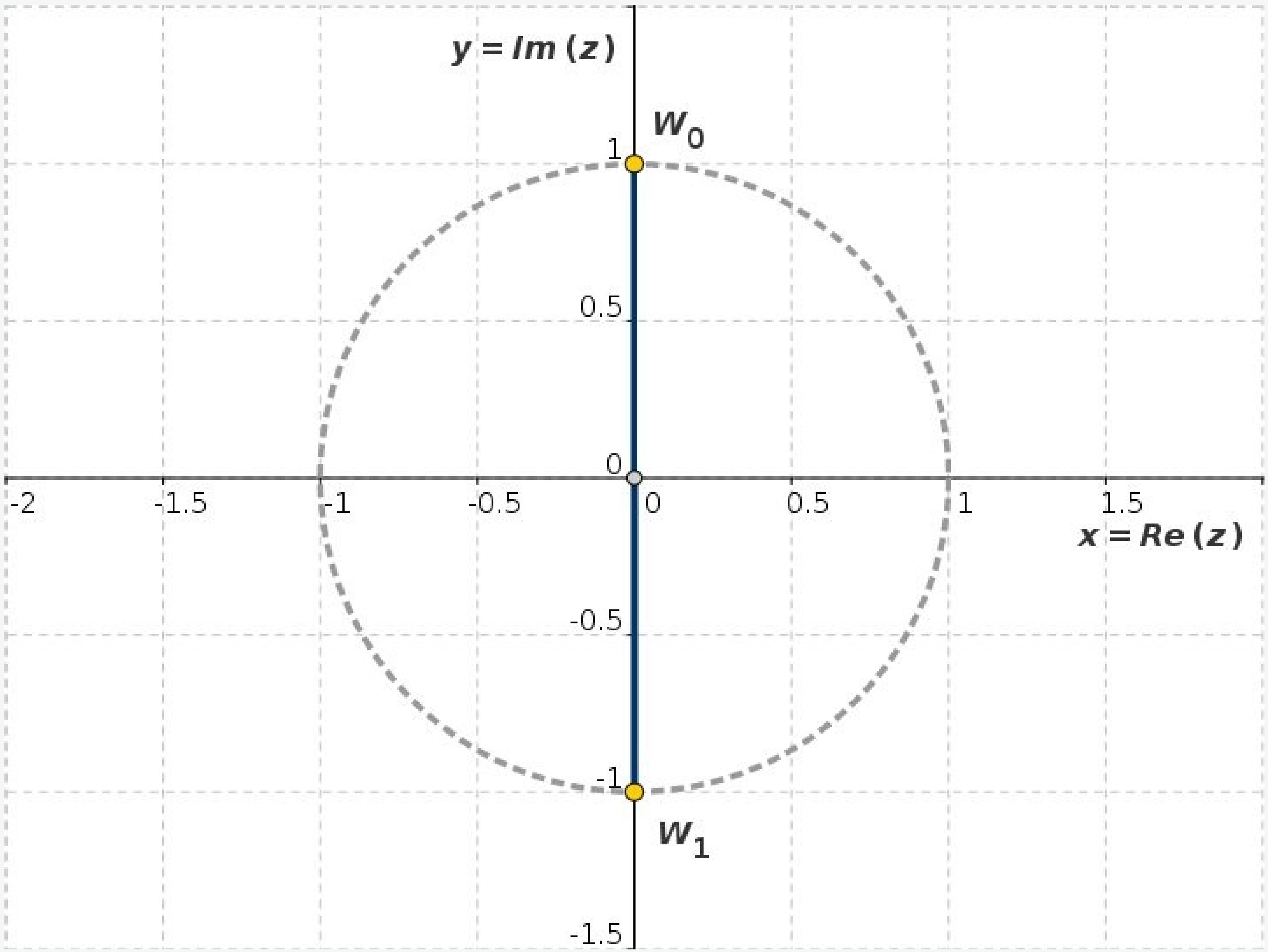

Abb. L3a: Graphische Darstellung der 2. Wurzeln aus -1

$$\sqrt[3]{-1} = (-1)^{\frac{1}{3}} = e^{i\left(\frac{\pi}{3} + \frac{2k}{3}\pi\right)} \qquad (k = 0,\ 1,\ 2)$$

$$k = 0: \quad W_0 = e^{i\frac{\pi}{3}} = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{1 + \sqrt{3}\,i}{2}$$

$$k = 1: \quad W_1 = e^{i\left(\frac{\pi}{3} + \frac{2}{3}\pi\right)} = \cos\pi + i\sin\pi = -1$$

$$k = 2: \quad W_2 = e^{i\left(\frac{\pi}{3} + \frac{4}{3}\pi\right)} = \cos\left(\frac{5}{3}\pi\right) + i\sin\left(\frac{5}{3}\pi\right) =$$

$$= \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right) = \frac{1 - \sqrt{3}\,i}{2}$$

Die zweite Lösung der Gleichung $z^3 = -1$ ist reell.

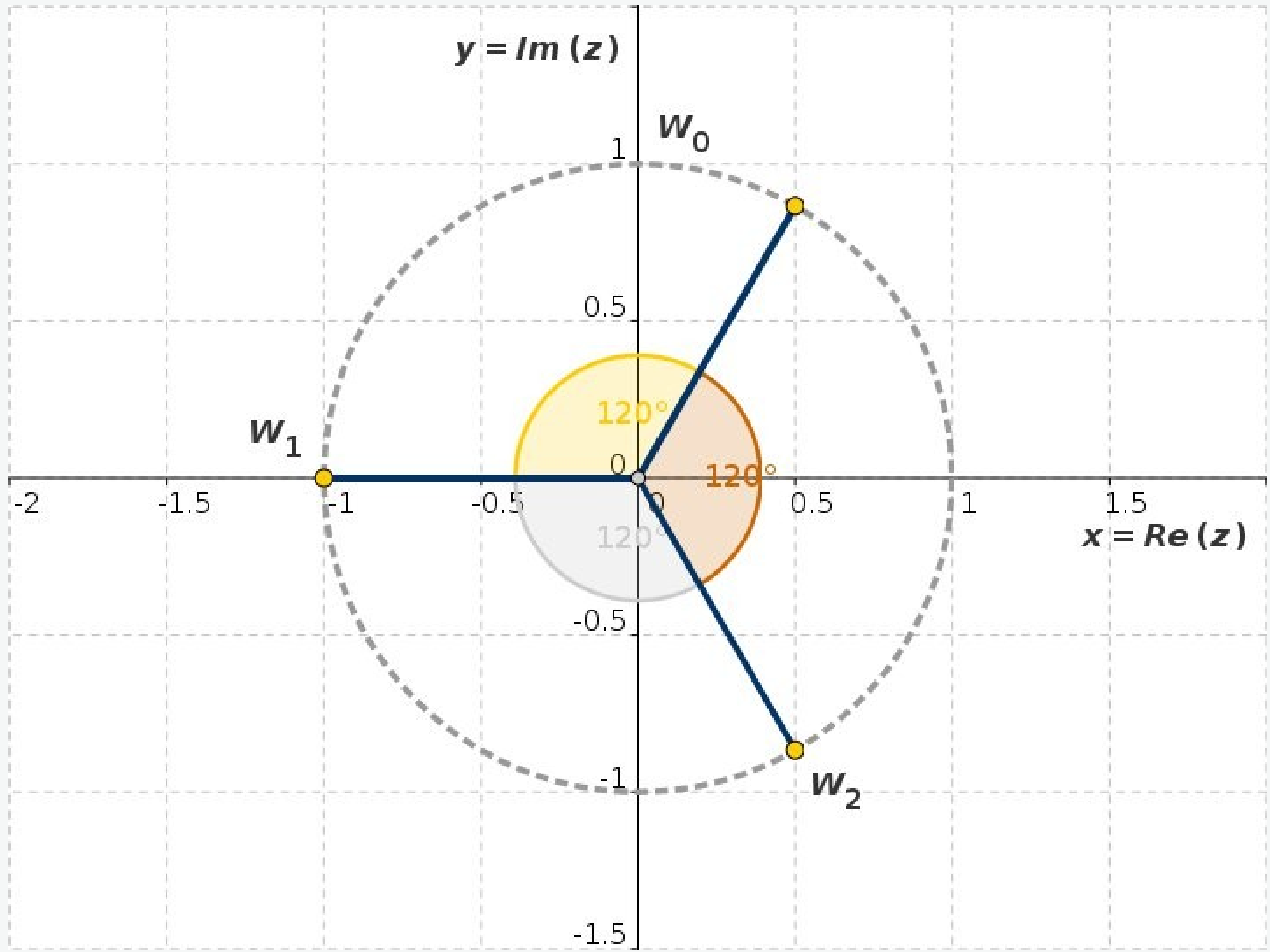

Abb. L3b: Graphische Darstellung der 3. Wurzeln aus -1

$$\sqrt[4]{-1} = (-1)^{\frac{1}{4}} = e^{i\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)} \qquad (k = 0,\ 1,\ 2,\ 3)$$

$$k = 0: \quad W_0 = e^{i\frac{\pi}{4}} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{1 + i}{\sqrt{2}}$$

$$k = 1: \quad W_1 = e^{i\left(\frac{\pi}{4} + \frac{\pi}{2}\right)} = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = \frac{-1 + i}{\sqrt{2}}$$

$$k = 2: \quad W_2 = e^{i\left(\frac{\pi}{4} + \pi\right)} = \cos\left(\frac{5}{4}\pi\right) + i\sin\left(\frac{5}{4}\pi\right) = -\frac{(1 + i)}{\sqrt{2}}$$

$$k = 3: \quad W_3 = e^{i\left(\frac{\pi}{4} + \frac{3\pi}{2}\right)} = \cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right) = \frac{1 - i}{\sqrt{2}}$$

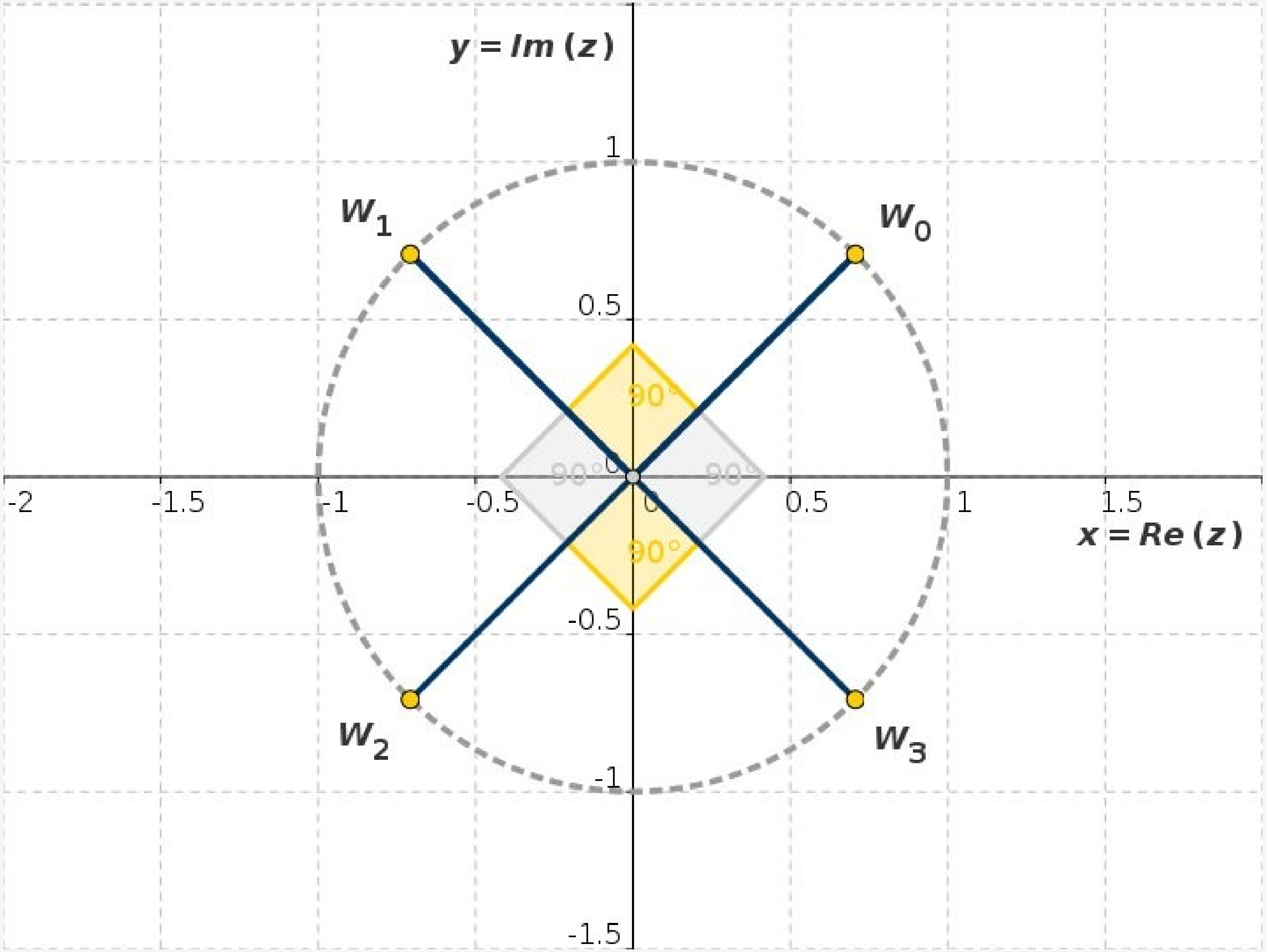

Abb. L3c: Graphische Darstellung der 4. Wurzeln aus -1

$$i = 0 + i\,, \qquad x = 0, \quad y = 1$$

$$(x = 0\,, \qquad y > 0)$$

$$r = |\,z\,| = \sqrt{x^2 + y^2} = \sqrt{0^2 + 1^2} = 1$$

$$\sin \varphi_0 = \frac{y}{\sqrt{x^2 + y^2}} = 1, \qquad \varphi_0 = \frac{\pi}{2}$$

$$\varphi = \varphi_0 + 2\,k\,\pi = \frac{\pi}{2} + 2\,k\,\pi$$

$$i = e^{\,i\left(\frac{\pi}{2} + 2\,k\,\pi\right)}$$

$$\sqrt{i} = i^{\frac{1}{2}} = e^{i\left(\frac{\pi}{4} + k\,\pi\right)} = e^{i\left(\frac{\pi}{2} + \pi\,k\right)} \qquad (k = 0,\ 1)$$

$$k = 0: \qquad W_0 = e^{i\,\frac{\pi}{4}} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{1 + i}{\sqrt{2}}$$

$$k = 1: \qquad W_2 = e^{i\left(\frac{\pi}{4} + \pi\right)} = \cos\left(\frac{5}{4}\pi\right) + i\sin\left(\frac{5}{4}\pi\right) = -\frac{(1 + i)}{\sqrt{2}}$$

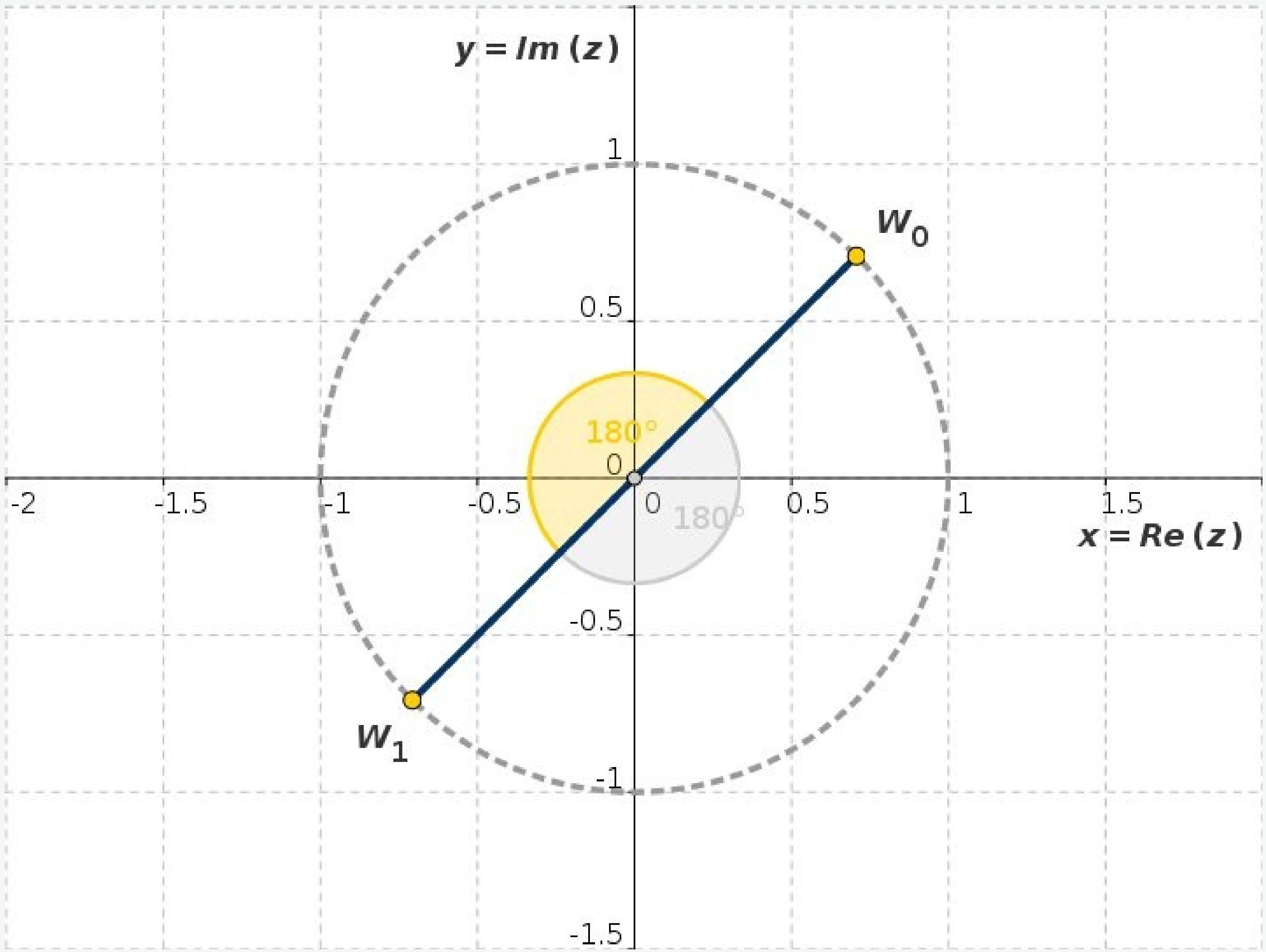

Abb. 4La: Graphische Darstellung der 2. Wurzeln aus i

$$\sqrt[3]{i} = i^{\frac{1}{3}} = e^{i\left(\frac{\pi}{6} + \frac{2k}{3}\pi\right)} \qquad (k = 0,\ 1,\ 2)$$

$$k = 0: \quad W_0 = e^{i\frac{\pi}{6}} = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3} + i}{2}$$

$$k = 1: \quad W_1 = e^{i\frac{5\pi}{6}} = \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) = \frac{-\sqrt{3} + i}{2}$$

$$k = 2: \quad W_2 = e^{i\frac{3\pi}{2}} = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = -i$$

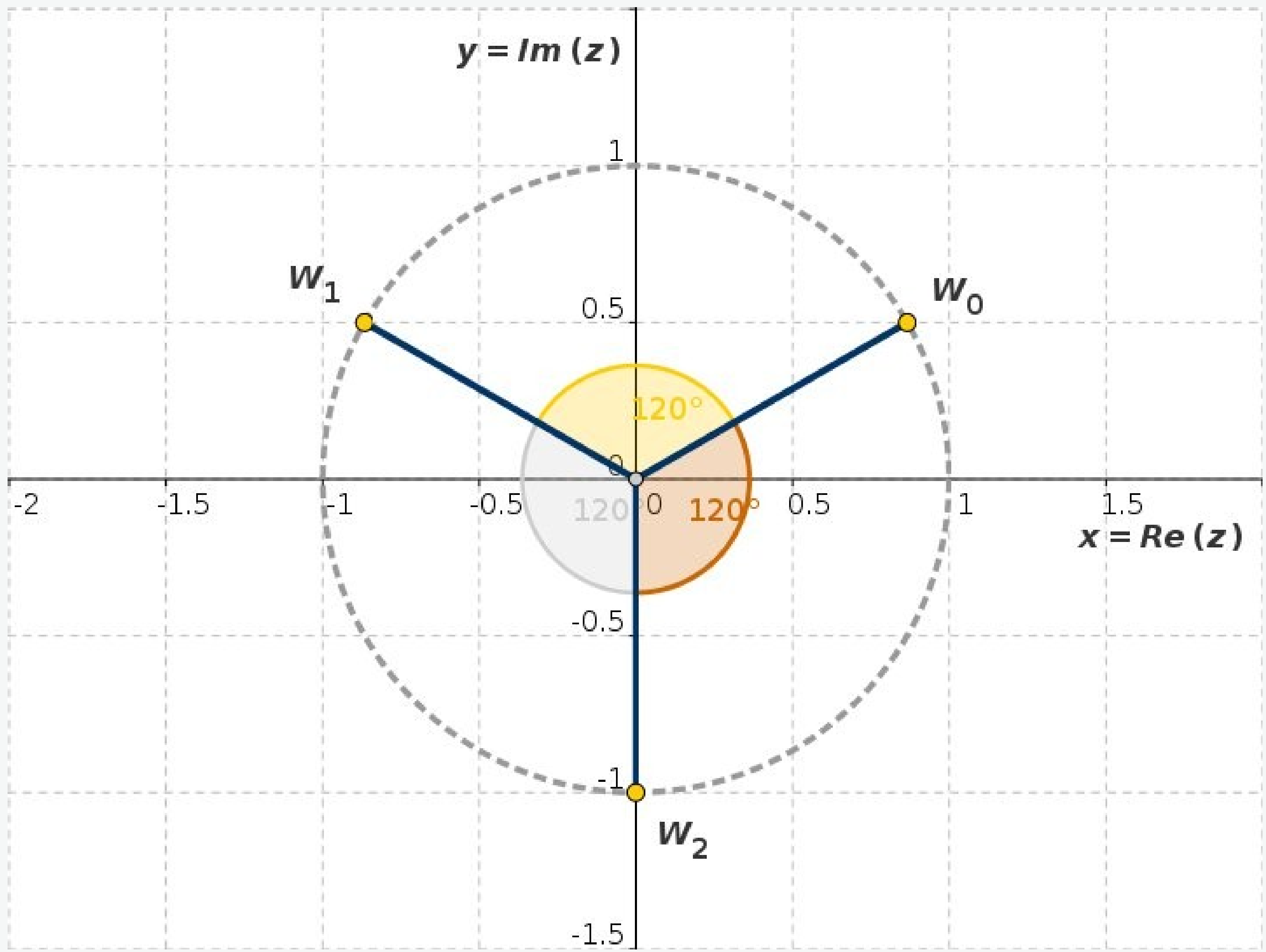

Abb. L4b: Graphische Darstellung der 3. Wurzeln aus i

$$\sqrt[4]{i} = i^{\frac{1}{4}} = e^{i\left(\frac{\pi}{8} + \frac{k\pi}{2}\right)} \qquad (k = 0,\ 1,\ 2,\ 3)$$

$$k = 0: \quad W_0 = e^{i\frac{\pi}{8}} = \cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right) \simeq 0.924 + 0.383\,i$$

$$k = 1: \quad W_1 = e^{i\frac{5\pi}{8}} = \cos\left(\frac{5\pi}{8}\right) + i\sin\left(\frac{5\pi}{8}\right) \simeq -0.383 + 0.924\,i$$

$$k = 2: \quad W_1 = e^{i\frac{9\pi}{8}} = \cos\left(\frac{9\pi}{8}\right) + i\sin\left(\frac{9\pi}{8}\right) \simeq -0.924 - 0.383\,i$$

$$k = 3: \quad W_3 = e^{i\frac{13}{8}\pi} = \cos\left(\frac{13}{8}\pi\right) + i\sin\left(\frac{13}{8}\pi\right) \simeq -0.383 + 0.924\,i$$

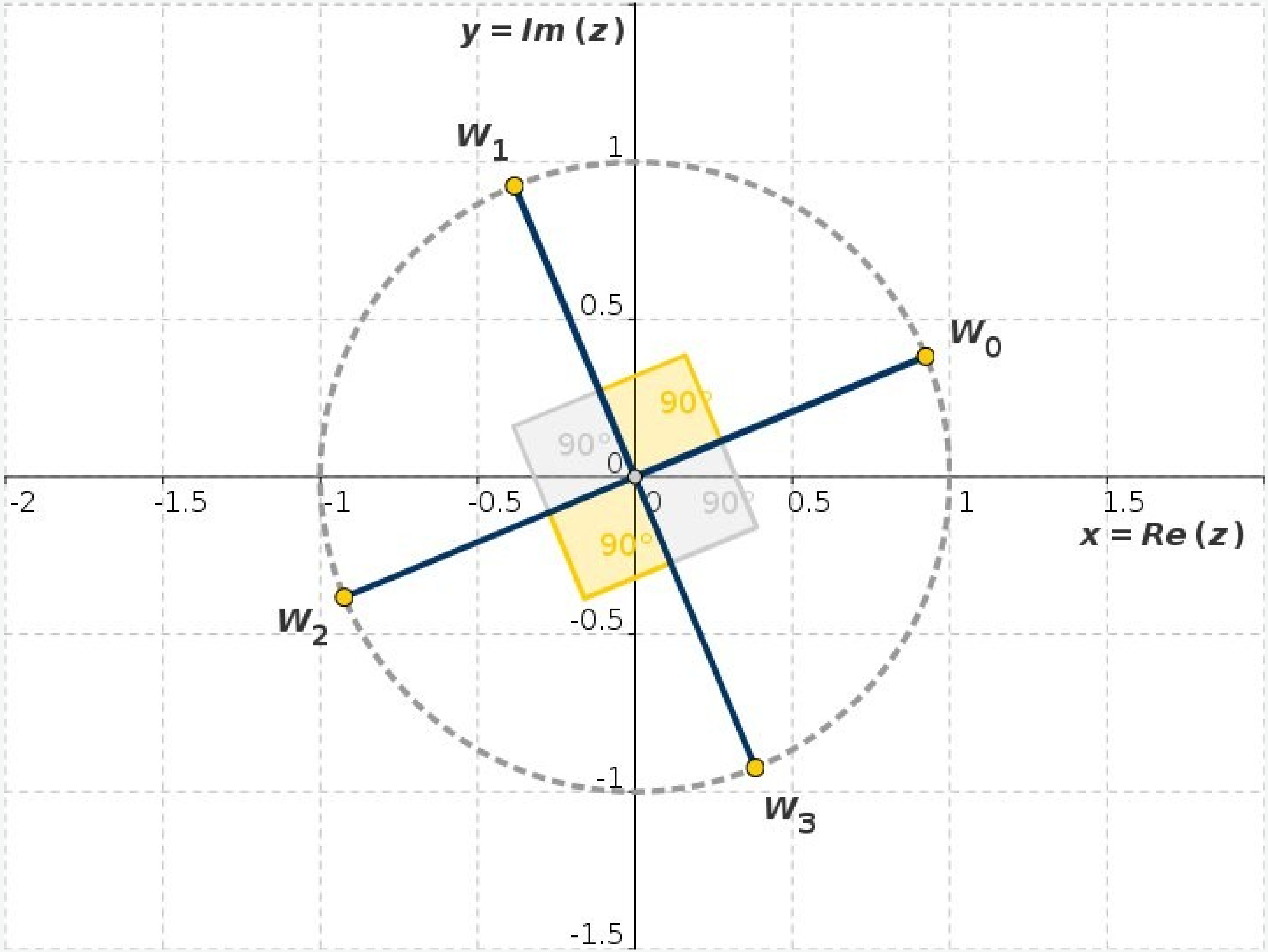

Abb. L4c: Graphische Darstellung der 4. Wurzeln aus i