



Integralrechnung: Aufgaben

Jörg Gayler, Lubov Vassilevskaya

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1. Unbestimmtes Integral: Aufgaben

1.1. Grund- oder Stammintegrale (Tabelle 1)

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1),$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int e^x dx = e^x + C,$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x dx = -\cos x + C,$$

$$\int \cos x dx = \sin x + C$$

$$\int \frac{dx}{\cos^2 x} = \tan x + C,$$

$$\int \frac{dx}{\sin^2 x} = -\cot x + C$$

$$\int \sinh x dx = \cosh x + C,$$

$$\int \cosh x dx = \sinh x + C$$

$$\int \frac{dx}{\cosh^2 x} = \tanh x + C,$$

$$\int \frac{dx}{\sinh^2 x} = -\coth x + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a} + C \quad (a \neq 0)$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C \quad (a \neq 0, |x| \neq |a|)$$

$$\int \frac{x dx}{a^2 \pm x^2} = \pm \frac{1}{2} \ln |a^2 \pm x^2| + C$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C \quad (a > 0)$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln \left| x + \sqrt{x^2 \pm a^2} \right| + C \quad (a > 0)$$

$$\int \frac{x dx}{\sqrt{a^2 \pm x^2}} = \pm \sqrt{a^2 \pm x^2} + C \quad (a > 0)$$

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} + C \quad (a > 0)$$

$$\int \sqrt{x^2 \pm a^2} dx = \frac{x}{2} \sqrt{x^2 \pm a^2} \pm \frac{a^2}{2} \ln \left| x + \sqrt{x^2 \pm a^2} \right| \quad (a > 0)$$

1.2. Unbestimmte Integrale (Tabelle 2)

$$\int \sin(ax) dx = -\frac{1}{a} \cos(ax) + C, \quad \int \cos(ax) dx = \frac{1}{a} \sin(ax) + C \quad (a \neq 0)$$

2. Einige Regeln

2.1. Binomische Formeln

Binome zweiten Grades

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(a + b)(a - b) = a^2 - b^2$$

Binome höheren Grades

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

2.2. Rechenregeln mit Potenzen

$$a^n \cdot a^m = a^{n+m}, \quad \frac{a^n}{a^m} = a^{n-m}, \quad (a^n)^m = a^{n \cdot m}$$

$$a^n \cdot b^n = (a \cdot b)^n, \quad \frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$$

$$a^0 = 1, \quad a^{-n} = \frac{1}{a^n}, \quad a^n = \frac{1}{a^{-n}}$$

2.3. Rechenregeln mit Logarithmen

$$\log_a(x \cdot y) = \log_a x + \log_a y$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a(x^r) = r \log_a x$$

$$\log_a \frac{1}{x} = -\log_a x$$

2.4. Trigonometrische Formeln

$$\sin^2 \alpha + \cos^2 \alpha = 1 \quad (1)$$

$$\sin(-\alpha) = -\sin \alpha, \quad \cos(-\alpha) = \cos \alpha \quad (2)$$

$$\sin \alpha = \cos\left(\frac{\pi}{2} - \alpha\right), \quad \cos \alpha = \sin\left(\frac{\pi}{2} - \alpha\right) \quad (3)$$

$$\sin \alpha = \sin(\pi - \alpha), \quad \cos \alpha = -\cos(\pi - \alpha) \quad (4)$$

$$\sin(2\alpha) = 2 \sin \alpha \cos \alpha, \quad \cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha \quad (5)$$

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta, \quad (6)$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta \quad (7)$$

$$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)], \quad (8)$$

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)] \quad (9)$$

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha - \beta) + \sin(\alpha + \beta)] \quad (10)$$

$$\sin^2 \alpha = \frac{1}{2} (1 - \cos 2\alpha), \quad \cos^2 \alpha = \frac{1}{2} (1 + \cos 2\alpha) \quad (11)$$

$$\sin^3 \alpha = \frac{1}{4} (3 \sin \alpha - \sin(3\alpha)), \quad \cos^3 \alpha = \frac{1}{4} (\cos(3\alpha) + 3 \cos \alpha) \quad (12)$$

2.5. Integration von Potenzfunktionen

A1

Beispiel:

$$\begin{aligned} I &= \int \left(5\sqrt[4]{x} + \frac{1}{x\sqrt[3]{x}} - \frac{1}{x^2} \right) dx = \int (5x^{1/4} + x^{-4/3} - x^{-2}) dx = 4x^{5/4} - \frac{3}{x^{1/3}} - \frac{1}{x} + C \\ &= 4x\sqrt[4]{x} - \frac{3}{\sqrt[3]{x}} - \frac{1}{x} + C \end{aligned}$$

$$a) \quad I_1 = \int (2x - 3) dx, \quad I_2 = \int (4x^2 - 2x - 3) dx, \quad I_3 = \int (x^2 - 3x) \cdot (2x - 1) dx$$

$$b) \quad I_1 = \int \left(3x^2 + 2x - \frac{6}{x^2} \right) dx, \quad I_2 = \int \left(\frac{4}{x^3} + \frac{6}{x^4} - \frac{10}{x^6} \right) dx$$

$$c) \quad I_1 = \int \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right) dx, \quad I_2 = \int (\sqrt{x} + \sqrt[3]{x} + \sqrt[5]{x}) dx, \quad I_3 = \int x(5\sqrt{x} - 7\sqrt[3]{x}) dx$$

$$d) \quad I_1 = \int \left(\frac{1}{x\sqrt{x}} - \frac{1}{x^2} \right) dx, \quad I_2 = \int \left(\frac{1}{\sqrt{x}} - \frac{6}{\sqrt[3]{x}} + \frac{20}{\sqrt[5]{x}} \right) dx$$

$$e) \quad I_1 = \int \left(\frac{1}{\sqrt[3]{x}} + x\sqrt{x} + 2 \right) dx, \quad I_2 = \int \left(\frac{3}{4\sqrt[4]{x}} - 7x^2\sqrt{x} \right) dx$$

$$f) \quad I_1 = \int \left(\sqrt{x}\sqrt{x} - \frac{2}{\sqrt{x}} \right) dx, \quad I_2 = \int \sqrt[3]{x^2}\sqrt{x} dx$$

A2

$$a) \quad I_1 = \int (x^3 + 1)^2 dx, \quad I_2 = \int (x^2 - 1)^3 dx, \quad I_3 = \int x(x^4 + 1)^2 dx$$

$$b) \quad I_1 = \int \frac{x^3 + 3x - 1}{x} dx, \quad I_2 = \int \frac{x(x^2 - 1)(x + 2)}{x + 1} dx, \quad I_3 = \int \frac{x^2 - 3x + 4}{\sqrt{x}} dx$$

$$c) \quad I_1 = \int \frac{(x - 1)^3}{\sqrt{x}} dx, \quad I_2 = \int \frac{(\sqrt{x} + 2)^2}{\sqrt{x}} dx, \quad I_3 = \int \frac{x - 1}{\sqrt{x + 1}} dx$$

2.6. Elementarintegrale

A3

$$a) \quad I_1 = \int (e^x - 2 \sin x) dx, \quad I_2 = \int \left(2^x + \sqrt{\frac{1}{x}} \right) dx$$

$$b) \quad I_1 = \int \left(\cos x + \frac{3}{\sqrt{4 - 4x^2}} \right) dx, \quad I_2 = \int \left(\sin x - \frac{5}{\sqrt{9 - 9x^2}} \right) dx$$

$$c) \quad I_1 = \int \left(\frac{1}{\sqrt{2 - 2x^2}} - 3^{-x} \right) dx, \quad I_2 = \int \left(10^{-x} + \frac{x^2 + 2}{1 + x^2} \right) dx$$

A4

Beispiel:

$$\begin{aligned}
 I &= \int \frac{x^2 - 3}{2x^2 - 2} dx = \frac{1}{2} \int \frac{x^2 - 3}{x^2 - 1} dx = \frac{1}{2} \int \frac{x^2 - 1 - 2}{x^2 - 1} dx = \frac{1}{2} \left[\int dx - 2 \int \frac{dx}{x^2 - 1} \right] = \\
 &= \frac{1}{2} \int dx + \int \frac{dx}{1 - x^2} = \frac{x}{2} + \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| + C
 \end{aligned}$$

$$I_1 = \int \frac{x^2}{2(1+x^2)} dx, \quad I_2 = \int \frac{x^2}{x^2 - 1} dx, \quad I_3 = \int \frac{x^2 + 2}{x^2 - 1} dx$$

A5

Beispiel:

$$I = \int \frac{5}{2x^2 + 7} dx = \frac{5}{2} \int \frac{1}{x^2 + \frac{7}{2}} dx = \frac{5}{2} \int \frac{1}{x^2 + \left(\sqrt{\frac{7}{2}}\right)^2} dx = \frac{5}{2} \int \frac{dx}{x^2 + a^2} = \frac{5}{\sqrt{14}} \arctan \frac{\sqrt{2}x}{\sqrt{7}} + C,$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + C, \quad a = \sqrt{\frac{7}{2}}$$

$$I_1 = \int \frac{dx}{x^2 + 4}, \quad I_2 = \int \frac{dx}{x^2 + 3}, \quad I_3 = \int \frac{3}{4x^2 + 3} dx$$

2.7. Integration durch Substitution

$$\int f(g(x))g'(x)dx = \int f(u)du, \quad u = g(x), \quad du = g'(x)dx \quad (13)$$

Das Integral eines Produktes lässt sich immer dann berechnen, wenn ein Faktor $f(g(x))$ eine Funktion einer inneren Funktion und der andere Faktor die Ableitung $g'(x)$ der inneren Funktion ist, sofern für die Funktion f unter Beachtung der Substitution $u = g(x)$ das Integral $\int f(u)du$ gelöst werden kann.

A6

Beispiel 1:

$$I = \int (2x+3)^5 dx, \quad u = 2x+3, \quad \frac{du}{dx} = 2, \quad dx = \frac{1}{2} du$$

$$I = \int (2x+3)^5 dx = \frac{1}{2} \int u^5 du = \frac{1}{2} \frac{u^6}{6} + C = \frac{1}{12} (2x+3)^6 + C$$

Beispiel 2:

$$I = \int x^2 (6-x^3)^4 dx, \quad u = 6-x^3, \quad \frac{du}{dx} = -3x^2, \quad x^2 dx = -\frac{du}{3}$$

$$I = \int x^2 (6-x^3)^4 dx = -\frac{1}{3} \int u^4 du = -\frac{1}{3} \frac{u^5}{5} + C = -\frac{1}{15} (6-x^3)^5 + C$$

$$a) \quad I_1 = \int (2+x)^3 dx, \quad I_2 = \int (1+3x)^4 dx, \quad I_3 = \int (4-2x)^5 dx$$

$$b) \quad I_1 = \int x(x^2+1)^2 dx, \quad I_2 = \int x(2x^2-3)^3 dx, \quad I_3 = \int 2x(3-x^2)^6 dx$$

$$c) \quad I_1 = \int 3x^2(1+x^3)^2 dx, \quad I_2 = \int 5x^3(7-x^4)^4 dx, \quad I_3 = \int x^5(5-2x^6)^3 dx$$

A7

Beispiel:

$$I = \int (6x-9)(x^2-3x+11)^2 dx, \quad u = x^2-3x+11, \quad \frac{du}{dx} = 2x-3, \quad (2x-3)dx = du$$

$$I = \int (6x-9)(x^2-3x+11)^2 dx = 3 \int u^2 du = u^3 + C = (x^2-3x+11)^3 + C$$

$$I_1 = \int (x-1)(x^2 - 2x + 1) dx$$

$$I_2 = \int (3x^2 + 2)(2x^3 + 4x - 7) dx$$

$$I_3 = \int x(2x^2 - x + 1) \left(x^4 - \frac{2}{3}x^3 + x^2 - 12\right)^2 dx$$

$$I_4 = \int x^2(2x^3 - 1)(x^6 - x^3 + 9)^3 dx$$

A8

Beispiel:

$$\begin{aligned} I &= \int \frac{x}{2x-6} dx = \frac{1}{2} \int \frac{x}{x-3} dx = \frac{1}{2} \int \frac{x-3+3}{x-3} dx = \frac{1}{2} \int dx + \frac{3}{2} \int \frac{dx}{x-3} = \\ &= \frac{x}{2} + \frac{3}{2} \ln|x-3| + C \end{aligned}$$

$$I_1 = \int \frac{dx}{2+x}, \quad I_2 = \int \frac{dx}{x-3}, \quad I_3 = \int \frac{x}{x+4} dx$$

A9

Beispiel:

$$I = \int \frac{x(9x-4)dx}{(3x^3-2x^2+7)^5} = \int \frac{du}{u^5} = -\frac{1}{4u^4} + C = -\frac{1}{4(3x^3-2x^2+7)^4} + C$$

$$u = 3x^3 - 2x^2 + 7, \quad \frac{du}{dx} = 9x^2 - 4x = x(9x-4), \quad dx = \frac{du}{x(9x-4)}$$

$$a) \quad I_1 = \int \frac{(x+1)dx}{(x^2+2x-3)^2}, \quad I_2 = \int \frac{(3x-2)dx}{(3x^2-4x-11)^3}, \quad I_3 = \int \frac{x(2x^2-3)dx}{(x^4-3x^2+12)^4}$$

A10

Beispiel:

$$I = \int (4-x^2) \sqrt[3]{12x-x^3} dx = \frac{1}{3} \int u^{1/3} du = \frac{1}{4} x(12-x^2) \sqrt[3]{12x-x^3}$$

$$u = 12x - x^3, \quad \frac{du}{dx} = 12 - 3x^2 = 3(4-x^2), \quad dx = \frac{du}{3(4-x^2)}$$

$$a) \quad I_1 = \int \sqrt{x-2} dx, \quad I_2 = \int \sqrt{3x-6} dx, \quad I_3 = \int \sqrt[3]{2x+7}$$

$$b) \quad I_1 = \int \sqrt[3]{(4-3x)^2} dx, \quad I_2 = \int x \sqrt{x^2+3} dx, \quad I_3 = \int (6x^2-4) \sqrt[4]{x^3-2x+11} dx,$$

A11

Beispiel 1:

$$I = \int e^{7-3x} dx = -\frac{1}{3} \int e^u du = -\frac{1}{3} e^u + C$$

$$u = 7 - 3x, \quad \frac{du}{dx} = -3, \quad dx = -\frac{1}{3} du$$

Beispiel 2:

$$I = \int \frac{dx}{4e^x - 2} = \frac{1}{2} \int \frac{dx}{2e^x - 1} = \frac{1}{2} \int \frac{du}{u(u+1)} = \frac{1}{2} \int \left(\frac{1}{u} - \frac{1}{u+1} \right) du =$$

$$= \frac{1}{2} \left(\int \frac{du}{u} - \int \frac{du}{u+1} \right) = \frac{1}{2} \left(\int \frac{du}{u} - \int \frac{dv}{v} \right) = \frac{1}{2} (\ln u - \ln v) + C_1 = \frac{1}{2} (\ln u - \ln(u+1)) + C_1$$

$$= \frac{1}{2} (\ln(2e^x - 1) - \ln(2e^x)) + C_1 = \frac{1}{2} \ln(2e^x - 1) - \frac{1}{2} (\ln 2 + \ln(e^x)) + C_1 =$$

$$= \frac{1}{2} \ln(2e^x - 1) - \frac{1}{2} x + C_2, \quad C_2 = -\frac{1}{2} \ln 2 + C_1$$

$$u = 2e^x - 1, \quad \frac{du}{dx} = 2e^x = u + 1, \quad dx = \frac{du}{u+1}$$

$$v = u + 1, \quad dv = du$$

$$a) \quad I_1 = \int e^{2x} dx, \quad I_2 = \int e^{3-2x} dx, \quad I_3 = \int e^{mx+n} dx \quad (m, n \neq 0)$$

$$b) \quad I_1 = \int x e^{x^2} dx, \quad I_2 = \int x^2 e^{2x^3} dx, \quad I_3 = \int (x+1) e^{x^2+2x-3} dx$$

$$c) \quad I_1 = \int \frac{dx}{e^x - 1}, \quad I_2 = \int \frac{dx}{2e^x - 3}, \quad I_3 = \int \frac{e^x dx}{3e^x + 5}$$

A12

$$a) \quad I_1 = \int_0^{\pi} \sin(2x) \cos x dx, \quad I_2 = \int_0^{\pi/2} \sin(3x) \cos x dx$$

$$b) \quad I_1 = \int_0^{\pi/4} \sin(2x) \sin x dx, \quad I_2 = \int_0^{\pi/3} \sin(4x) \sin(2x) dx$$

$$c) \quad I_1 = \int_0^{\pi/4} \cos(2x) \cos x dx, \quad I_2 = \int_0^{\pi/6} \cos(4x) \cos(2x) dx$$

A13

$$I_a = \int \sin^2(ax) dx, \quad I_b = \int \cos^2(ax) dx$$

A14

Beispiel:

$$\begin{aligned} I &= \int \frac{\sin(4x)}{12 + 8 \cos(4x)} dx = \frac{1}{4} \int \frac{\sin(4x)}{3 + 2 \cos(4x)} dx = \frac{1}{4} \int \frac{\sin(4x)}{u} \left(-\frac{1}{8}\right) \frac{du}{\sin(4x)} = \\ &= -\frac{1}{32} \int \frac{du}{u} = -\frac{1}{32} \ln|u| + C = -\frac{1}{32} \ln|3 + 2 \cos(4x)| + C, \end{aligned}$$

$$u = 3 + 2 \cos(4x), \quad \frac{du}{dx} = -2 \cdot \sin(4x) \cdot 4 = -8 \sin(4x), \quad dx = -\frac{1}{8} \frac{du}{\sin(4x)}$$

$$a) \quad I_1 = \int \frac{\cos x}{1 + \sin x} dx, \quad I_2 = \int \frac{\sin x}{1 + \cos x} dx, \quad I_3 = \int \frac{\cos x}{3 + 2 \sin x} dx$$

$$b) \quad I_1 = \int \frac{\sin x}{7 - 3 \cos x} dx, \quad I_2 = \int \frac{\sin(2x)}{1 + \cos(2x)} dx, \quad I_3 = \int \frac{\cos(3x)}{4 + 2 \sin(3x)} dx$$

A15

Beispiel 1:

$$I = \int \sin^3 x \cos x dx = \int u^3 \cos x \frac{du}{\cos x} = \int u^3 du = \frac{u^4}{4} + C = \frac{1}{4} \sin^4 x + C$$

$$u = \sin x, \quad \frac{du}{dx} = \cos x, \quad dx = \frac{du}{\cos x}$$

Beispiel 2:

$$\begin{aligned} I &= \int \cos^6(2x) \sin(4x) dx = 2 \int \cos^6(2x) \sin(2x) \cos(2x) dx = 2 \int \cos^7(2x) \sin(2x) dx = \\ &= 2 \int u^7 \sin(2x) \left(-\frac{du}{2 \sin(2x)}\right) = -\int u^7 du = -\frac{u^8}{8} + C = -\frac{1}{8} \cos^8(2x) + C \end{aligned}$$

$$u = \cos(2x), \quad \frac{du}{dx} = -2 \sin(2x), \quad dx = -\frac{du}{2 \sin(2x)}$$

$$a) \quad I_1 = \int \sin^5 x \cos x dx, \quad I_2 = \int \cos^5 x \sin x dx, \quad I_3 = \int \sin^7 x \cos x dx$$

$$b) \quad I_1 = \int \sin^3 x \sin(2x) dx, \quad I_2 = \int \cos^5 x \sin(2x) dx, \quad I_3 = \int \sin^8(2x) \sin(4x) dx$$

Hinweis: $\sin(2x) = 2 \sin x \cos x$

A16

Beispiel:

$$I = \int \frac{\cos x}{1 + \sin^2 x} dx = \int \frac{(\sin x)'}{1 + \sin^2 x} dx = \int \frac{\cos x}{1 + u^2} \cdot \frac{du}{\cos x} = \int \frac{du}{1 + u^2} = \arctan u + C =$$

$$= \arctan(\sin x) + C,$$

$$u = \sin x, \quad \frac{du}{dx} = \cos x, \quad dx = \frac{du}{\cos x}$$

Grundintegral: $\int \frac{dx}{1 + x^2} = \arctan x + C$

$$a) \quad I_1 = \int \frac{\sin x}{1 + \cos^2 x} dx, \quad I_2 = \int \frac{\cos x}{4 + \sin^2 x} dx, \quad I_3 = \int \frac{3 \sin x}{6 + 2 \cos^2 x} dx$$

$$b) \quad I_1 = \int \frac{\sin(2x)}{1 + \cos^2(2x)} dx, \quad I_2 = \int \frac{\cos(2x)}{4 + \sin^2(2x)} dx, \quad I_3 = \int \frac{\sin(3x)}{2 + 5 \cos^2(3x)} dx$$

A17

Beispiel:

$$I = \int \frac{dx}{1 + \sqrt{x}} = 2 \int \frac{u du}{1 + u} = 2 \int \frac{(u + 1 - 1) du}{1 + u} = 2 \int \left(1 - \frac{1}{1 + u}\right) du =$$

$$= 2 \left(\int du - \int \frac{dv}{v} \right) = 2(u - \ln|v|) + C = 2(u - \ln|1 + u|) + C =$$

$$= 2\sqrt{x} - 2 \ln|1 + \sqrt{x}| + C$$

$$u = \sqrt{x}, \quad \frac{du}{dx} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}, \quad dx = 2\sqrt{x} du = 2u du$$

$$v = 1 + u, \quad dv = du$$

$$a) \quad I_1 = \int \frac{dx}{2 + \sqrt{x+1}}, \quad I_2 = \int \frac{dx}{4 + \sqrt{x-3}}, \quad I_3 = \int \frac{dx}{1 + \sqrt{3x-2}}$$

$$b) \quad I_1 = \int \frac{\sqrt{x} dx}{x+4}, \quad I_2 = \int \frac{\sqrt{x} dx}{x+a^2}, \quad I_3 = \int \frac{x \sqrt{x} dx}{x+9}$$

A18

$$a) \quad I_1 = \int \frac{\sqrt[3]{x+1}}{\sqrt{x}} dx, \quad I_2 = \int \frac{\sqrt{x+1}}{\sqrt[3]{x+1}} dx, \quad I_3 = \int \frac{\sqrt{2x+5}}{\sqrt[4]{2x+5}} dx$$

$$b) \quad I_1 = \int \frac{4\sqrt{x+2}-3}{\sqrt[3]{x+2}} dx, \quad I_2 = \int \frac{dx}{x\sqrt{x+1}}, \quad I_3 = \int \frac{dx}{x\sqrt{2x-3}}$$

A19

$$I_1 = \int x \cdot \sqrt{x+2} dx, \quad I_2 = \int x \cdot \sqrt{x+5} dx, \quad I_3 = \int x^2 \cdot \sqrt{2x-5} dx.$$

2.8. Partielle Integration

$$\int u v' dx = u v - \int u' v dx \quad (14)$$

A20

Beispiel 1:

$$I = \int x \ln x dx = \frac{x^2}{2} \ln x - \int \frac{1}{x} \frac{x^2}{2} dx = \frac{x^2}{2} \left(\ln x - \frac{1}{2} \right) + C,$$

$$u = \ln x, \quad u' = \frac{1}{x}, \quad v' = x, \quad v = \int x dx = \frac{x^2}{2}$$

Beispiel 2:

$$I = \int \ln^2 x dx = x \ln^2 x - 2 \int \ln x dx = x \ln^2 x - 2x \ln x + 2x + C$$

$$\int \ln x dx = x(\ln x - 1) + C$$

$$u = \ln^2 x, \quad u' = \frac{2}{x} \ln x, \quad v' = 1, \quad v = x$$

Beispiel 2:

$$I = \int \sqrt{x} \ln x dx = \frac{2}{3} x^{3/2} \ln x - \frac{2}{3} \int x^{1/2} dx = \frac{2}{3} x^{3/2} \left(\ln x - \frac{2}{3} \right) + C =$$

$$= \frac{2}{3} x \sqrt{x} \left(\ln x - \frac{2}{3} \right) + C$$

$$u = \ln x, \quad u' = \frac{1}{x}, \quad v' = \sqrt{x}, \quad v = \int \sqrt{x} dx = \int x^{1/2} dx = \frac{2}{3} x^{3/2}$$

$$a) \quad I_1 = \int \ln^3 x dx, \quad I_2 = \int \ln^4 x dx, \quad I_3 = \int \ln^5 x dx$$

$$b) \quad I_1 = \int x^2 \ln x dx, \quad I_2 = \int x^3 \ln x dx, \quad I_3 = \int x \ln(x^2) dx$$

$$c) \quad I_1 = \int (x+1) \ln x dx, \quad I_2 = \int (x+1)^2 \ln x dx, \quad I_3 = \int x \ln(x+1) dx$$

$$d) \quad I_1 = \int \sqrt{x} \ln(\sqrt{x}) dx, \quad I_2 = \int \sqrt[3]{x} \ln x dx, \quad I_3 = \int \sqrt[5]{x^2} \ln x dx$$

A21

$$I_1 = \int \frac{\ln x}{x} dx, \quad I_2 = \int \frac{\ln x}{x^2} dx, \quad I_3 = \int \frac{\ln x}{x^3} dx$$

A22

Beispiel 1:

$$I = \int x e^x dx = x e^x - \int e^x dx = (x - 1) e^x + C,$$

$$u = x, \quad u' = 1, \quad v' = e^x, \quad v = \int e^x dx = e^x$$

$$I_1 = \int x^2 e^x dx, \quad I_2 = \int x^3 e^x dx, \quad I_3 = \int x^4 e^x dx$$

A23

$$a) \quad I_1 = \int x \sin x dx, \quad I_2 = \int x \cos x dx, \quad I_3 = \int x \sin(2x) dx, \quad I_4 = \int x \cos(2x) dx$$

$$b) \quad I_1 = \int x^2 \sin x dx, \quad I_2 = \int x^2 \cos x dx, \quad I_3 = \int x^2 \cos(2x) dx, \quad I_4 = \int x^2 \sin(3x) dx$$

3. Anwendungen der Integralrechnung

3.1. Flächen zwischen Kurven: Aufgaben

3.1.1. Eingeschlossene Flächen

Berechnen Sie die von den Kurven $f(x)$ und $g(x)$ eingeschlossene Fläche

A24 $f(x) = x^2 - 2$, $g(x) = 2$

A25 $f(x) = 2x - x^2$, $g(x) = -3$

A26 $f(x) = x^2 - 2x$, $g(x) = x$

3.1.2. Flächen zwischen Kurven in einem Intervall

Welche Fläche schließen die Kurven im Intervall I

A27 $f(x) = x^2 - 2$, $g(x) = 2$, $I = [-2, 3]$

A28 $f(x) = x^2 - 2x$, $g(x) = x$, $I = [-1, 3]$

A29 $f(x) = x^2 - 2x$, $g(x) = x + 1$, $I = [-1, 2]$

4. Unbestimmtes Integral: Lösungen

4.1. Integration von Potenzfunktionen

L1

$$a) \quad I_1 = \int (2x - 3) dx = x^2 - 3x + C, \quad I_2 = \int (4x^2 - 2x - 3) dx = \frac{4}{3}x^3 - x^2 - 3x + C$$

$$I_3 = \int (x^2 - 3x) \cdot (2x - 1) dx = \frac{1}{2}x^4 - \frac{7}{3}x^3 + \frac{3}{2}x^2 + C$$

$$b) \quad I_1 = \int \left(3x^2 + 2x - \frac{6}{x^2} \right) dx = x^3 + x^2 + \frac{6}{x} + C,$$

$$I_2 = \int \left(\frac{4}{x^3} + \frac{6}{x^4} - \frac{10}{x^6} \right) dx = -\frac{2}{x^2} \left(1 + \frac{1}{x} - \frac{1}{x^3} \right) + C$$

$$c) \quad I_1 = \int \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right) dx = \int (x^{1/2} + x^{-1/2}) dx = \frac{2}{3} \sqrt{x}(3 + x) + C,$$

$$I_2 = \int (\sqrt{x} + \sqrt[3]{x} + \sqrt[5]{x}) dx = \frac{2}{3}x^{3/2} + \frac{3}{4}x^{4/3} + \frac{5}{6}x^{6/5} + C = \frac{2}{3}x\sqrt{x} + \frac{3}{4}x\sqrt[3]{x} + \frac{5}{6}x\sqrt[5]{x} + C,$$

$$I_3 = \int x(5\sqrt{x} - 7\sqrt[3]{x}) dx = 2x^{5/2} - 3x^{7/3} + C = x^2(2\sqrt{x} - 3\sqrt[3]{x}) + C$$

$$d) \quad I_1 = \int \left(\frac{1}{x\sqrt{x}} - \frac{1}{x^2} \right) dx = -\frac{2}{\sqrt{x}} + \frac{1}{x} + C,$$

$$I_2 = \int \left(\frac{1}{\sqrt{x}} - \frac{6}{\sqrt[3]{x}} + \frac{20}{\sqrt[5]{x}} \right) dx = 2\sqrt{x} - 9x^{2/3} + 25x^{4/5} + C$$

$$e) \quad I_1 = \int \left(\frac{1}{\sqrt[3]{x}} + x\sqrt{x} + 2 \right) dx = \frac{3}{2}x^{2/3} + \frac{2}{5}x^{5/2} + 2x + C = \frac{3}{2}\sqrt[3]{x^2} + \frac{2}{5}x^2\sqrt{x} + 2x + C,$$

$$I_2 = \int \left(\frac{3}{4\sqrt[4]{x}} - 7x^2\sqrt{x} \right) dx = x^{3/4} - 2x^{7/2} + C = x^{3/4} - 2x^3\sqrt{x} + C$$

$$f) \quad I_1 = \int \left(\sqrt{x}\sqrt{x} - \frac{2}{\sqrt{x}} \right) dx = \int (x^{3/4} - 2x^{-1/2}) dx = \frac{4}{7}x \cdot x^{3/4} - 4\sqrt{x} = \frac{4}{7}x\sqrt[4]{x^3} - 4\sqrt{x}$$

$$I_2 = \int \sqrt[3]{x^2}\sqrt{x} dx = \frac{6}{11}x \cdot x^{5/6} = \frac{6}{11}x\sqrt[6]{x^5}$$

L2

$$a) I_1 = \int (x^3 + 1)^2 dx = \frac{1}{7} x^7 + \frac{1}{2} x^4 + x + C,$$

$$I_2 = \int (x^2 - 1)^3 dx = \frac{1}{7} x^7 - \frac{3}{5} x^5 + x^3 - x + C,$$

$$I_3 = \int x(x^4 + 1)^2 dx = \frac{1}{10} x^{10} + \frac{1}{3} x^6 + \frac{1}{2} x^2$$

$$b) I_1 = \int \frac{x^3 + 3x - 1}{x} dx = \frac{1}{3} x^3 + 3x - \ln x + C,$$

$$I_2 = \int \frac{x(x^2 - 1)(x + 2)}{x + 1} dx = \int x(x - 1)(x + 2) = \frac{1}{4} x^4 + \frac{1}{3} x^3 - x^2 + C,$$

$$I_3 = \int \frac{x^2 - 3x + 4}{\sqrt{x}} dx = \frac{2}{5} \sqrt{x}(20 - 5x + x^2) + C,$$

$$c) I_1 = \int \frac{(x - 1)^3}{\sqrt{x}} dx = \frac{2}{35} \sqrt{x}(-35 + 35x - 21x^2 + 5x^3) + C,$$

$$I_2 = \int \frac{(\sqrt{x} + 2)^2}{\sqrt{x}} dx = \frac{2}{3} x^{3/2} + 4x + 8\sqrt{x} + C = \frac{2}{3} x \sqrt{x} + 4x + 8\sqrt{x} + C,$$

$$I_3 = \int \frac{x - 1}{\sqrt{x + 1}} dx = \int (\sqrt{x} - 1) dx = \frac{2}{3} x^{3/2} - x + C$$

4.2. Elementarintegrale

L3

$$a) I_1 = \int (e^x - 2 \sin x) dx = e^x + 2 \cos x + C,$$

$$I_2 = \int \left(2^x + \sqrt{\frac{1}{x}} \right) dx = \frac{2^x}{\ln 2} + 2\sqrt{x} + C$$

$$b) I_1 = \int \left(\cos x + \frac{3}{\sqrt{4 - 4x^2}} \right) dx = \int \left(\cos x + \frac{3}{2\sqrt{1 - x^2}} \right) dx = \sin x + \frac{3}{2} \arcsin x + C,$$

$$I_2 = \int \left(\sin x - \frac{5}{\sqrt{9 - 9x^2}} \right) dx = \int \left(\sin x - \frac{5}{3\sqrt{1 - x^2}} \right) dx = -\cos x - \frac{5}{3} \arcsin x + C$$

$$c) I_1 = \int \left(\frac{1}{\sqrt{2 - 2x^2}} - 3^{-x} \right) dx = \frac{1}{\sqrt{2}} \arcsin x + \frac{3^{-x}}{\ln 3} + C,$$

$$I_2 = \int \left(10^{-x} + \frac{x^2 + 2}{1 + x^2} \right) dx = -\frac{10^{-x}}{\ln(10)} + x + \arctan x + C$$

L4

$$I_1 = \int \frac{x^2}{2(1+x^2)} dx = \frac{1}{2} \int \frac{x^2+1-1}{1+x^2} dx = \frac{x}{2} - \frac{1}{2} \arctan x + C$$

$$I_2 = \int \frac{x^2}{x^2-1} dx = \int \frac{x^2-1+1}{x^2-1} dx = x - \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| + C$$

$$I_3 = \int \frac{x^2+2}{x^2-1} dx = \int \frac{(x^2-1+1)+2}{x^2-1} dx = x - \frac{3}{2} \ln \left| \frac{1+x}{1-x} \right| + C$$

L5

$$I_1 = \int \frac{dx}{x^2+4} = \frac{1}{2} \arctan \left(\frac{x}{2} \right) + C,$$

$$I_2 = \int \frac{dx}{x^2+3} = \frac{1}{\sqrt{3}} \arctan \left(\frac{x}{\sqrt{3}} \right) + C,$$

$$I_3 = \int \frac{3 dx}{4x^2+3} = \frac{\sqrt{3}}{2} \arctan \left(\frac{2x}{\sqrt{3}} \right) + C$$

4.3. Integration durch Substitution

L6

$$\begin{aligned}
 a) \quad I_1 &= \int (2+x)^3 dx = \frac{(2+x)^4}{4} + C, & u &= 2+x, \\
 I_2 &= \int (1+3x)^4 dx = \frac{(1+3x)^5}{15} + C, & u &= 1+3x, \\
 I_3 &= \int (4-2x)^5 dx = -\frac{(4-2x)^6}{12} + C, & u &= 4-2x
 \end{aligned}$$

$$\begin{aligned}
 b) \quad I_1 &= \int x(x^2+1)^2 dx = \frac{1}{2} \int u^2 du = \frac{(x^2+1)^3}{6} + C, & u &= x^2+1, \\
 I_2 &= \int x(2x^2-3)^3 dx, & u &= 2x^2-3, \\
 I_3 &= \int 2x(3-x^2)^6 dx, & u &= 3-x^2
 \end{aligned}$$

$$\begin{aligned}
 c) \quad I_1 &= \int 3x^2(1+x^3)^2 dx, & u &= 1+x^3, \\
 I_2 &= \int 5x^3(7-x^4)^4 dx, & u &= 7-x^4, \\
 I_3 &= \int x^5(5-2x^6)^3 dx, & u &= 5-2x^6
 \end{aligned}$$

L7

$$I_1 = \int (x-1)(x^2-2x+1) dx = \frac{1}{4}(x^2-2x+1)^2 + C, \quad u = x^2-2x+1, \quad dx = \frac{du}{2(x-1)}$$

$$I_2 = \int (3x^2+2)(2x^3+4x-7) dx = \frac{1}{4}(2x^3+4x-7)^2 + C,$$

$$u = 2x^3+4x-7, \quad dx = \frac{du}{2(3x^2+2)}$$

$$I_3 = \int x(2x^2-x+1) \left(x^4 - \frac{2}{3}x^3 + x^2 - 12\right)^2 dx = \frac{1}{6} \left(x^4 - \frac{2}{3}x^3 + x^2 - 12\right)^3,$$

$$u = x^4 - \frac{2}{3}x^3 + x^2 - 12, \quad dx = \frac{du}{2x(2x^2-x+1)}$$

$$I_4 = \int x^2(2x^3-1)(x^6-x^3+9)^3 dx = \frac{1}{12}(x^6-x^3+9)^4,$$

$$u = x^6 - x^3 + 9, \quad dx = \frac{du}{3x^2(2x^3-1)}$$

L8

$$I_1 = \int \frac{dx}{2+x} = \ln|2+x| + C,$$

$$I_2 = \int \frac{dx}{x-3} = \ln|x-3| + C,$$

$$I_3 = \int \frac{x}{x+4} dx = \int \frac{x+4-4}{x+4} dx = \int dx - 4 \int \frac{dx}{x+4} = x - 4 \ln|x+4| + C$$

L9

$$a) \quad I_1 = \int \frac{(x+1)dx}{(x^2+2x-3)^2} = -\frac{1}{2(x^2+2x-3)} + C, \quad u = x^2+2x-3,$$

$$I_2 = \int \frac{(3x-2)dx}{(3x^2-4x-11)^3} = -\frac{1}{4(3x^2-4x-11)^2} + C, \quad u = 3x^2-4x-11,$$

$$I_3 = \int \frac{x(2x^2-3)dx}{(x^4-3x^2+12)^4} = -\frac{1}{6(x^4-3x^2+12)^3} + C, \quad u = x^4-3x^2+12$$

L10

$$a) \quad I_1 = \int \sqrt{x-2} dx = \frac{2}{3}(x-2)^{3/2} + C = \frac{2}{3}(x-2)\sqrt{x-2} + C, \quad u = x-2,$$

$$I_2 = \int \sqrt{3x-6} dx = \frac{2}{3}(x-2)\sqrt{3x-6} + C, \quad u = 3x-6,$$

$$I_3 = \int \sqrt[3]{2x+7} = \frac{3}{8}(2x+7)^{4/3} + C = \frac{3}{8}(2x+7)\sqrt[3]{2x+7} + C, \quad u = 2x+7$$

$$b) \quad I_1 = \int \sqrt[3]{(4-3x)^2} dx = \frac{1}{5}(3x-4)\sqrt[3]{(4-3x)^2} + C, \quad u = 4-3x,$$

$$I_2 = \int x\sqrt{x^2+3} dx = \frac{1}{3}(x^2+3)^{3/2} + C = \frac{1}{3}(x^2+3)\sqrt{x^2+3} + C, \quad u = x^2+3,$$

$$I_3 = \int (6x^2-4)\sqrt[4]{x^3-2x+11} dx = \frac{8}{5}(x^3-2x+11)^{5/4} + C = \\ = \frac{8}{5}(x^3-2x+11)\sqrt[4]{x^3-2x+11} + C, \quad u = x^3-2x+11$$

L11

$$a) I_1 = \int e^{2x} dx = \frac{1}{2} e^{2x} + C, \quad u = 2x,$$

$$I_2 = \int e^{3-2x} dx = -\frac{1}{2} e^{3-2x} + C, \quad u = 3 - 2x,$$

$$I_3 = \int e^{mx+n} dx = \frac{1}{m} e^{mx+n} + C, \quad u = mx + n \quad (m, n \neq 0)$$

$$b) I_1 = \int x e^{x^2} dx = \frac{1}{2} e^{x^2} + C, \quad u = x^2,$$

$$I_2 = \int x^2 e^{2x^3} dx = \frac{1}{6} e^{2x^3} + C, \quad u = 2x^3,$$

$$I_3 = \int (x+1) e^{x^2+2x-3} dx = \frac{1}{2} e^{x^2+2x-3} + C, \quad u = x^2 + 2x - 3$$

$$c) I_1 = \int \frac{dx}{e^x - 1} = \ln|e^x - 1| - \ln(e^x) + C =$$

$$= \ln|e^x - 1| - x + C, \quad \ln(e^x) = x, \quad u = e^x - 1,$$

$$I_2 = \int \frac{dx}{2e^x - 3} = \int \frac{du}{u(u+3)} = \frac{1}{3} \int \left(\frac{1}{u} - \frac{1}{u+3} \right) du =$$

$$= \frac{1}{3} \left(\int \frac{du}{u} - \frac{du}{u+3} \right) = \frac{1}{3} (\ln|u| - \ln|u+3|) + C_1,$$

$$= \frac{1}{3} \ln|2e^x - 3| - \frac{1}{3} \ln(2e^x) + C_1 = \frac{1}{3} \ln|2e^x - 3| - \frac{1}{3} (\ln 2 + \ln(e^x)) + C_1 =$$

$$= \frac{1}{3} \ln|2e^x - 3| - \frac{1}{3} x - \frac{1}{3} \ln 2 + C_1 = \frac{1}{3} \ln|2e^x - 3| - \frac{1}{3} x + C_2,$$

$$u = 2e^x - 3, \quad \frac{du}{dx} = 2e^x, \quad dx = \frac{du}{2e^x} = \frac{du}{u+3}, \quad \ln(e^x) = x, \quad C_2 = C_1 - \frac{1}{3} \ln 2,$$

$$I_3 = \int \frac{e^x dx}{3e^x + 5} = \frac{1}{3} \int \frac{du}{u} = \frac{1}{3} \ln(u) + C = \frac{1}{3} \ln(3e^x + 5) + C,$$

$$u = 3e^x + 5, \quad \frac{du}{dx} = 3e^x, \quad e^x dx = \frac{du}{3}$$

L12

$$a) I_1 = \int_0^{\pi} \sin(2x) \cos x dx$$

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha - \beta) + \sin(\alpha + \beta)] \quad (\text{Eq. (10), Seite 3})$$

$$\sin(2x) \cos x = \frac{1}{2} (\sin x + \sin(3x))$$

$$I_1 = \int_0^{\pi} \sin(2x) \cos x dx = \frac{1}{2} \int_0^{\pi} (\sin x + \sin(3x)) dx =$$

$$-\frac{1}{2} \left[\frac{1}{3} \cos(3x) + \cos x \right]_0^{\pi} = \frac{4}{3}$$

$$I_2 = \int_0^{\pi/2} \sin(3x) \cos x dx = \frac{1}{2} \int_0^{\pi/2} (\sin(2x) + \sin(4x)) dx =$$

$$= -\frac{1}{4} \left[\frac{1}{2} \cos(4x) + \cos(2x) \right]_0^{\pi/2} = \frac{1}{2}$$

$$\text{Variante 2:} \quad \sin^3 \alpha = \frac{1}{4} (3 \sin \alpha - \sin(3\alpha)) \quad (\text{Eq. (12), Seite 3})$$

$$\sin(3x) = 3 \sin x - 4 \sin^3 x,$$

$$I_2 = \int_0^{\pi/2} \sin(3x) \cos x dx = \int_0^{\pi/2} (3 \sin x - 4 \sin^3 x) \cos x dx =$$

$$= 3 \int_0^{\pi/2} \sin x \cos x dx - 4 \int_0^{\pi/2} \sin^3 x \cos x dx =$$

$$= \left[-\frac{3}{2} \cos^2 x - \sin^4 x \right]_0^{\pi/2} = \frac{4}{3}$$

$$b) I_1 = \int_0^{\pi/4} \sin(2x) \sin x dx =$$

$$1) = \frac{2}{3} \left[\sin^3 x \right]_0^{\pi/4} = \frac{1}{3\sqrt{2}},$$

$$2) = \frac{1}{6} [3 \sin x - \sin(3x)]_0^{\pi/4} = \frac{1}{3\sqrt{2}},$$

$$I_2 = \int_0^{\pi/3} \sin(4x) \sin(2x) dx = \frac{1}{4} \left[\sin(2x) - \frac{1}{3} \sin(6x) \right]_0^{\pi/3} = \frac{\sqrt{3}}{8}$$

$$c) \quad I_1 = \int_0^{\pi/4} \cos(2x) \cos x \, dx = \frac{1}{2} \left[\sin x + \frac{1}{3} \sin(3x) \right]_0^{\pi/4} = \frac{\sqrt{2}}{3}$$

$$I_2 = \int_0^{\pi/6} \cos(4x) \cos(2x) \, dx = \frac{1}{4} \left[\sin(2x) + \frac{1}{3} \sin(6x) \right]_0^{\pi/6} = \frac{\sqrt{3}}{8}$$

L13

$$I_a = \int \sin^2(ax) \, dx = \frac{x}{2} - \frac{1}{4a} \sin(2ax) + C, \quad I_b = \int \cos^2(ax) \, dx = \frac{x}{2} + \frac{1}{4a} \sin(2ax) + C$$

L14

$$a) \quad I_1 = \int \frac{\cos x}{1 + \sin x} \, dx = \ln |1 + \sin x| + C, \quad u = 1 + \sin x,$$

$$I_2 = \int \frac{\sin x}{1 + \cos x} \, dx = -\ln |1 + \cos x| + C, \quad u = 1 + \cos x,$$

$$I_3 = \int \frac{\cos x}{3 + 2 \sin x} \, dx = \frac{1}{2} \ln |3 + 2 \sin x| + C, \quad u = 3 + 2 \sin x$$

$$b) \quad I_1 = \int \frac{\sin x}{7 - 3 \cos x} \, dx = \frac{1}{3} \ln |7 - 3 \cos x| + C, \quad u = 7 - 3 \cos x,$$

$$I_2 = \int \frac{\sin(2x)}{1 + \cos(2x)} \, dx = -\frac{1}{2} \ln |1 + \cos(2x)| + C, \quad u = 1 + \cos(2x),$$

$$I_3 = \int \frac{\cos(3x)}{4 + 2 \sin(3x)} \, dx = \frac{1}{6} \ln |2 + \sin(3x)| + C_1 = \frac{1}{6} \ln |4 + 2 \sin(3x)| + C_2,$$

$$C_1 = C_2 + \frac{\ln 2}{6}, \quad u = 2 + \sin(3x)$$

L15

$$a) \quad I_1 = \int \sin^5 x \cos x \, dx = \frac{1}{6} \sin^6 x + C, \quad u = \sin x,$$

$$I_2 = \int \cos^5 x \sin x \, dx = -\frac{1}{6} \cos^6 x + C, \quad u = \cos x,$$

$$I_3 = \int \sin^7 x \cos x \, dx = \frac{1}{8} \sin^8 x + C, \quad u = \sin x$$

$$b) \quad I_1 = \int \sin^3 x \sin(2x) \, dx = 2 \int \sin^4 x \cos x \, dx = \frac{2}{5} \sin^5 x + C, \quad u = \sin x,$$

$$I_2 = \int \cos^5 x \sin(2x) \, dx = 2 \int \cos^6 x \sin x \, dx = -\frac{2}{7} \cos^7 x + C, \quad u = \cos x,$$

$$I_3 = \int \sin^8(2x) \sin(4x) \, dx = 2 \int \sin^9(2x) \cos(2x) \, dx = \frac{1}{10} \sin^{10}(2x) + C, \quad u = \sin(2x)$$

L16

$$\begin{aligned}
 a) \quad I_1 &= \int \frac{\sin x}{1 + \cos^2 x} dx = -\arctan(\cos x) + C, & u &= \cos x, \\
 I_2 &= \int \frac{\cos x}{4 + \sin^2 x} dx = \frac{1}{2} \arctan\left(\frac{1}{2} \sin x\right) + C, & u &= \sin x, \\
 I_3 &= \int \frac{3 \sin x}{6 + 2 \cos^2 x} dx = -\frac{\sqrt{3}}{2} \arctan\left(\frac{1}{\sqrt{3}} \cos x\right) + C, & u &= \cos x, \\
 \\
 b) \quad I_1 &= \int \frac{\sin(2x)}{1 + \cos^2(2x)} dx = -\frac{1}{2} \arctan(\cos(2x)), & u &= \cos(2x), \\
 I_2 &= \int \frac{\cos(2x)}{4 + \sin^2(2x)} dx = \frac{1}{4} \arctan\left(\frac{1}{2} \sin(2x)\right) + C, & u &= \sin(2x), \\
 I_3 &= \int \frac{\sin(3x)}{2 + 5 \cos^2(3x)} dx = -\frac{1}{3\sqrt{10}} \arctan\left(\sqrt{\frac{5}{2}} \cos(3x)\right) + C, & u &= \cos(3x)
 \end{aligned}$$

L17

$$\begin{aligned}
 a) \quad I_1 &= \int \frac{dx}{2 + \sqrt{x+1}} = 2\sqrt{x+1} - 4 \ln|2 + \sqrt{x+1}| + C, & u &= \sqrt{x+1}, \\
 I_2 &= \int \frac{dx}{4 + \sqrt{x-3}} = 2\sqrt{x-3} - 8 \ln|\sqrt{x-3} + 4| + C, & u &= \sqrt{x-3}, \\
 I_3 &= \int \frac{dx}{1 + \sqrt{3x-2}} = \frac{2}{3}\sqrt{3x-2} - \frac{2}{3} \ln|1 + \sqrt{3x-2}| + C, & u &= \sqrt{3x-2} \\
 \\
 b) \quad I_1 &= \int \frac{\sqrt{x} dx}{x+4} = 2\sqrt{x} - 4 \arctan\left(\frac{\sqrt{x}}{2}\right) + C, & u &= \sqrt{x}, \\
 I_2 &= \int \frac{\sqrt{x} dx}{x+a^2}, & u &= \sqrt{x}, \quad du = \frac{dx}{2\sqrt{x}}, \\
 I_2 &= \int \frac{\sqrt{x} dx}{x+a^2} = 2 \int \frac{u^2 du}{a^2+u^2} = 2 \int \frac{(u^2+a^2-a^2) du}{a^2+u^2} = 2u - a^2 \\
 &= 2 \left(\int du - a^2 \int \frac{du}{a^2+u^2} \right) = 2u - 2a^2 \int \frac{du}{a^2+u^2} \\
 I_2 &= \int \frac{\sqrt{x} dx}{x+a^2} = 2\sqrt{x} - 2a \arctan\left(\frac{\sqrt{x}}{a}\right) + C, & u &= \sqrt{x}, \\
 I_3 &= \int \frac{x\sqrt{x} dx}{x+9} = \int \frac{(x+9-9)\sqrt{x}}{x+9} dx = \int \sqrt{x} dx - 9 \int \frac{\sqrt{x} dx}{x+9} = \\
 &= \frac{2}{3} x \sqrt{x} - 18 \sqrt{x} + 54 \arctan\left(\frac{\sqrt{x}}{3}\right)
 \end{aligned}$$

L18

$$\begin{aligned}
 a) \quad I_1 &= \int \frac{\sqrt[3]{x+1}}{\sqrt{x}} dx = \frac{5}{6} x^{5/6} + 2\sqrt{x} + C = \frac{5}{6} \sqrt[6]{x^5} + 2\sqrt{x} + C, \\
 I_2 &= \int \frac{\sqrt{x+1}}{\sqrt[3]{x+1}} dx = \frac{6}{7} (x+1)^{7/6} + C = \frac{6}{7} (x+1) \sqrt[6]{x+1} + C, \quad u = \sqrt{x+1}, \\
 I_3 &= \int \frac{\sqrt{2x+5}}{\sqrt[4]{2x+5}} dx = \frac{2}{5} (2x+5)^{5/4} + C = \frac{2}{5} (2x+5) \sqrt[4]{2x+5} + C, \quad u = 2x+5 \\
 \\
 b) \quad I_1 &= \int \frac{4\sqrt{x+2}-3}{\sqrt[3]{x+2}} dx = \frac{24}{7} (x+2)^{7/6} - \frac{9}{2} (x+2)^{2/3} + C = \\
 &= \frac{24}{7} (x+2) \sqrt[6]{x+2} - \frac{9}{2} \sqrt[3]{(x+2)^2} + C, \quad u = \sqrt{x+2}, \\
 I_2 &= \int \frac{dx}{x\sqrt{x+1}} = \ln \left| \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1} \right| + C \quad u = \sqrt{x+1} \\
 I_3 &= \int \frac{dx}{x\sqrt{2x-3}} = \frac{2}{\sqrt{3}} \arctan \left(\frac{\sqrt{2x-3}}{\sqrt{3}} \right) + C, \quad u = \sqrt{2x-3}
 \end{aligned}$$

L19

$$\begin{aligned}
 I_1 &= \int x \cdot \sqrt{x+2} dx = \frac{2}{15} (x+2)^{3/2} (3x-4) + C, \\
 I_2 &= \int x \cdot \sqrt{x+5} dx = \frac{2}{15} (x+5)^{3/2} (3x-10) + C, \\
 I_3 &= \int x^2 \cdot \sqrt{2x-5} dx = \frac{1}{21} (2x-5)^{3/2} (3x^2+6x+10) + C.
 \end{aligned}$$

4.4. Partielle Integration

L20

$$a) \quad I_1 = \int \ln^3 x \, dx = x \ln^3 x - 3x \ln^2 x + 6x \ln x - 6x + C =$$

$$= x(\ln^3 x - 3 \ln^2 x + 6 \ln x - 6) + C$$

$$u = \ln^3 x, \quad u' = \frac{3}{x} \quad v' = 1, \quad v = x$$

$$\int \ln^2 x \, dx = x(\ln^2 x - 2 \ln x + 2) + C, \quad \text{aus dem Beispiel zur Aufgabe}$$

$$I_2 = \int \ln^4 x \, dx = x(\ln^4 x - 4 \ln^3 x + 12 \ln^2 x - 24 \ln x + 24) + C,$$

$$u = \ln^4 x, \quad v' = 1$$

$$I_3 = \int \ln^5 x \, dx = x(\ln^5 x - 5 \ln^4 x + 20 \ln^3 x - 60 \ln^2 x + 120 \ln x - 120) + C,$$

$$u = \ln^5 x, \quad v' = 1$$

$$b) \quad I_1 = \int x^2 \ln x \, dx = \frac{x^3}{9} (3 \ln x - 1) + C, \quad u = \ln x, \quad v' = x^2,$$

$$I_2 = \int x^3 \ln x \, dx = \frac{x^4}{16} (4 \ln x - 1) + C, \quad u = \ln x, \quad v' = x^3,$$

$$I_3 = \int x \ln(x^2) \, dx =$$

$$c) \quad I_1 = \int (x+1) \ln x \, dx = \int \ln x \, dx + \int x \ln x \, dx = x \ln x \left(\frac{x}{2} + 1 \right) - \frac{x^2}{4} - x + C,$$

$$I_2 = \int (x+1)^2 \ln x \, dx = \int (x^2 + 2x + 1) \ln x \, dx =$$

$$= \int x^2 \ln x \, dx + 2 \int x \ln x \, dx + \int \ln x \, dx =$$

$$= x \ln x \left(\frac{x^2}{3} + x + 1 \right) - \frac{x^3}{9} - \frac{x^2}{2} - x + C,$$

$$I_3 = \int x \ln(x+1) \, dx = (x+1) \ln(x+1) \left(\frac{1}{2}(x+1) - 1 \right) - \frac{x^2}{4} + \frac{x}{2} + \frac{3}{4} + C$$

$$d) \quad I_1 = \int \sqrt{x} \ln(\sqrt{x}) \, dx = \frac{1}{3} x^{3/2} \left(\ln x - \frac{2}{3} \right) + C = \frac{1}{3} x \sqrt{x} \left(\ln x - \frac{2}{3} \right) + C$$

$$I_2 = \int \sqrt[3]{x} \ln x \, dx = \frac{3}{4} x^{4/3} \left(\ln x - \frac{3}{4} \right) + C = \frac{3}{4} x \sqrt[3]{x} \left(\ln x - \frac{3}{4} \right) + C$$

$$I_3 = \int \sqrt[5]{x^2} \ln x \, dx = \frac{5}{7} x^{7/5} \left(\ln x - \frac{5}{7} \right) + C = \frac{5}{7} x \sqrt[5]{x^2} \left(\ln x - \frac{5}{7} \right) + C$$

L21

$$I_1 = \int \frac{\ln x}{x} dx = \frac{1}{2} \ln^2 x + C,$$

$$I_2 = \int \frac{\ln x}{x^2} dx = -\frac{1}{x} (\ln x + 1) + C,$$

$$I_3 = \int \frac{\ln x}{x^3} dx = -\frac{1}{4x^2} (2 \ln x + 1) + C$$

L22

$$I_1 = \int x^2 e^x dx = e^x (x^2 - 2x + 2) + C,$$

$$I_2 = \int x^3 e^x dx = e^x (x^3 - 3x^2 + 6x - 6) + C,$$

$$I_3 = \int x^4 e^x dx = e^x (x^4 - 4x^3 + 12x^2 - 24x + 24) + C$$

L23

a)

$$I_1 = \int x \sin x dx = \sin x - x \cos x + C,$$

$$I_2 = \int x \cos x dx = \cos x + x \sin x + C,$$

$$I_3 = \int x \sin(2x) dx = \frac{1}{4} \sin(2x) - \frac{x}{2} \cos(2x) + C,$$

$$I_4 = \int x \cos(2x) dx = \frac{1}{4} \cos(2x) + \frac{x}{2} \sin(2x) + C$$

b)

$$I_1 = \int x^2 \sin x dx = -x^2 \cos x + 2 \cos x + 2x \sin x + C,$$

$$I_2 = \int x^2 \cos x dx = x^2 \sin x - 2 \sin x + 2x \cos x + C,$$

$$I_3 = \int x^2 \cos(2x) dx = \frac{1}{2} \sin(2x) \left(x^2 - \frac{1}{2}\right) + \frac{x}{2} \cos(2x) + C,$$

$$I_4 = \int x^2 \sin(3x) dx = \frac{1}{3} \cos(3x) \left(-x^2 + \frac{2}{9}\right) + \frac{2}{9} x \sin(3x) + C$$

5. Anwendungen der Integralrechnung

5.1. Flächen zwischen Kurven: Lösungen

Berechnen Sie die von den Kurven $f(x)$ und $g(x)$ eingeschlossene Fläche

L24

$$f(x) = x^2 - 2, \quad g(x) = 2$$

$$f(x) = g(x) : \quad x^2 - 2 = 2 \Leftrightarrow x^2 = 4 \Rightarrow x_1 = -2, \quad x_2 = 2,$$

$$A = \int_{-2}^2 (2 - (x^2 - 2)) dx = 2 \int_{-2}^2 (4 - x^2) dx = \frac{32}{3} \approx 10.67$$

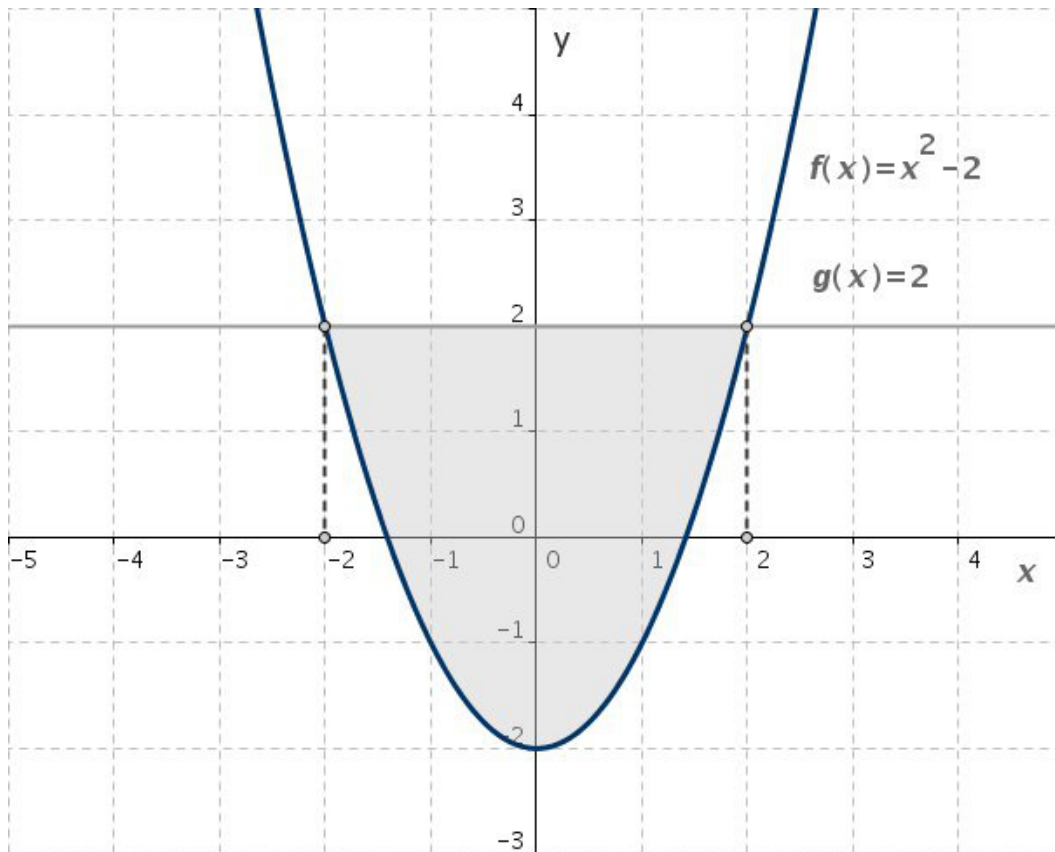


Fig. 1: Die Fläche zwischen der Parabel $f(x) = x^2 - 2$ und der Geraden $g(x) = 2$

L25

$$f(x) = 2x - x^2, \quad g(x) = -3$$

$$f(x) = g(x) : 2x - x^2 = -3 \Leftrightarrow x^2 - 2x - 3 = 0 \Rightarrow x_1 = -1, \quad x_2 = 3,$$

$$A = \int_{-1}^3 (2x - x^2 + 3) dx = \frac{32}{3} \approx 10.67$$

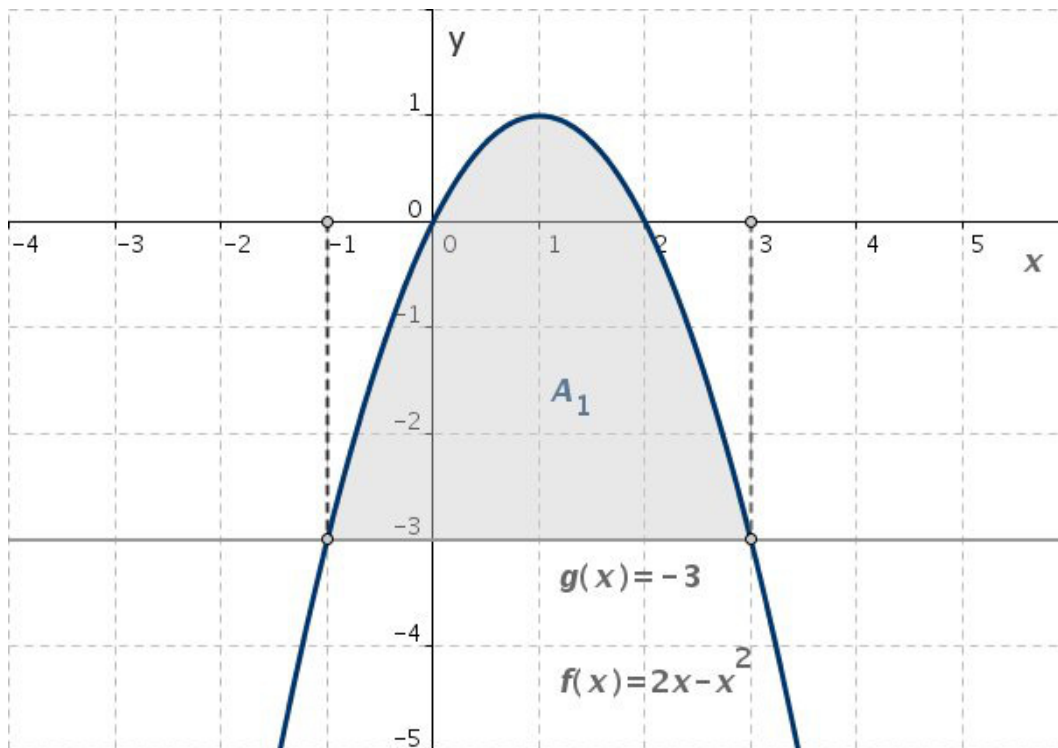


Fig. 2: Die Fläche zwischen der Parabel $f(x) = 2x - x^2$ und der Geraden $g(x) = -3$

L26

$$f(x) = x^2 - 2x, \quad g(x) = x$$

$$f(x) = g(x) : \quad x^2 - 2x = x \Leftrightarrow x^2 - 3x = x(x - 3) = 0 \Rightarrow x_1 = 0, \quad x_2 = 3,$$

$$A = \int_0^3 (g(x) - f(x)) dx = \int_0^3 (3x - x^2) dx = \frac{9}{2} = 4.5$$

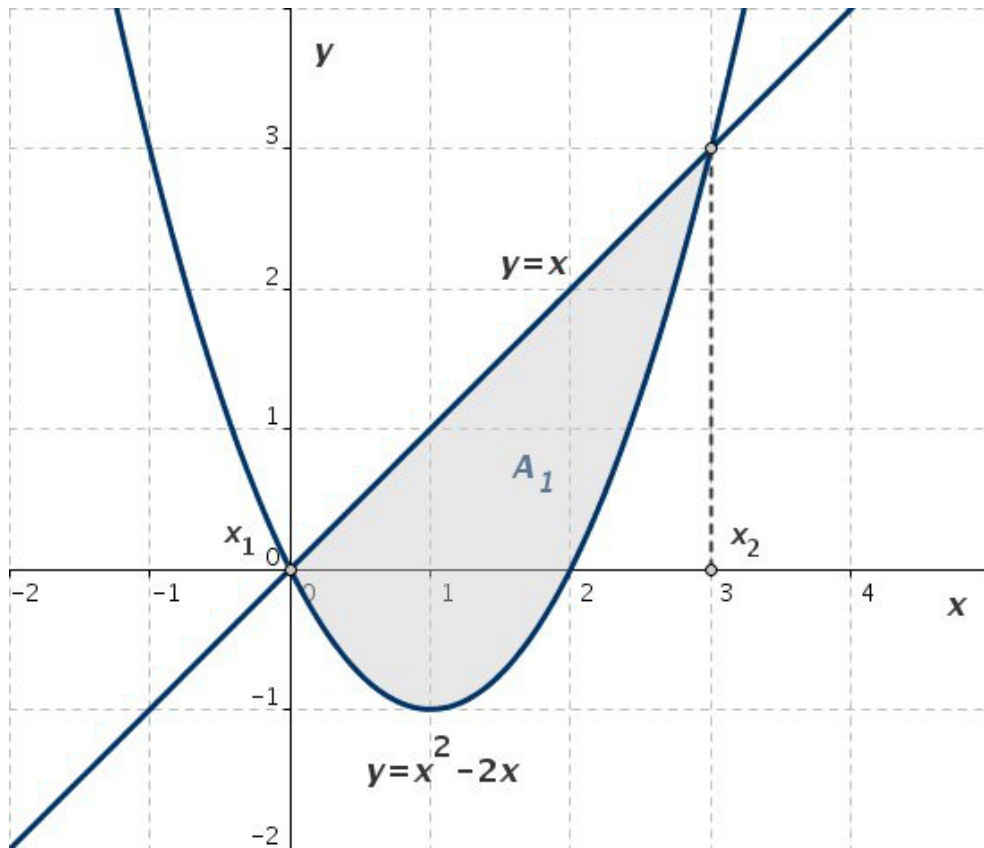


Fig. 3: Die Fläche zwischen der Parabel $f(x) = x^2 - 2x$ und der Geraden $g(x) = x$

L27

$$f(x) = x^2 - 2, \quad g(x) = 2, \quad I = [-2, 3]$$

$$f(x) = g(x) : x^2 - 2 = 2 \Leftrightarrow x^2 = 4 \Rightarrow x_1 = -2, \quad x_2 = 2,$$

$$x_1 \in I, \quad x_2 \in I, \quad I = I_1 + I_2, \quad I_1 = [-2, 2], \quad I_2 = [2, 3]$$

$$\begin{aligned} A &= A_1 + A_2 = \int_{-2}^2 (2 - (x^2 - 2)) dx + \int_2^3 (x^2 - 2 - 2) dx = 2 \int_0^2 (4 - x^2) dx + \\ &+ \int_2^3 (x^2 - 4) dx = \frac{32}{3} + \frac{7}{3} = 13 \end{aligned}$$

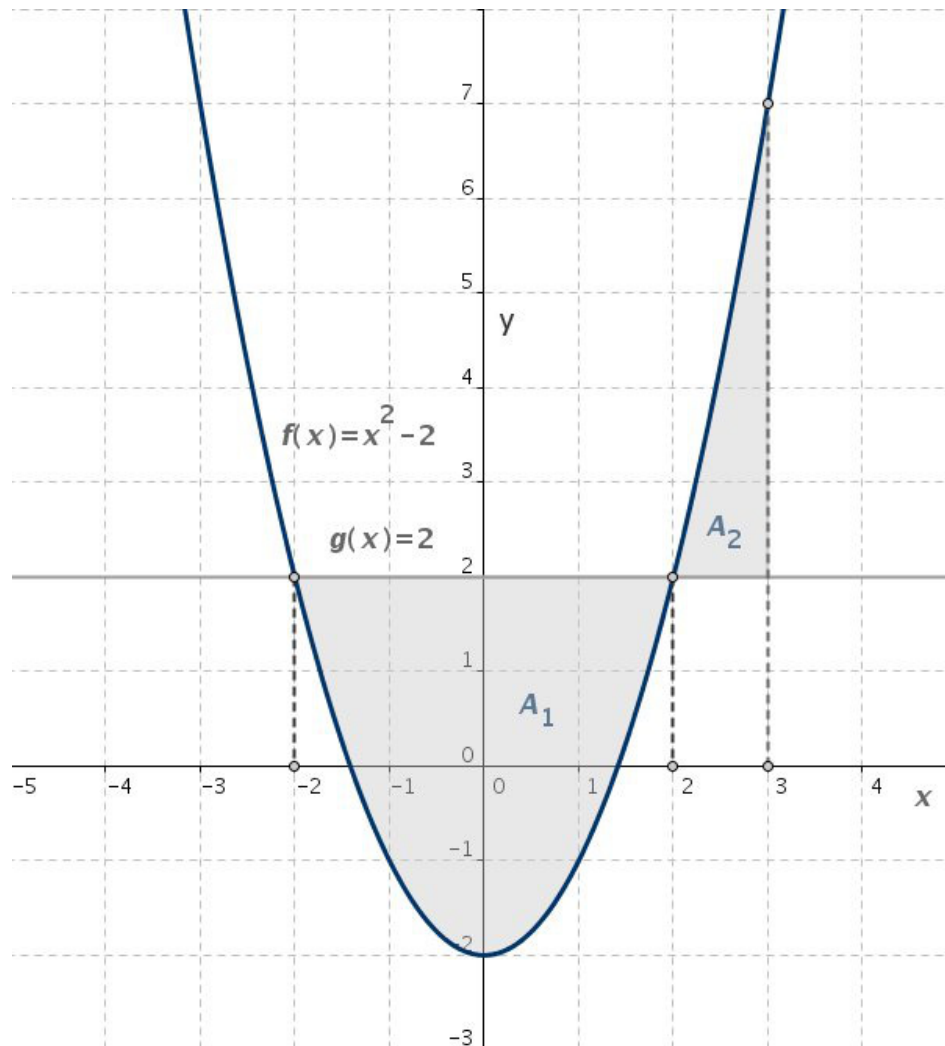


Fig. 4: Die Fläche zwischen der Parabel $f(x) = x^2 - 2$ und der Geraden $g(x) = 2$ im Intervall $I = [-2, 3]$

L28

$$f(x) = x^2 - 2x, \quad g(x) = x, \quad I = [-1, 3]$$

$$f(x) = g(x) : \quad x^2 - 2x = x \Leftrightarrow x^2 - 3x = x(x-3) = 0 \Rightarrow x_1 = 0, \quad x_2 = 3,$$

$$x_1 \in I, \quad x_2 \in I, \quad I = I_1 + I_2, \quad I_1 = [-1, 0], \quad I_2 = [0, 3]$$

$$\begin{aligned} A &= \int_{-1}^0 (f(x) - g(x)) dx + \int_0^3 (g(x) - f(x)) dx = \int_{-1}^0 (x^2 - 3x) dx + \int_0^3 (3x - x^2) dx = \\ &= \frac{11}{6} + \frac{9}{2} = \frac{38}{6} \approx 6.33 \end{aligned}$$

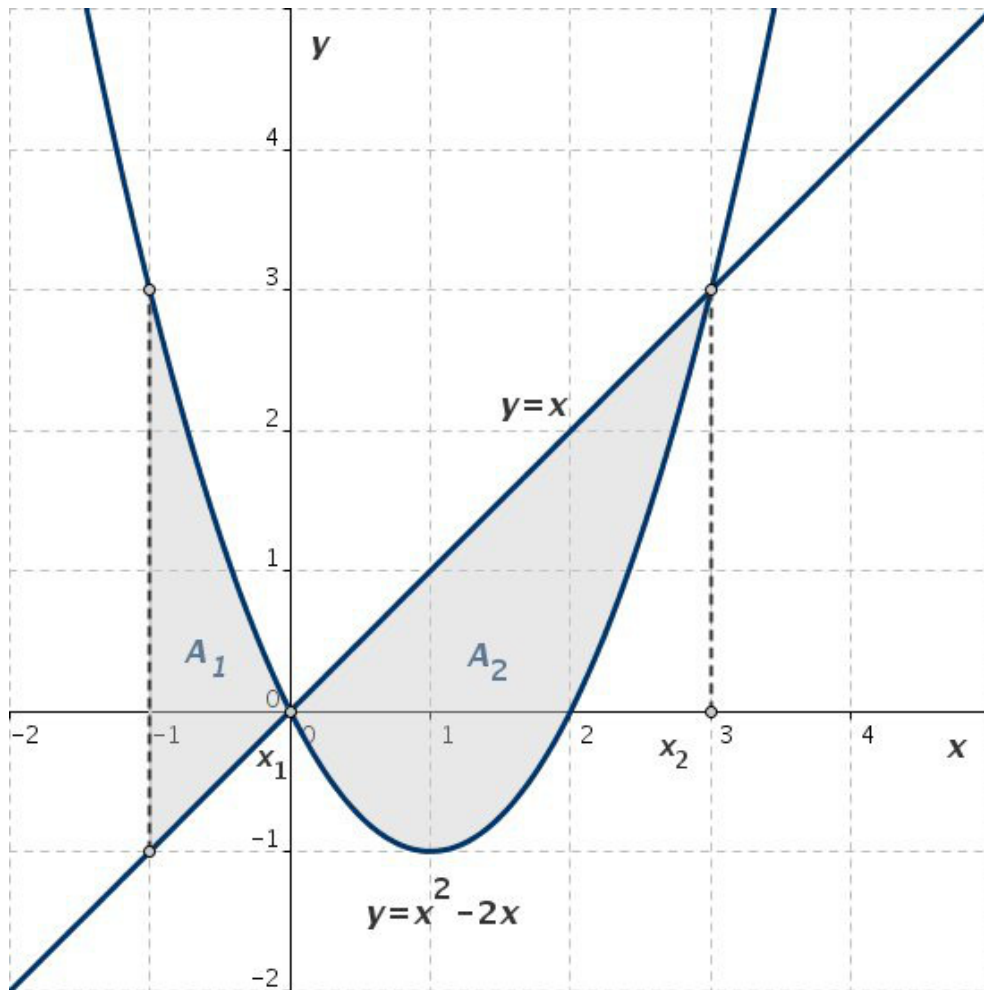


Fig. 5: Die Fläche zwischen der Parabel $f(x) = x^2 - 2x$ und der Geraden $g(x) = x$ im Intervall $[-1, 3]$

L29

$$f(x) = x^2 - 2x, \quad g(x) = x + 1, \quad I = [-1, 2]$$

$$f(x) = g(x) : \quad x^2 - 2x = x + 1 \Leftrightarrow x^2 - 3x - 1 = 0 \Rightarrow x_1 = -0.3, \quad x_2 = 3.3,$$

$$x_1 \in I, \quad x_2 \notin I, \quad I = I_1 + I_2, \quad I_1 = [-1, -0.3], \quad I_2 = [-0.3, 2]$$

$$\begin{aligned} A &= \int_{-1}^{-0.3} (f(x) - g(x)) dx + \int_{-0.3}^2 (g(x) - f(x)) dx = \int_{-1}^{-0.3} (x^2 - 3x - 1) dx + \int_{-0.3}^2 (1 + 3x - x^2) dx = \\ &= 0.99 + 5.49 \quad (??) \end{aligned}$$

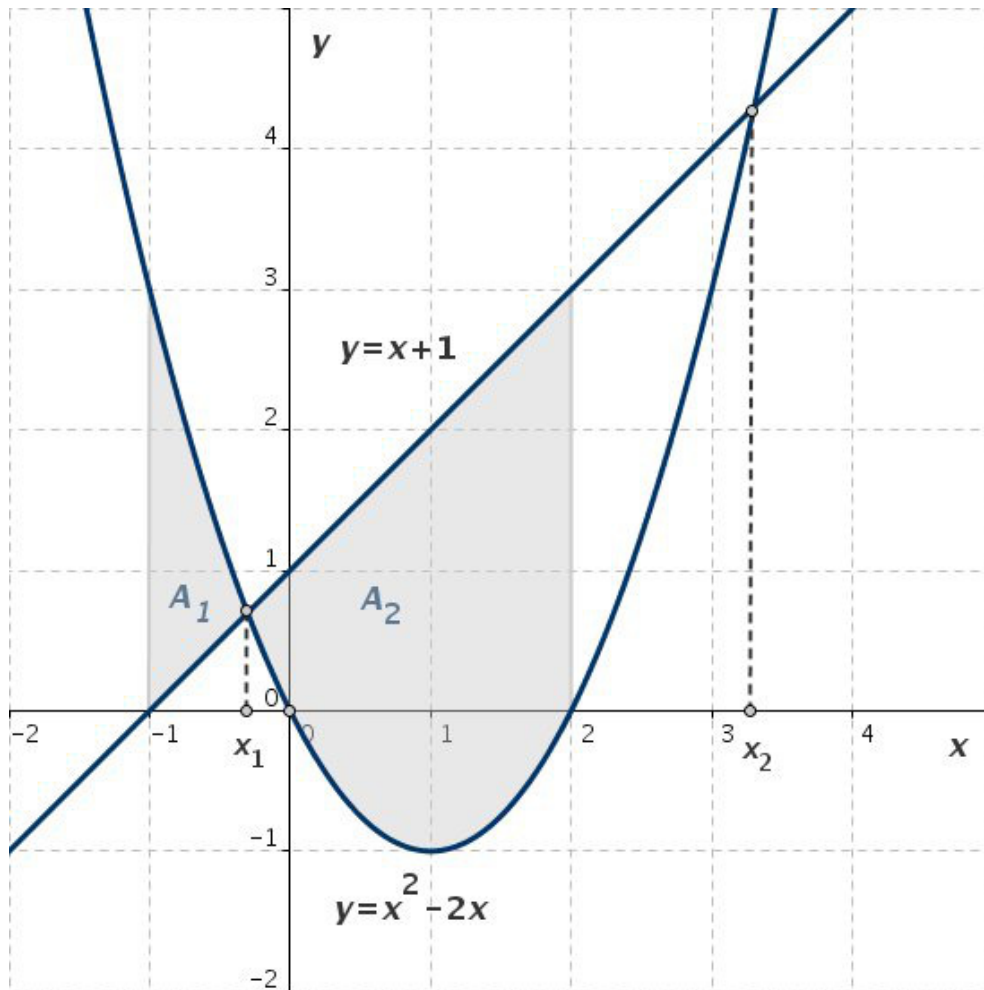


Fig. 6: Die Fläche zwischen der Parabel $f(x) = x^2 - 2x$ und der Geraden $g(x) = x + 1$ im Intervall $[-1, 2]$

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