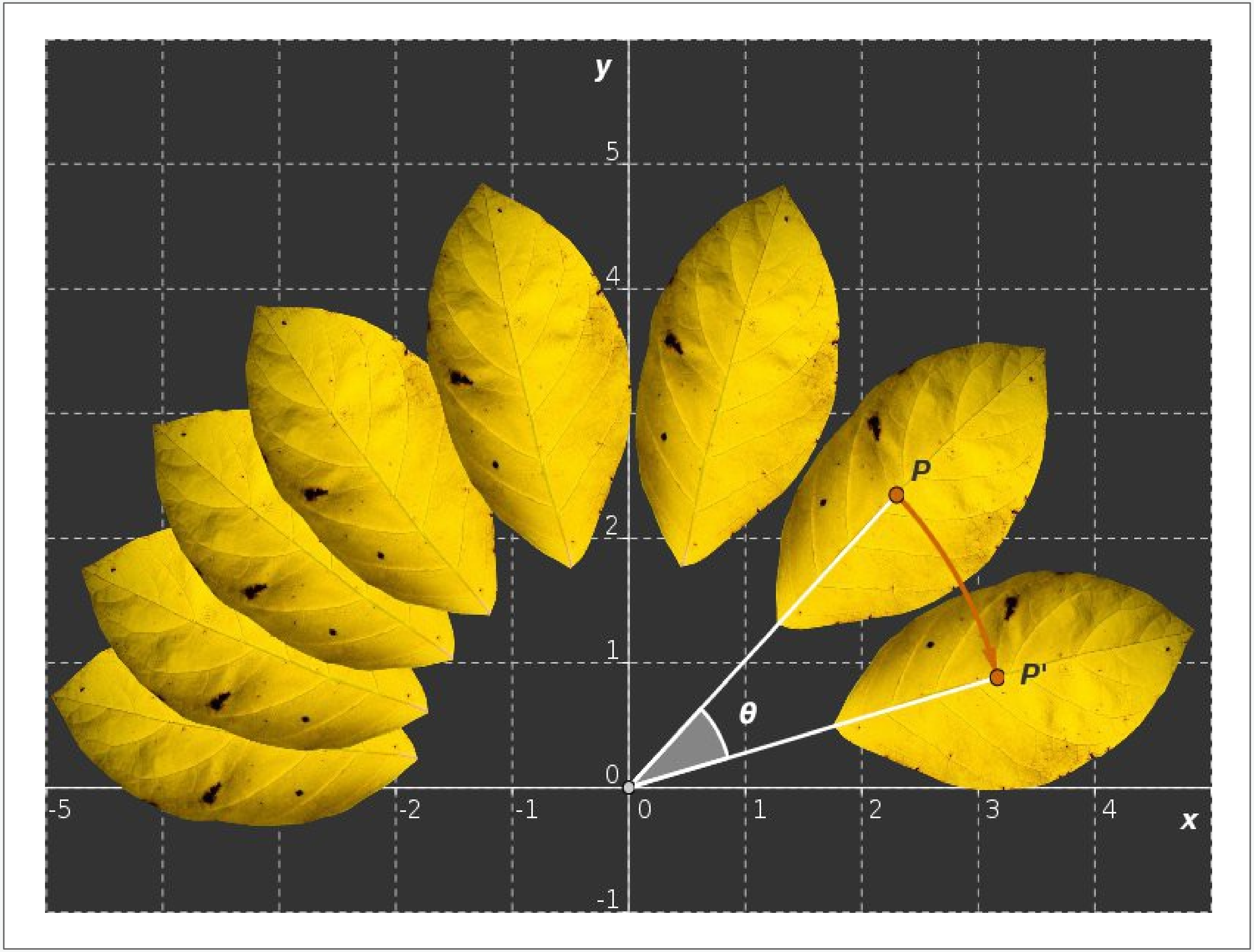
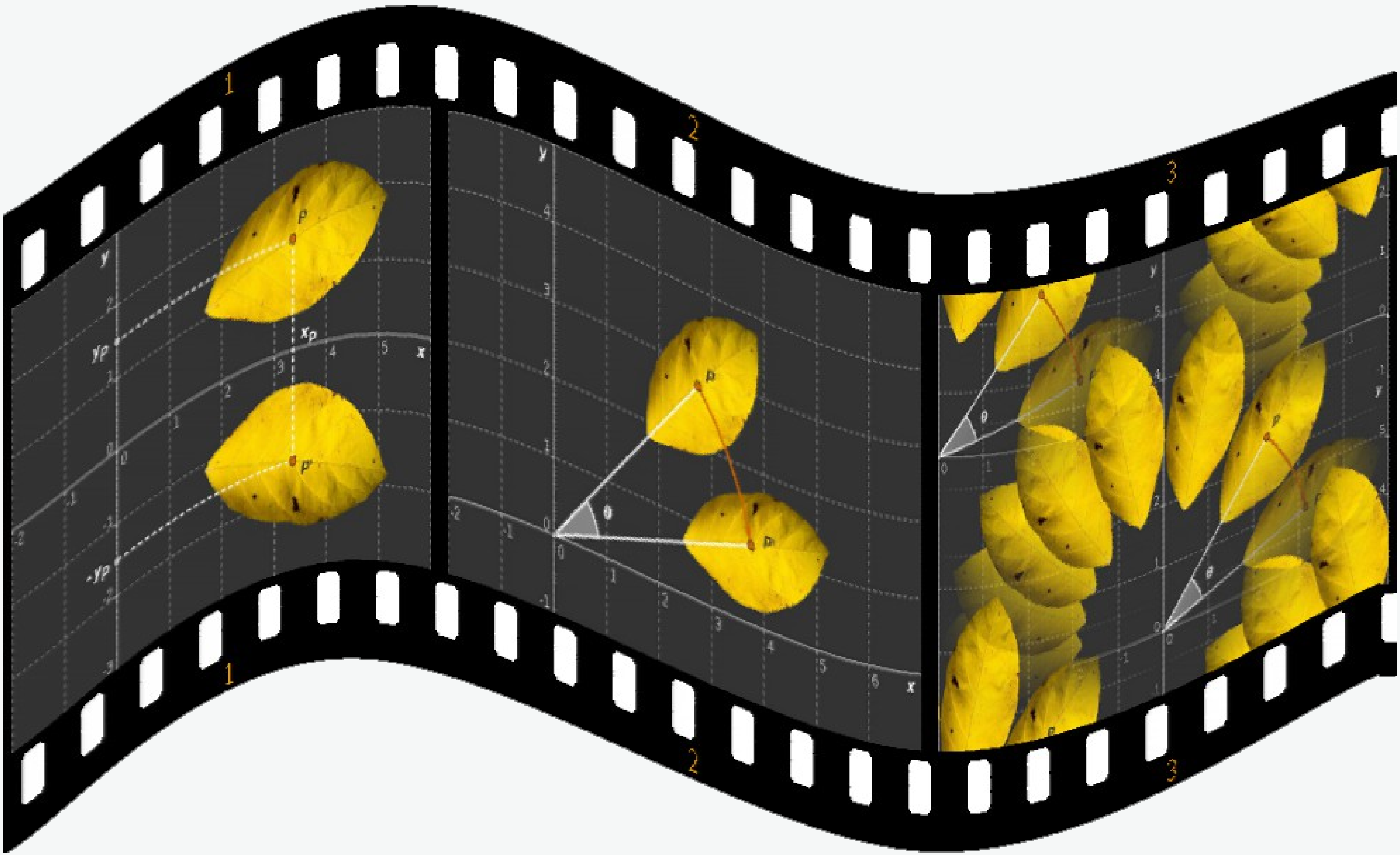


Spiegelungen, Drehungen


1
2
3
P
P'
x_P
y_P
$-y_P$
θ
x
y

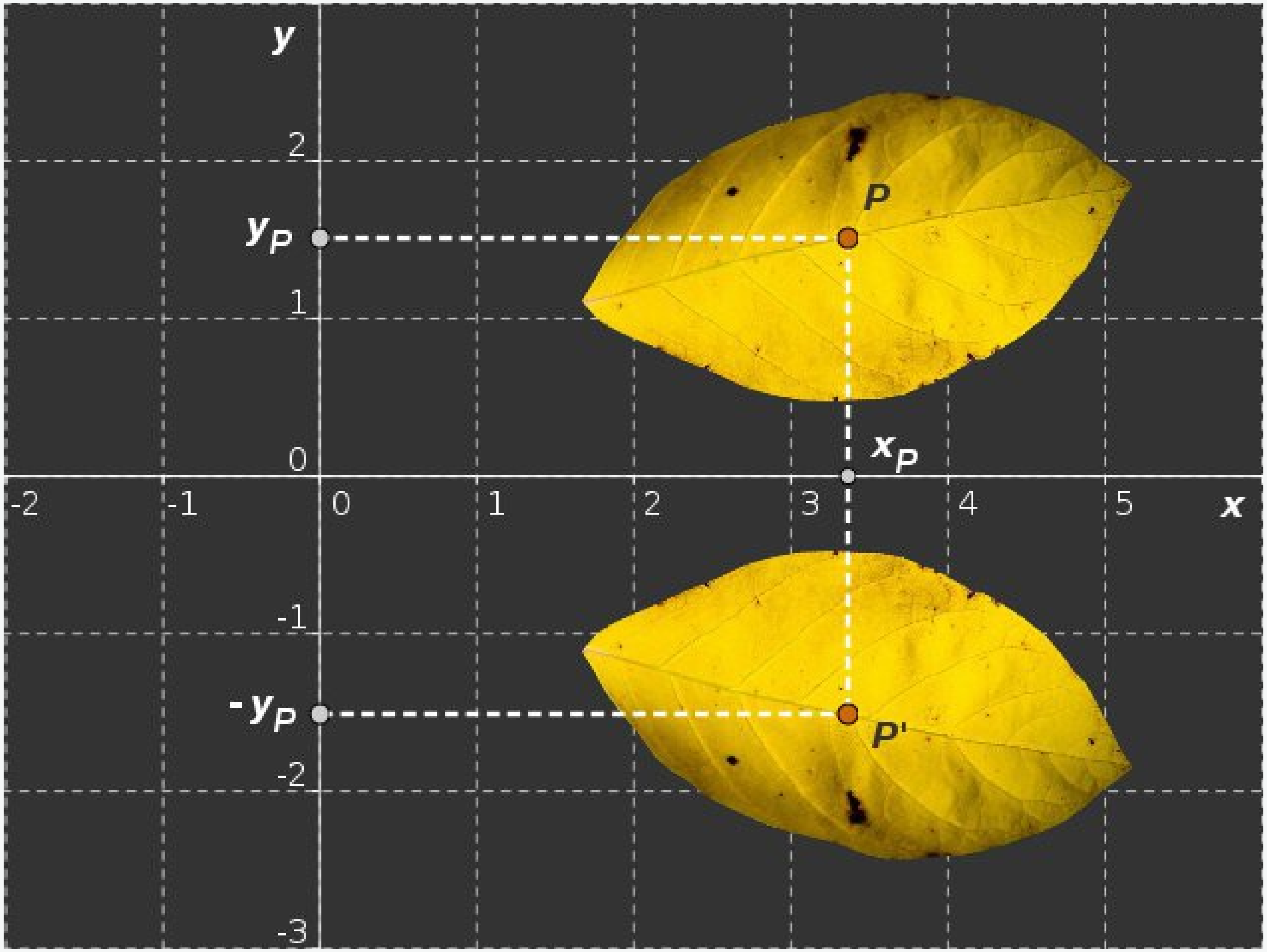


Abb. 1-1: Spiegelung an der x-Achse

$$P(x, y) \rightarrow P'(x', y'), \qquad x' = x, \qquad y' = -y$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

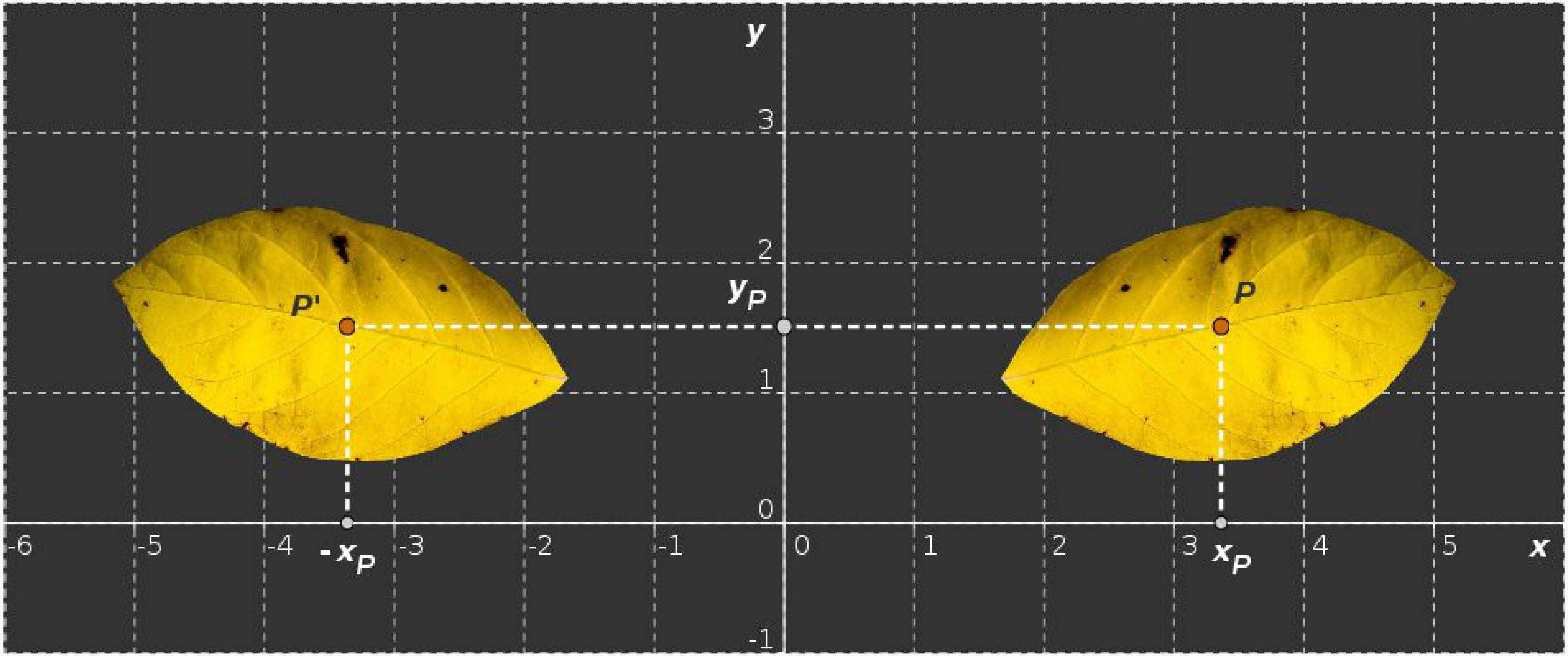


Abb. 1-2: Spiegelung an der y-Achse

$$P\,(x\,,\; y) \quad \rightarrow \quad P\,'(x\,',\; y\,')\,, \qquad x\,' = -\,x\,, \qquad y\,' = y$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} x\,' \\ y\,' \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

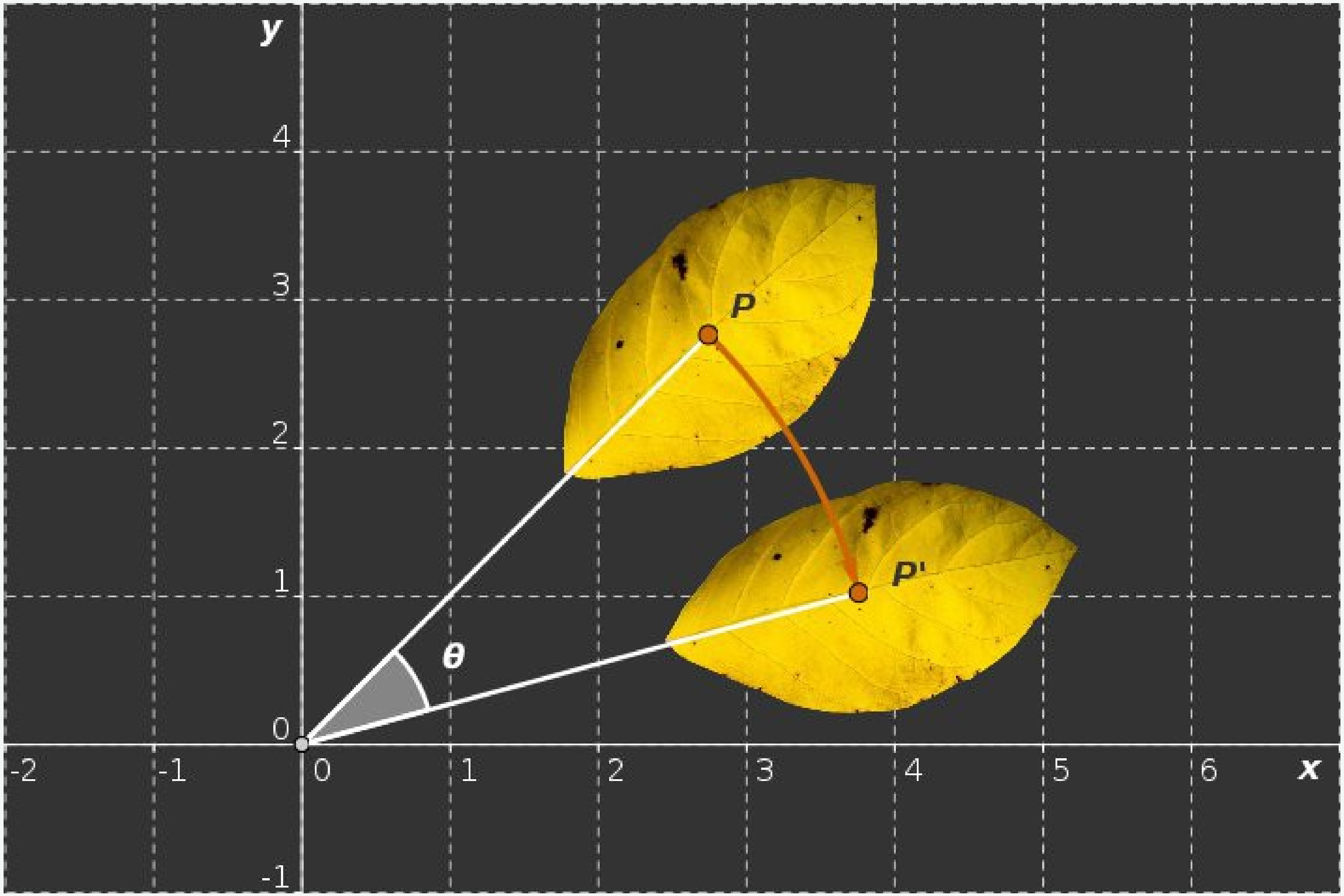


Abb. 1-3a: Drehung um den Winkel θ

Eine Drehung in der xy-Ebene ist durch ein Drehzentrum O und einen Drehwinkel θ festgelegt.

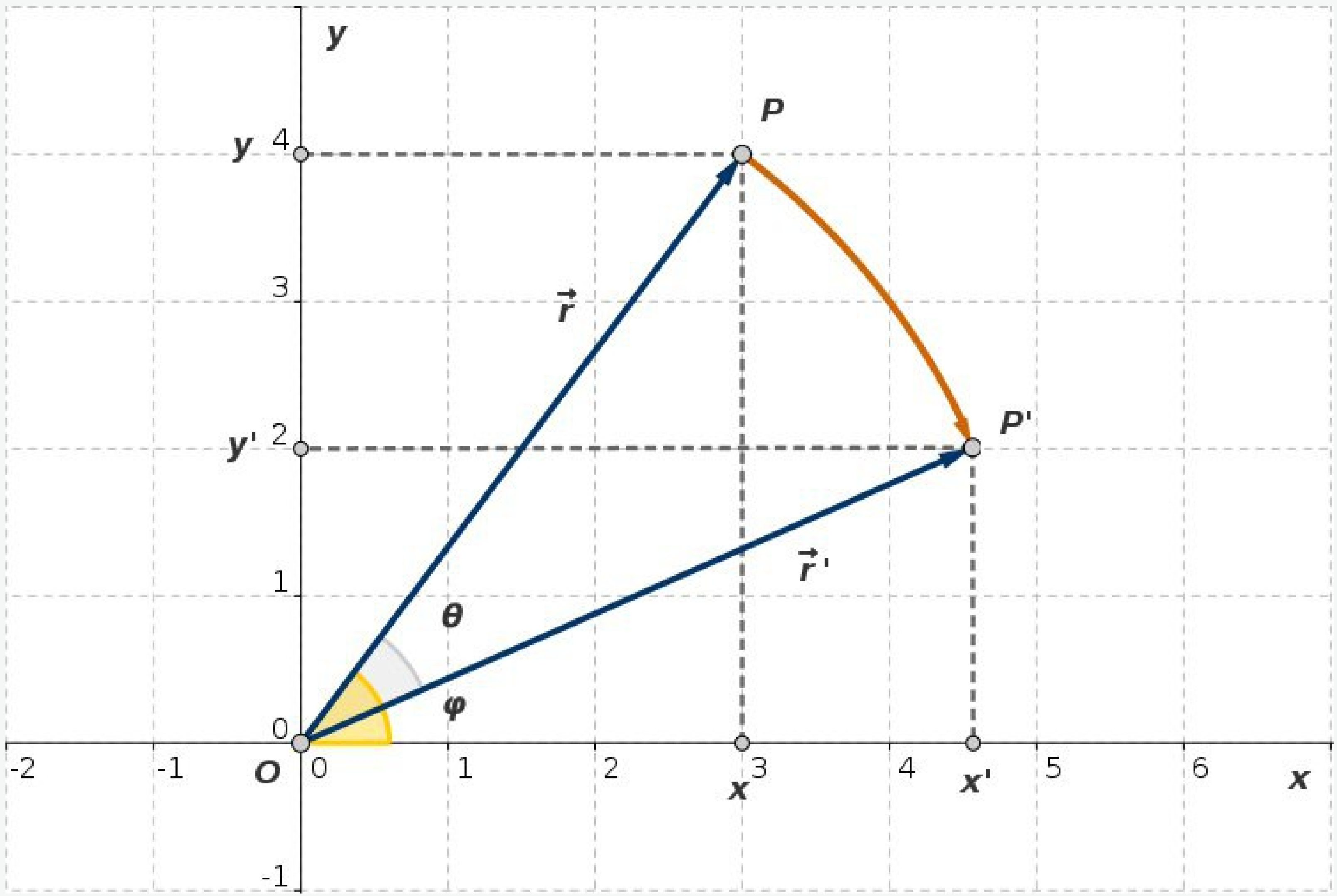


Abb. 1-3b: Drehung um den Winkel θ

$$P\,(x\,,\;y)\;\;\rightarrow\;\;P\,'(x\,',\;y\,')\,,\qquad \vec{r}\,=\,\overrightarrow{OP}\,=\begin{pmatrix} x \\ y \end{pmatrix},\qquad \vec{r}\,'=\overrightarrow{OP\,'}=\begin{pmatrix} x\,' \\ y\,' \end{pmatrix}$$

Um die Drehung um einen Winkel θ zu beschreiben, stellen wir die Koordinaten der Ortsvektoren in Polarform dar

$$\vec{r} = \overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad x = r \cos\varphi, \qquad y = r \sin\varphi, \qquad r = |\vec{r}|$$

$$\vec{r}\,' = \overrightarrow{OP\,'} = \begin{pmatrix} x' \\ y' \end{pmatrix}, \qquad x' = r \cos(\varphi - \theta), \qquad y' = r \sin(\varphi - \theta)$$

$$x' = r \cos(\varphi - \theta) = r\,(\cos\varphi \cos\theta + \sin\varphi \sin\theta) =$$

$$= x\cos\theta + y\sin\theta = (\cos\theta, \ \sin\theta)\cdot\begin{pmatrix} x \\ y \end{pmatrix} \equiv \vec{v}_1 \cdot \vec{r}$$

$$y' = r \sin(\varphi - \theta) = r\,(\sin\varphi \cos\theta - \cos\varphi \sin\theta) =$$

$$= -x\sin\theta + y\cos\theta = (-\sin\theta, \ \cos\theta)\cdot\begin{pmatrix} x \\ y \end{pmatrix} \equiv \vec{v}_2 \cdot \vec{r}$$

$$\vec{v}_1 = (\cos\theta, \ \sin\theta), \qquad \vec{v}_2 = (-\sin\theta, \ \cos\theta)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{pmatrix} = \begin{pmatrix} \vec{v}_1 \cdot \vec{r} \\ \vec{v}_2 \cdot \vec{r} \end{pmatrix}$$

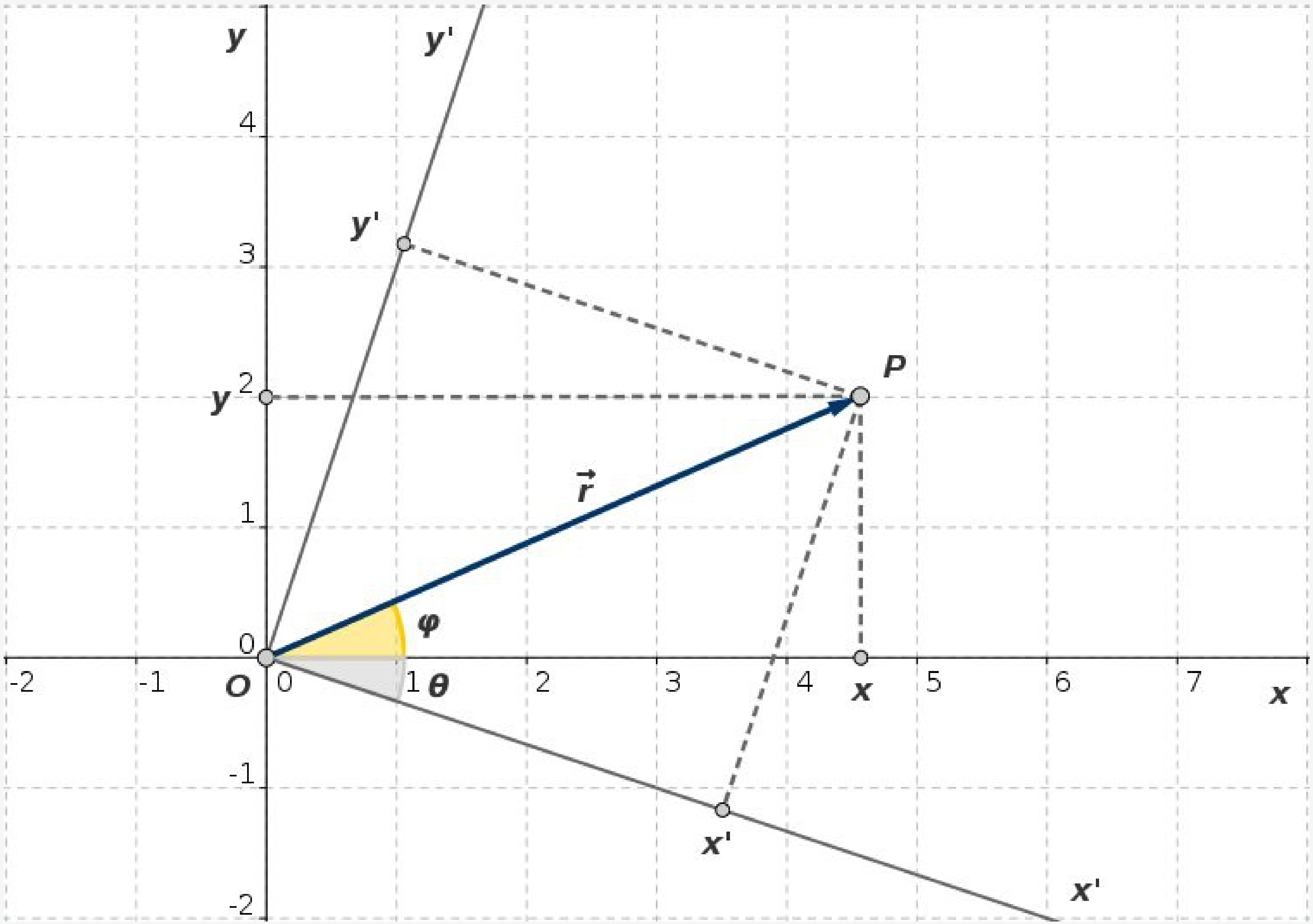


Abb. 1-4: Drehung des Koordinatensystems um den Winkel θ

Drehung um einen Winkel: Beispiel 4

$$xOy: \quad \vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad x'Oy': \quad \vec{r} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$xOy: \quad \vec{r} = \overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix} = r \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

$$x'Oy': \quad \vec{r} = \overrightarrow{OP} = \begin{pmatrix} x' \\ y' \end{pmatrix} = r \begin{pmatrix} \cos(\varphi + \theta) \\ \sin(\varphi + \theta) \end{pmatrix}$$

$$x' = r \cos(\varphi + \theta) = r \left(\cos\varphi \cos\theta - \sin\varphi \sin\theta\right) =$$

$$= x\cos\theta - y\sin\theta = (\cos\theta, \ -\sin\theta) \cdot \begin{pmatrix} x \\ y \end{pmatrix} \equiv \vec{v}_3 \cdot \vec{r}$$

$$y' = r \sin(\varphi + \theta) = r \left(\sin\varphi \cos\theta + \cos\varphi \sin\theta\right) =$$

$$= x\sin\theta + y\cos\theta = (\sin\theta, \ \cos\theta) \cdot \begin{pmatrix} x \\ y \end{pmatrix} \equiv \vec{v}_4 \cdot \vec{r}$$

$$\vec{v}_3 = (\cos\theta, \ -\sin\theta), \qquad \vec{v}_4 = (\sin\theta, \ \cos\theta)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix} = \begin{pmatrix} \vec{v}_3 \cdot \vec{r} \\ \vec{v}_4 \cdot \vec{r} \end{pmatrix}$$

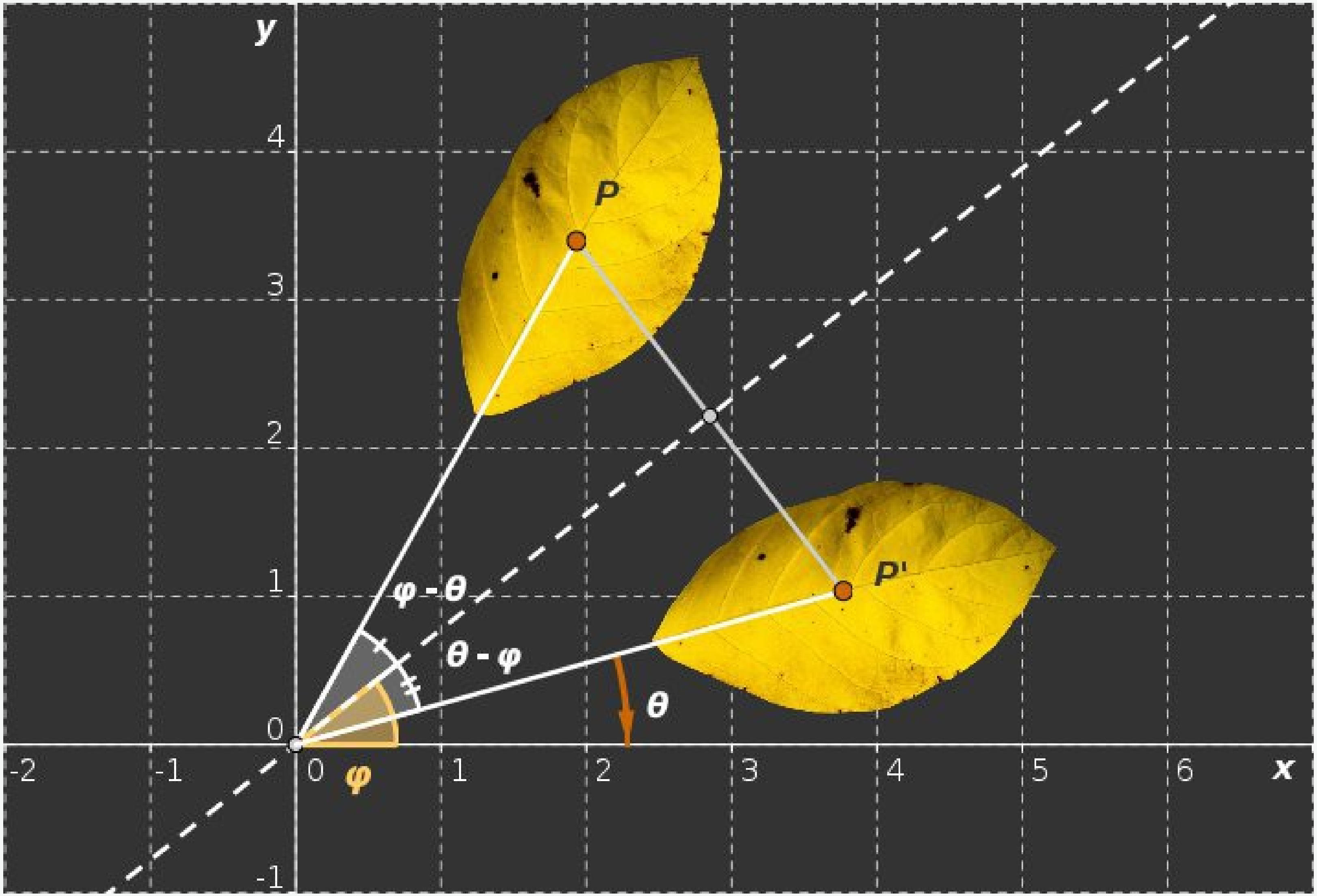


Abb. 1-5a: Spiegelung an einer Ursprungsgeraden

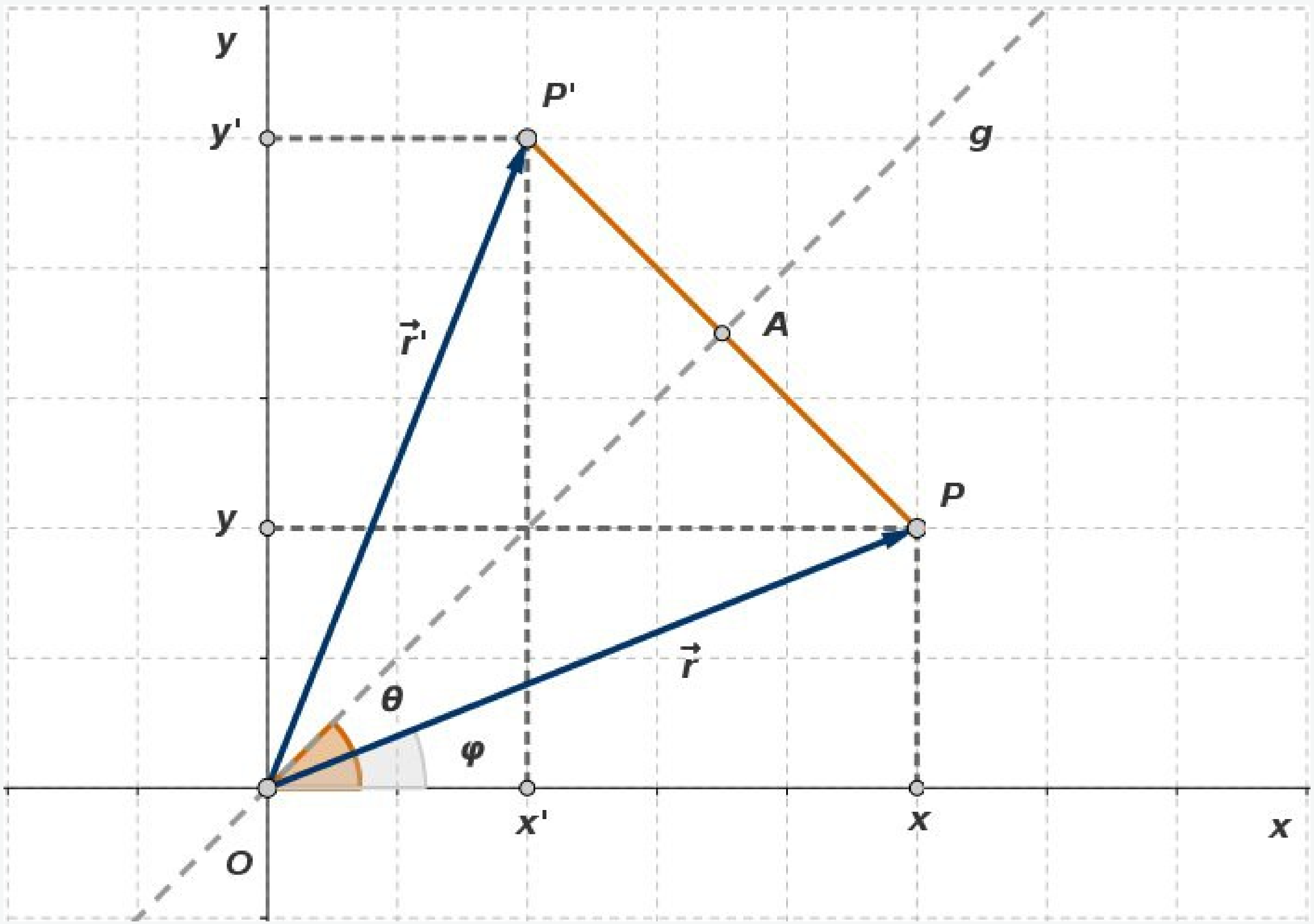


Abb. 1-5b: Spiegelung an einer Ursprungsgeraden

$$\sphericalangle\, POA = \sphericalangle\, AOP\,' = \theta - \varphi \quad \Rightarrow \quad \sphericalangle\, xOP\,' = \theta + \sphericalangle\, AOP\,' = 2\,\theta - \varphi$$

$$\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix} = r \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}, \qquad \vec{r}\,' = \begin{pmatrix} x\,' \\ y\,' \end{pmatrix} = r \begin{pmatrix} \cos(2\theta - \varphi) \\ \sin(2\theta - \varphi) \end{pmatrix}$$

$$x\,' = r\cos(2\theta - \varphi) = r\,(\cos(2\theta)\cos\varphi + \sin(2\theta)\sin\varphi) =$$

$$= x\cos(2\theta) + y\sin(2\theta) = (\cos(2\theta),\ \sin(2\theta)) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \vec{v}_5 \cdot \vec{r}$$

$$y\,' = r\sin(2\theta - \varphi) = r\,(\sin(2\theta)\cos\varphi - \cos(2\theta)\sin\varphi) =$$

$$= x\sin(2\theta) - y\cos(2\theta) = (\sin(2\theta),\ -\cos(2\theta)) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \vec{v}_6 \cdot \vec{r}$$

$$\vec{v}_5 = (\cos(2\theta),\ \sin(2\theta)), \qquad \vec{v}_6 = (\sin(2\theta),\ -\cos(2\theta))$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \quad \to \quad \begin{pmatrix} x\,' \\ y\,' \end{pmatrix} = r \begin{pmatrix} \cos(2\theta - \varphi) \\ \sin(2\theta - \varphi) \end{pmatrix} = \begin{pmatrix} x\cos(2\theta) + y\sin(2\theta) \\ x\sin(2\theta) - y\cos(2\theta) \end{pmatrix} =$$

$$= \begin{pmatrix} \vec{v}_5 \cdot \vec{r} \\ \vec{v}_6 \cdot \vec{r} \end{pmatrix}$$