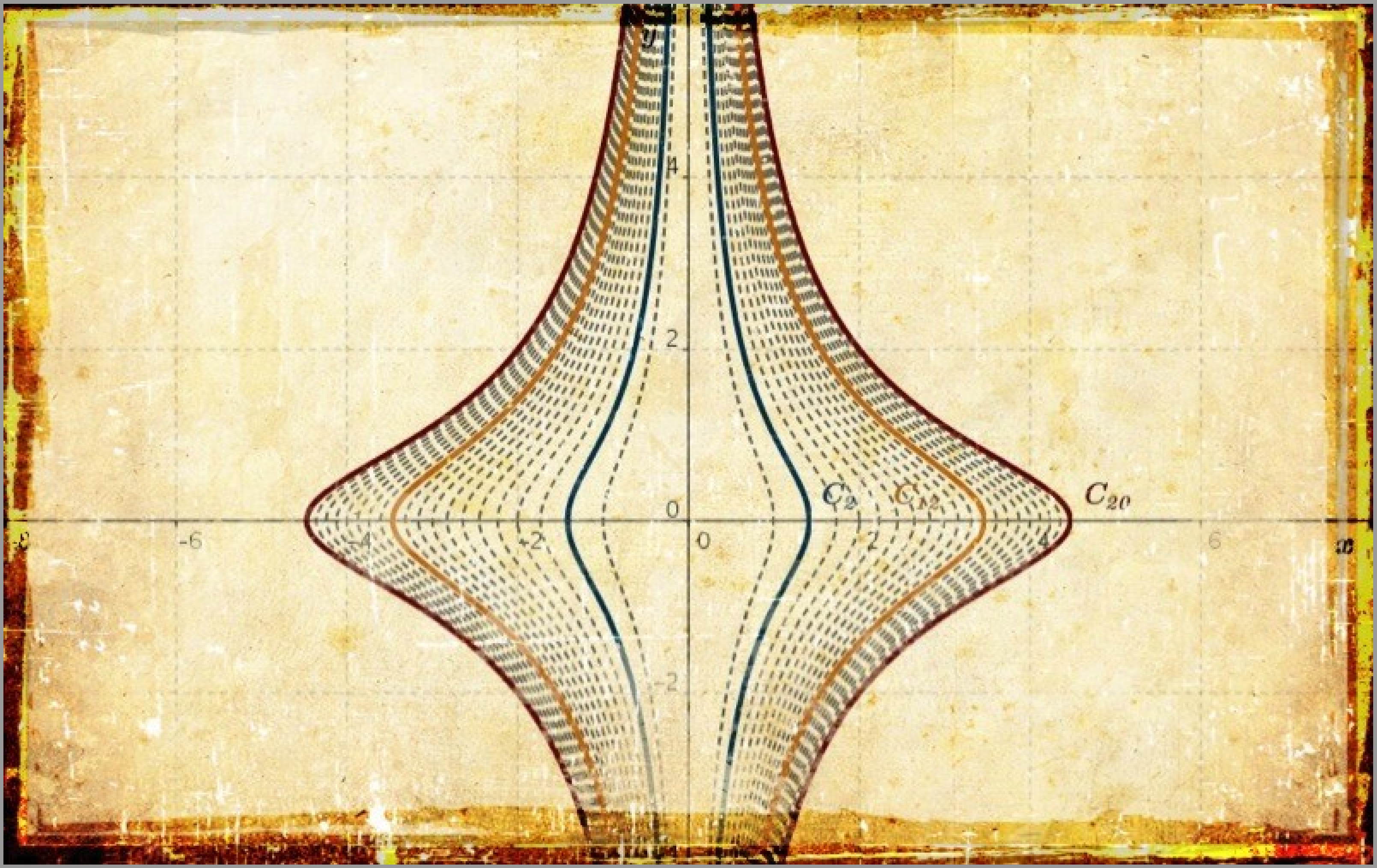
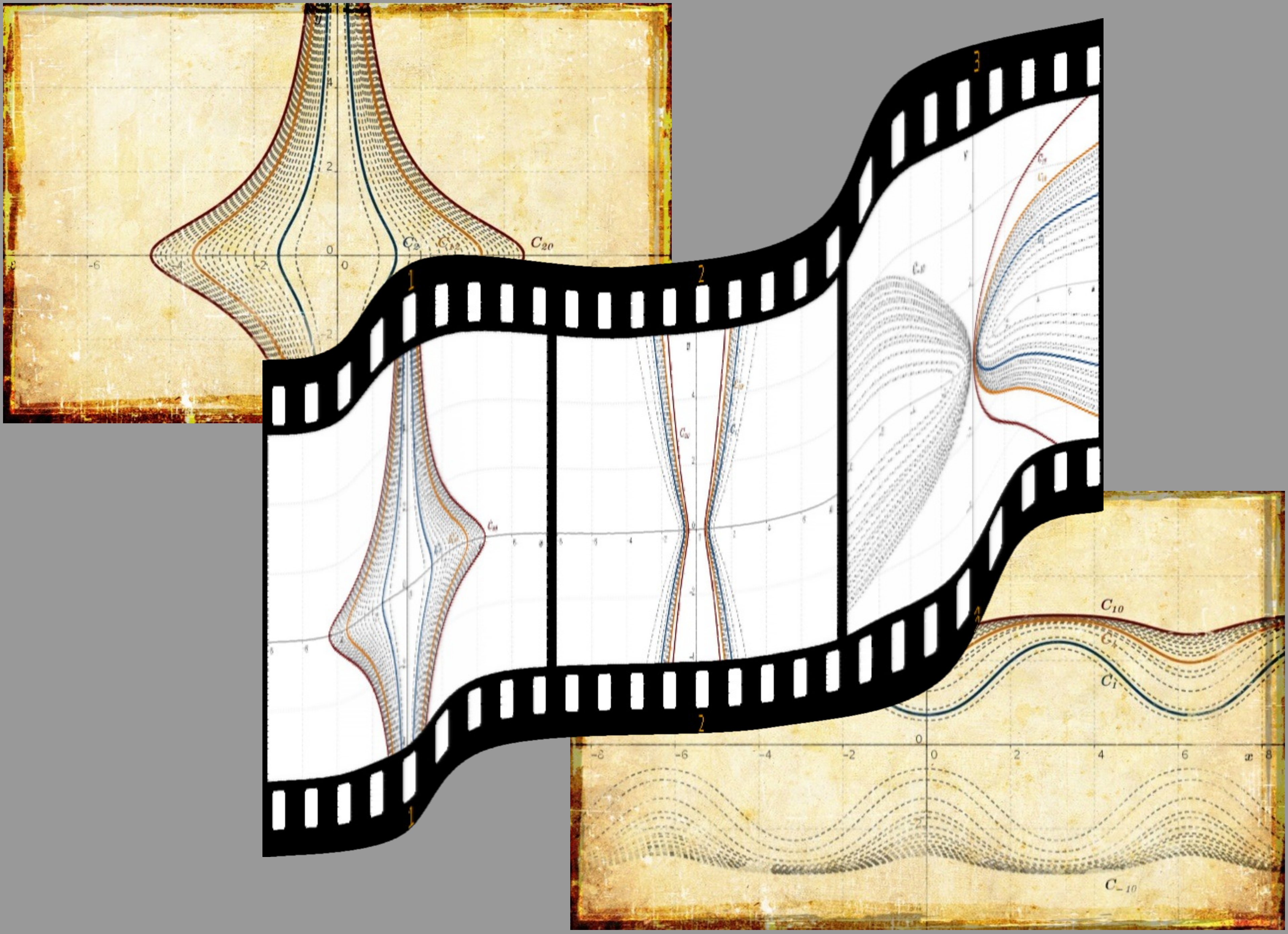
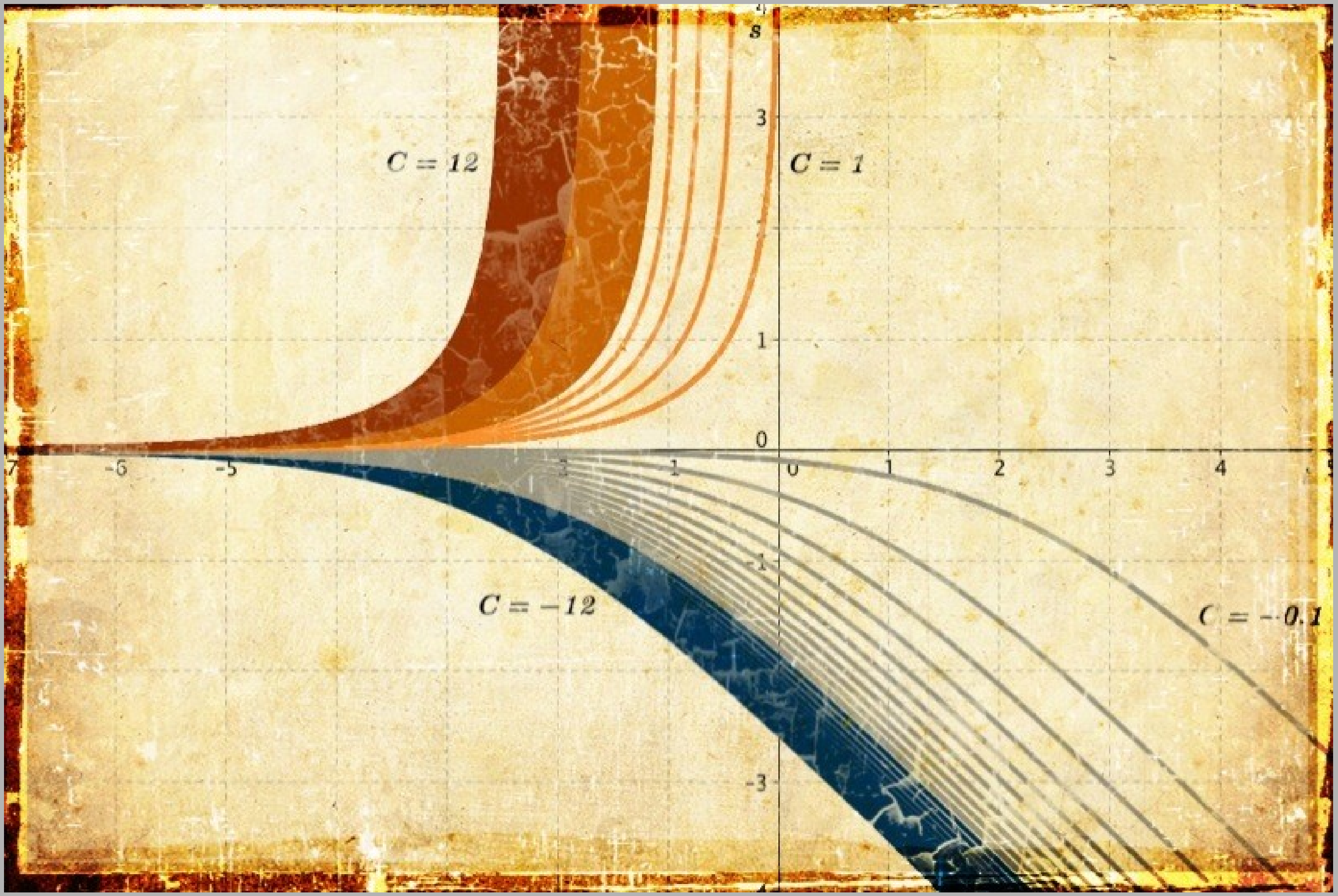



*Trennung der Variablen: Aufgaben 9-16*


$C_2$
$C_{10}$
$C_{20}$
$C_{10}$
$C_1$
$C_{-10}$



s
C = 12
C = 1
C = −12
C = −0.1

Lösen Sie die folgenden Differentialgleichungen 1. Ordnung bzw. Anfangswertprobleme durch Trennung der Variablen:

Aufgabe 1: $1 + y^2 + x\,y\,y' = 0$

Aufgabe 2: $1 + 4\,y^2 + a\,x\,y\,y' = 0, \qquad a \in \mathbb{R}, \qquad a \neq 0$

Aufgabe 3: $x\sqrt{1 + y^2} + y\;y'\sqrt{1 + x^2} = 0$

Aufgabe 4: $y' = \sin x \cdot \sin y$

Aufgabe 5: $y' = 3\cos(2\,x) - \dfrac{x}{2}, \qquad y\left(\dfrac{\pi}{4}\right) = 2, \qquad y\left(\dfrac{\pi}{8}\right) = 1$

Aufgabe 6: $e^{-s}\,(1 + s') = 1, \qquad s = s(t)$

$a)\;\; s(0) = -2, \qquad b)\;\; s(0) = 2$

Aufgabe 7: $(1 + e^{x})\;y' = e^{x}, \qquad y(0) = 0$

Aufgabe 8: $(1 + e^{x})\;y\,y' = e^{x}$

$$1 + y^2 + x\,y\,y\,' = 0$$

$$(1 + y^2)\;dx + x\,y\;dy = 0 \qquad \left(\times \;\frac{1}{x\,(1 + y^2)}\right)$$

$$u = 1 + y^2, \quad du = 2\,y\,dy\,, \qquad y\,dy = \frac{du}{2}$$

$$\int \frac{y\,dy}{1 + y^2} = -\int \frac{dx}{x} \quad \Rightarrow \quad \int \frac{du}{u} = -2\int \frac{dx}{x}$$

$$\ln|\,u\,| = -2\;\ln|\,x\,| + \ln|C| = \ln\left|\frac{C}{x^2}\right| \quad \Rightarrow \quad u = \frac{C}{x^2}$$

$$y^2 = \frac{C}{x^2} - 1, \qquad y^2\,x^2 + x^2 = C$$

$$y = \pm\sqrt{\frac{C}{x^2} - 1}$$

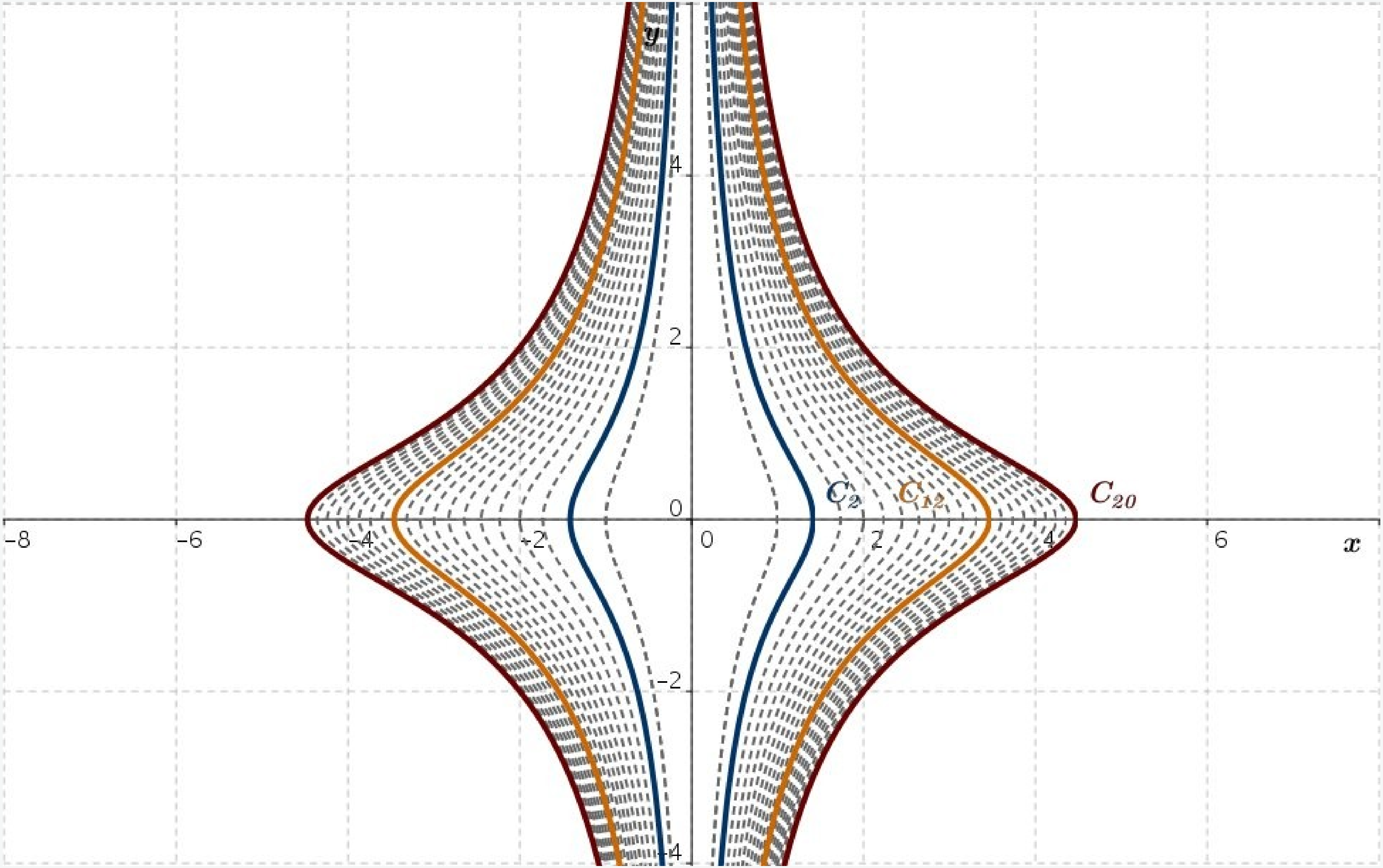



*Abb. L1: 20 Integralkurven der DGL. Die blaue Kurve entspricht dem Wert der Integrationskonstante C = 2, die orange Kurve – dem Wert C = 12, die rote Kurve – dem Wert C = 20*

$$y^2 x^2 + x^2 = C$$

$$1 + 4\,y^2 + a\,x\,y\,y\,' = 0, \qquad a \in \mathbb{R}\,, \qquad a \neq 0$$

$$(1 + 4\,y^2)\;dx + a\,x\,y\;dy = 0 \qquad \left( \times \quad \frac{1}{a\,x\,(1 + 4\,y^2)} \right)$$

$$\int \frac{y\,dy}{1 + 4\,y^2} = -\frac{1}{a} \int \frac{dx}{x}\,, \qquad \int \frac{du}{u} = -\frac{8}{a} \int \frac{dx}{x}$$

$$u = 1 + 4\,y^2, \quad du = 8\,y\,dy\,, \qquad y\,dy = \frac{du}{8}$$

$$\ln|\,u\,| = -\frac{8}{a}\ \ln|\,x\,| + \ln|C| = \ln\left|\,C\;x^{-\frac{8}{a}}\right|, \qquad u = C\;x^{-\frac{8}{a}}$$

$$1 + 4\,y^2 = C\;x^{-\frac{8}{a}}, \qquad x^{\frac{8}{a}} + 4\,y^2\;x^{\frac{8}{a}} = C$$

$$y^2 = \frac{1}{4}\left(C\;x^{-\frac{8}{a}} - 1\right), \qquad y = \pm\,\frac{1}{2}\sqrt{C\;x^{-\frac{8}{a}} - 1}$$

$$x^{\frac{8}{a}} + 4\,y^2\,x^{\frac{8}{a}} = C$$

Abbildung L2-1, Seite 4-2c:

$$a = -2: \qquad x^{-4} + 4\,y^2\,x^{-4} = C\,, \qquad 1 + 4\,y^2 = C\,x^4$$

Abbildung L2-2, Seite 4-2d:

$$a = -8: \qquad x^{-1} + 4\,y^2\,x^{-1} = C\,, \qquad 1 + 4\,y^2 = C\,x$$

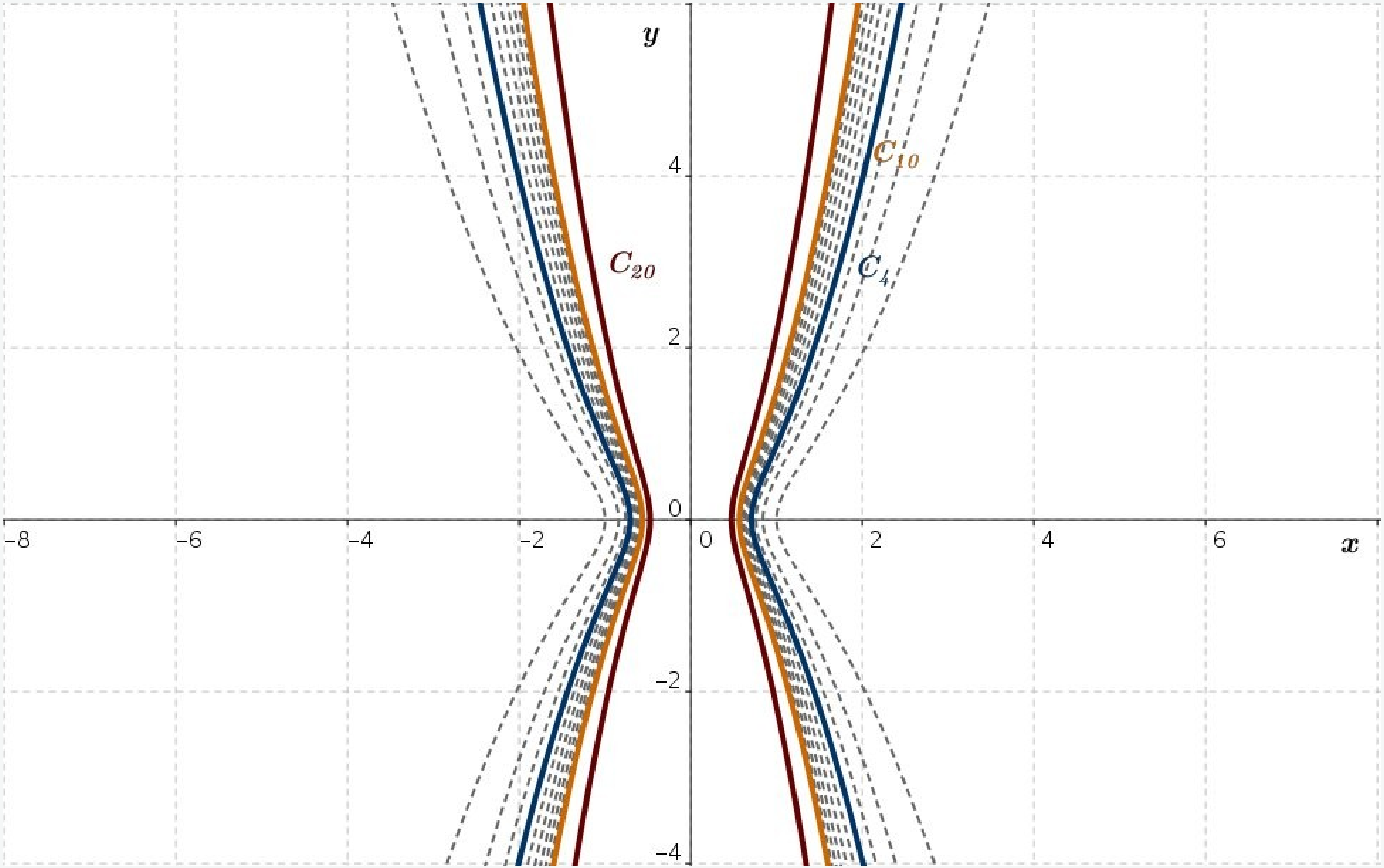



*Abb. L2-1: Integralkurven der DGL. Die blaue Kurve entspricht dem Wert der Integrationskonstante C = 4, die orange Kurve – dem Wert C = 10, die rote Kurve – dem Wert C = 20*

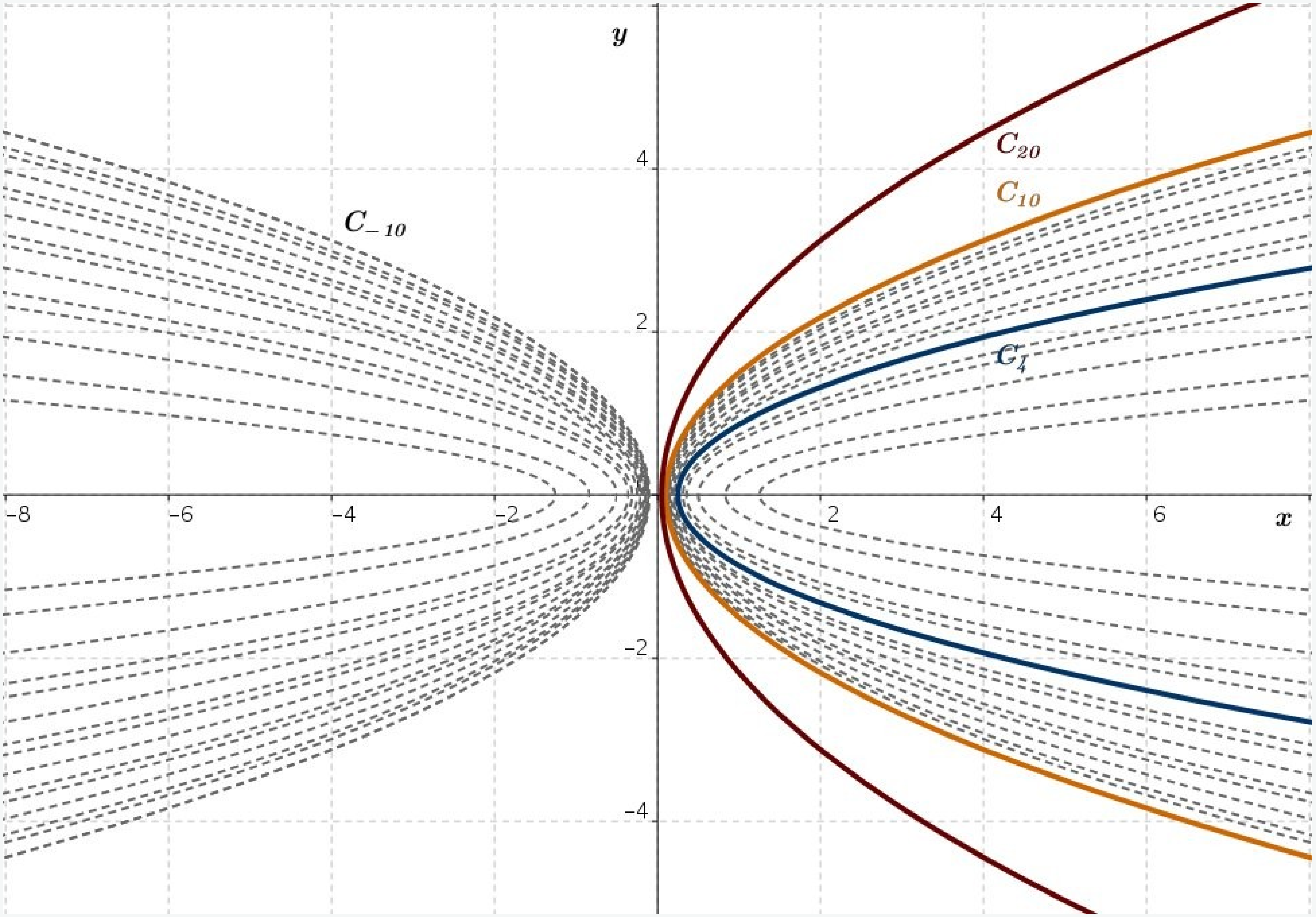



*Abb. L2-2: Integralkurven der DGL. Die blaue Kurve entspricht dem Wert der Integrationskonstante C = 4, die orange Kurve – dem Wert C = 10, die rote Kurve – dem Wert C = 20*

# Trennung der Variablen: Lösung 3

$$x\sqrt{1+y^2} + y\,y'\sqrt{1+x^2} = 0 \qquad \times \qquad \frac{1}{\sqrt{1+y^2}\,\sqrt{1+x^2}}$$

$$\int \frac{y\,dy}{\sqrt{1+y^2}} = -\int \frac{x\,dx}{\sqrt{1+x^2}}$$

$$u = 1+y^2, \quad du = 2\,y\,dy, \qquad v = 1+x^2, \quad dv = 2\,x\,dx$$

$$\int \frac{y\,dy}{\sqrt{1+y^2}} = -\int \frac{x\,dx}{\sqrt{1+x^2}} \quad \Rightarrow \quad \frac{1}{2}\int \frac{du}{\sqrt{u}} = -\frac{1}{2}\int \frac{dv}{\sqrt{v}} \quad \Leftrightarrow$$

$$\frac{1}{2}\int u^{-1/2}\,du = -\frac{1}{2}\int v^{-1/2}\,dv \quad \Rightarrow \quad u^{1/2} = -v^{1/2} + C \quad \Leftrightarrow$$

$$\sqrt{1+y^2} + \sqrt{1+x^2} = C$$

$$y' = \sin x \cdot \sin y\,, \qquad \frac{dy}{dx} = \sin x \cdot \sin y\,, \qquad \frac{dy}{\sin y} = \sin x \; dx$$

$$\boxed{\int \frac{dx}{\sin(a\,x)} = \frac{1}{a} \ln \left| \tan \frac{a\,x}{2} \right|}$$

Das kann man zeigen mit der Substitution $u = \tan\left(\frac{x}{2}\right)$

$$\int \frac{dy}{\sin y} = \int \sin x \; dx\,, \qquad \ln \left| \tan \frac{y}{2} \right| = -\cos x + \ln |\, C\,|$$

$$\tan \frac{y}{2} = C \; e^{-\cos x}\,, \qquad \frac{y}{2} = \arctan\left(C \; e^{-\cos x}\right)$$

$$y = 2 \arctan\left(C \; e^{-\cos x}\right)$$

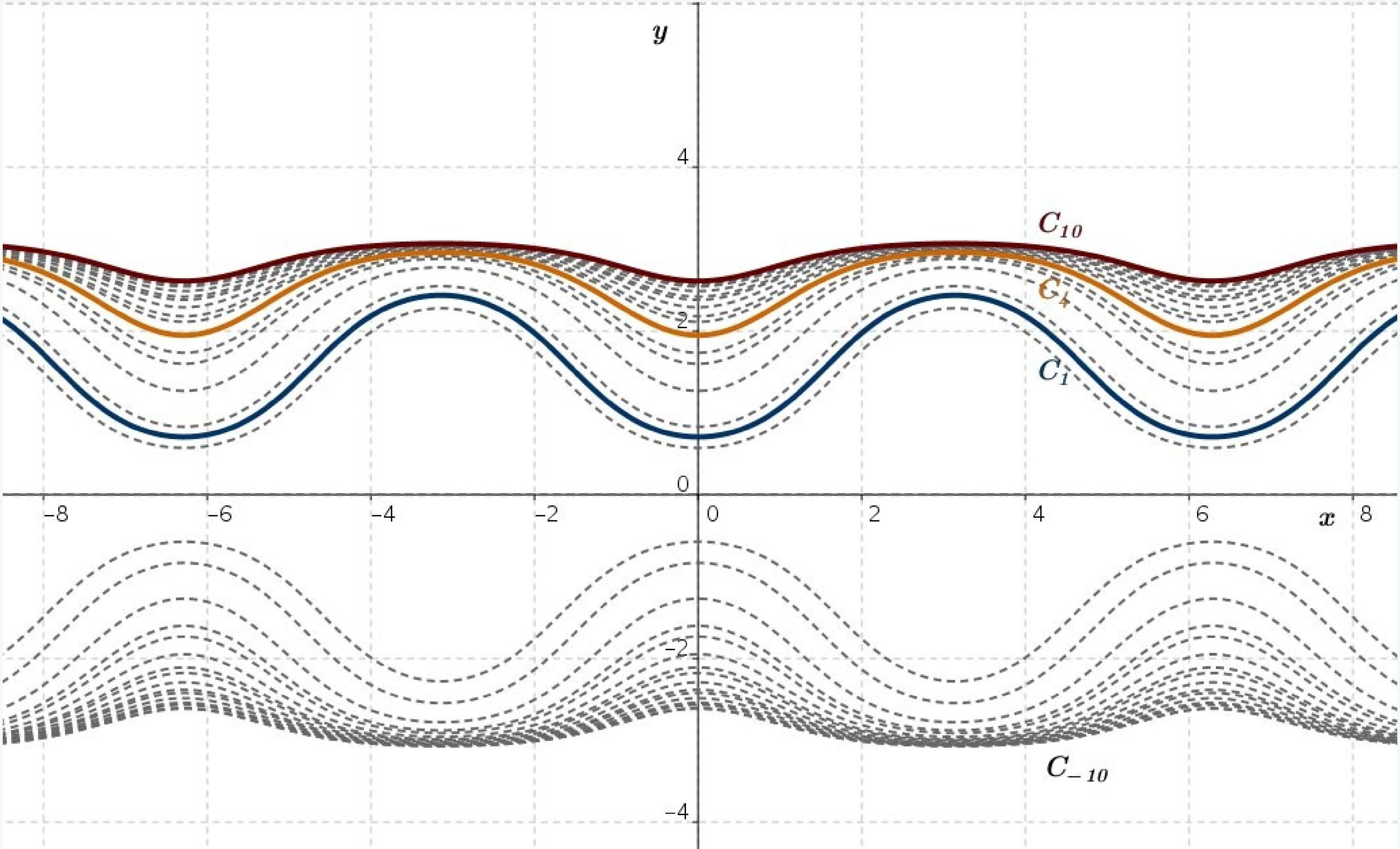



*Abb. L4: Integralkutven der DGL. Die blaue Kurve entspricht dem Wert der Integrationskonstante C = 1, die orange Kurve – dem Wert C = 4, die rote Kurve – dem Wert C = 10*

$$y' = 3\cos(2x) - \frac{x}{2}, \qquad y\left(\frac{\pi}{4}\right) = 2, \qquad y\left(\frac{\pi}{8}\right) = 1$$

<u>Allgemeine Lösung:</u> $y(x) = \frac{3}{2}\sin(2x) - \frac{x^2}{4} + C$

<u>Spezielle Lösungen:</u>

$$y\left(\frac{\pi}{4}\right) = 2 \qquad : y(x) = \frac{3}{2}\sin(2x) - \frac{x^2}{4} + \frac{1}{2} + \frac{\pi^2}{64}$$

$$\simeq \frac{3}{2}\sin(2x) - \frac{x^2}{4} + 0.654$$

$$y\left(\frac{\pi}{8}\right) = 1 \qquad : y(x) = \frac{3}{2}\sin(2x) - \frac{x^2}{4} + 1 - \frac{3}{2\sqrt{2}} + \frac{\pi^2}{256} \simeq$$

$$\simeq \frac{3}{2}\sin(2x) - \frac{x^2}{4} - 0.022$$

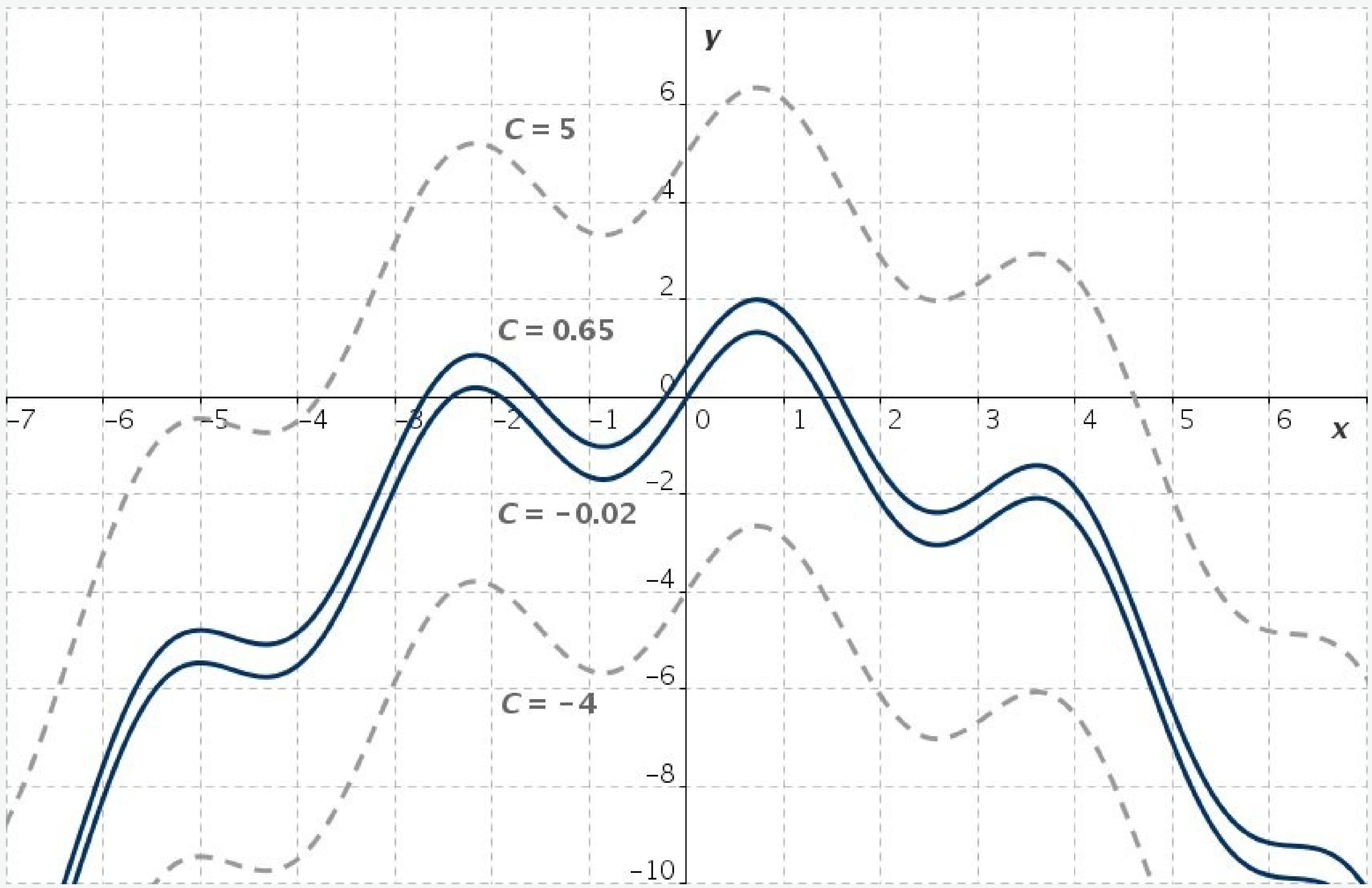



*Abb. L5: Integralkurven der DGL. Die blauen Kurven mit C = 0.65 bzw. C = -0.02 entsprechen den speziellen Lösungen der Gleichung mit y (π/4) = 2 bzw. y (π/8) = 1*

$$e^{-s}\left(1+\frac{ds}{dt}\right)=1, \qquad 1+\frac{ds}{dt}=e^{s}$$

$$\frac{ds}{dt}=e^{s}-1, \qquad \int\frac{ds}{e^{s}-1}=\int dt$$

$$u=e^{s}-1, \qquad du=e^{s}\,ds=(e^{s}-1+1)\,ds=(u+1)\,ds$$

$$ds=\frac{du}{u+1}, \qquad \int\frac{ds}{e^{s}-1}=\int\frac{du}{u(u+1)}$$

$$\int\frac{du}{u(u+1)}=\int\left(\frac{1}{u}-\frac{1}{u+1}\right)du=\ln|u|-\ln|u+1|=$$

$$=\ln\left|\frac{u}{u+1}\right|=\ln\left|\frac{e^{s}-1}{e^{s}}\right|$$

$$\ln\left|\frac{e^{s}-1}{e^{s}}\right|=t+\ln|C| \quad\Leftrightarrow\quad \ln\left|\frac{e^{s}-1}{C\,e^{s}}\right|=t$$

$$\ln\left|\frac{1-e^{-s}}{C}\right|=t, \qquad \frac{1-e^{-s}}{C}=e^{t}, \qquad 1-e^{-s}=C\,e^{t}$$

Allgemeine Lösung: $e^{-s}=1-C\,e^{t}, \qquad s(t)=-\ln\left(1-C\,e^{t}\right)$

$$e^{-s}\,(1 + s') = 1, \qquad a)\;\; s(0) = -2$$

Allgemeine Lösung: $\quad s(t) = -\ln\left(1 - C\,e^{t}\right)$

$$s(0) = -\ln\left(1 - C\,e^{0}\right) = -\ln(1 - C) = -2$$

$$-\ln(1 - C) = -2, \qquad \ln(1 - C) = 2, \qquad 1 - C = e^{2}$$

$$C = 1 - e^{2}$$

$$s(t) = -\ln\left(1 - C\,e^{t}\right) = -\ln\left(1 - \left(1 - e^{2}\right)e^{t}\right)$$

Spezielle Lösung *a*):

$$s(0) = -2: \qquad s(t) = -\ln\left(1 - \left(1 - e^{2}\right)e^{t}\right) = -\ln\left(\left(e^{2} - 1\right)e^{t} + 1\right)$$

Spezielle Lösung *b*):

$$C = 1 - e^{-2}$$

$$s(0) = 2: \qquad s(t) = -\ln\left(1 - \left(1 - e^{-2}\right)e^{t}\right) = -\ln\left(\left(e^{-2} - 1\right)e^{t} + 1\right)$$

Die beiden speziellen Lösungen sind in der Abb. 6L-1 auf der nächsten Seite dargestellt.

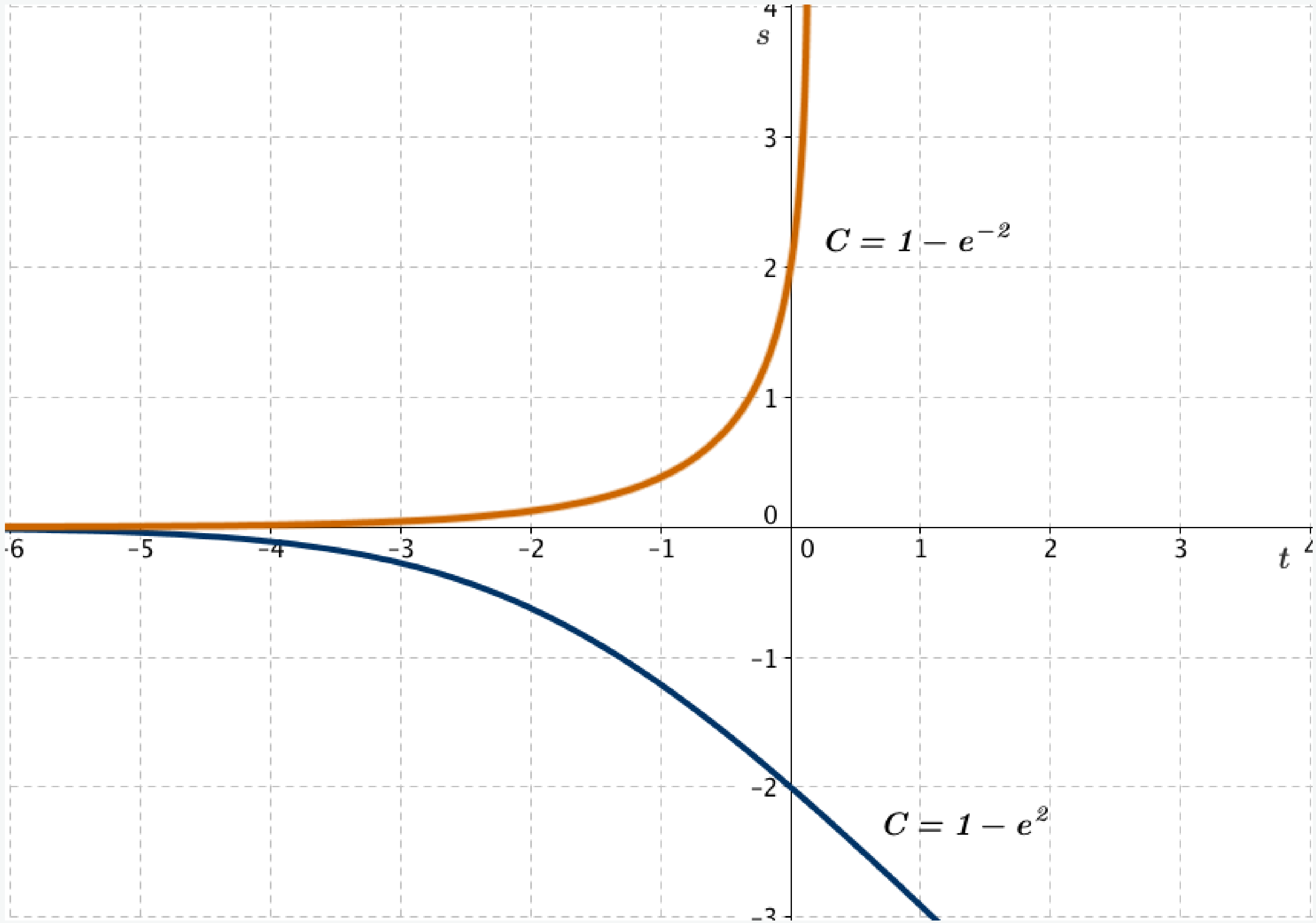



*Abb. 6L-1: Integralkurven der DGL, die blaue Kurve entspricht der speziellen Lösung a), die rote Kurve entspricht der speziellen Lösung b)*

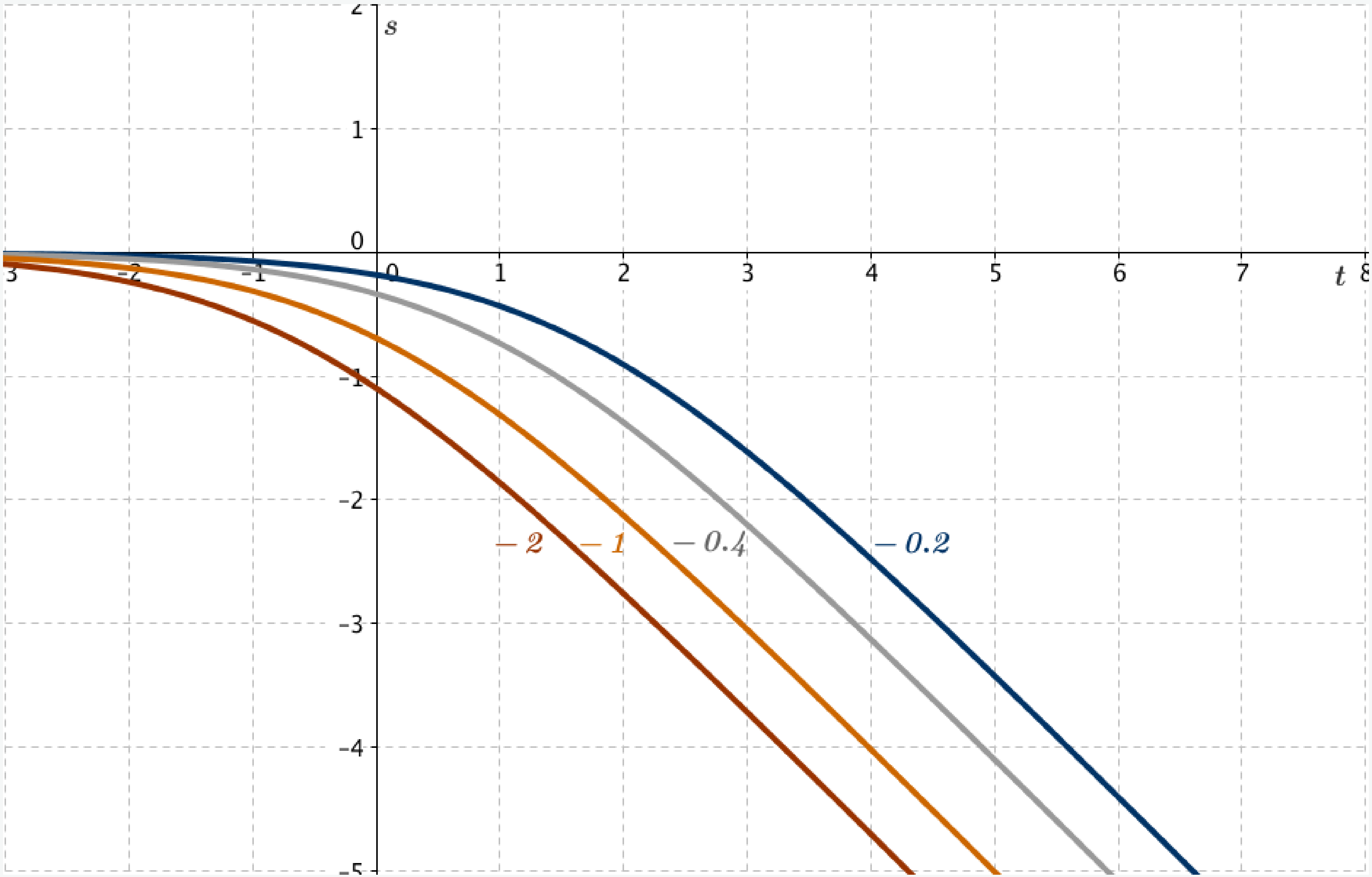



*Abb. 6L-2: Integralkurven der DGL, die den Werten -0.2, -04, -1 und -2 der Integrationskonstante C entsprechen*

$$s(t) = -\ln(1 - C e^{t})$$

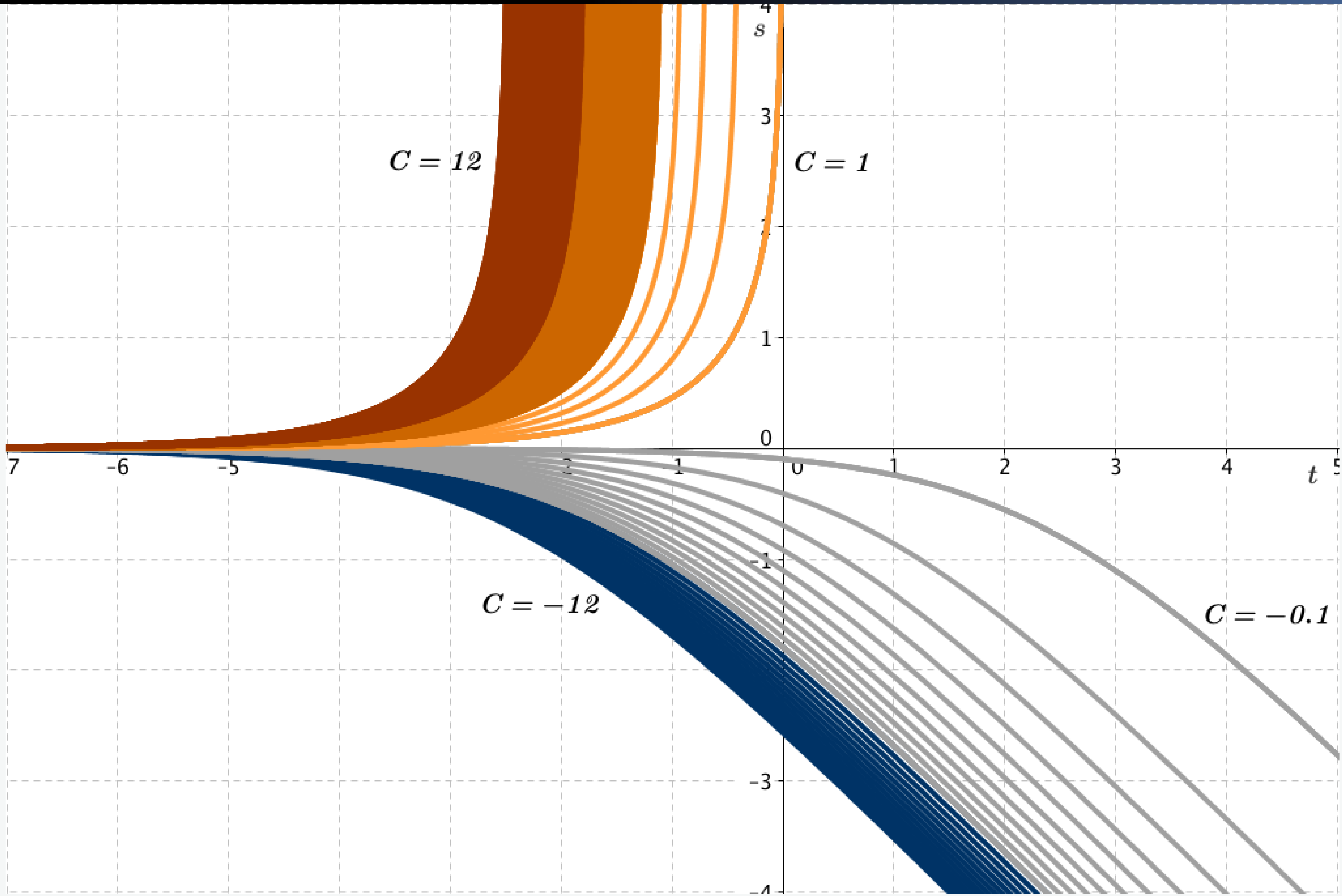
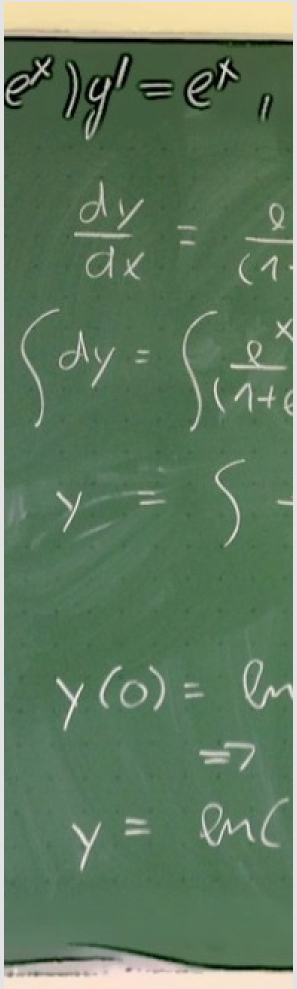



*Abb. 6L-3: Integralkurven der DGL*

$$s(t) = -\ln\left(1 - C e^{t}\right)$$

# Trennung der Variablen: Lösung 7



$$(1+e^{x})\, y' = e^{x}$$

$$\frac{dy}{dx} = \frac{e^{x}}{1+e^{x}}, \qquad \int dy = \int \frac{e^{x}\, dx}{1+e^{x}} = \int \frac{du}{u}$$

$$u = 1+e^{x}, \qquad \frac{du}{dx} = e^{x}, \qquad du = e^{x}\, dx$$

$$\int dy = \int \frac{du}{u}, \qquad y = \ln|u| + C = \ln(1+e^{x}) + C$$

<u>Allgemeine Lösung:</u> $\quad y = \ln(1+e^{x}) + C$

<u>Spezielle Lösung:</u> $\quad (1+e^{x})\, y' = e^{x}, \qquad y(0) = 0$

$$y(0) = \ln(1+e^{0}) + C = \ln(2) + C = 0, \qquad C = -\ln(2)$$

$$y = \ln(1+e^{x}) + C = \ln(1+e^{x}) - \ln(2) = \ln\left(\frac{1+e^{x}}{2}\right)$$

$$y = \ln\left(\frac{1+e^{x}}{2}\right)$$

# *Trennung der Variablen: Lösung 8*

$$\underline{(1+e^x)\, y\, y' = e^x}$$

$$y \cdot y' = \frac{e^x}{1+e^x}$$

$$y \frac{dy}{dx} = \frac{e^x}{1+e^x}$$

$$\int y\, dy = \int \frac{e^x}{1+e^x}\, dx$$

$$\int y\, dy = \int \frac{\cancel{e^x}}{u}\, \frac{du}{\cancel{e^x}}$$

$$\int y\, dy = \int \frac{du}{u}$$

$$\frac{y^2}{2} = \ln|u| + c$$

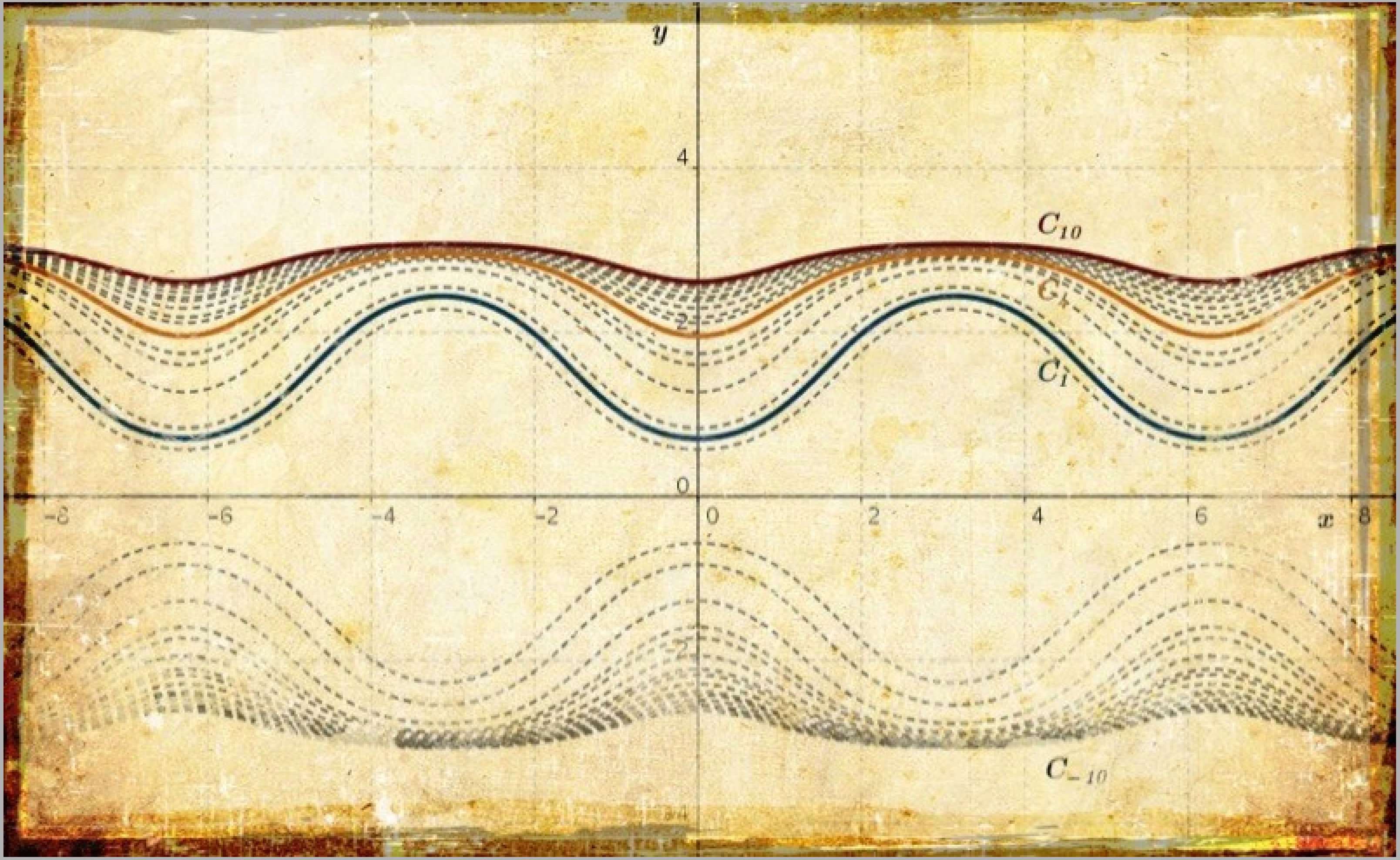
$$\frac{y^2}{2} = \ln(1+e^x) + c \quad \Big| \cdot 2$$

$$u = 1+e^x \qquad u' = e^x$$

$$dx = \frac{du}{u'} = \frac{du}{e^x}$$

$$C_1 = 2 \cdot C$$

$$y^2 = 2 \cdot \ln(1+e^x) + C_1$$



y
4
2
0
-2
-4
-8
-6
-4
-2
0
2
4
6
8
x
$C_{10}$
$C_1$
$C_{-10}$