

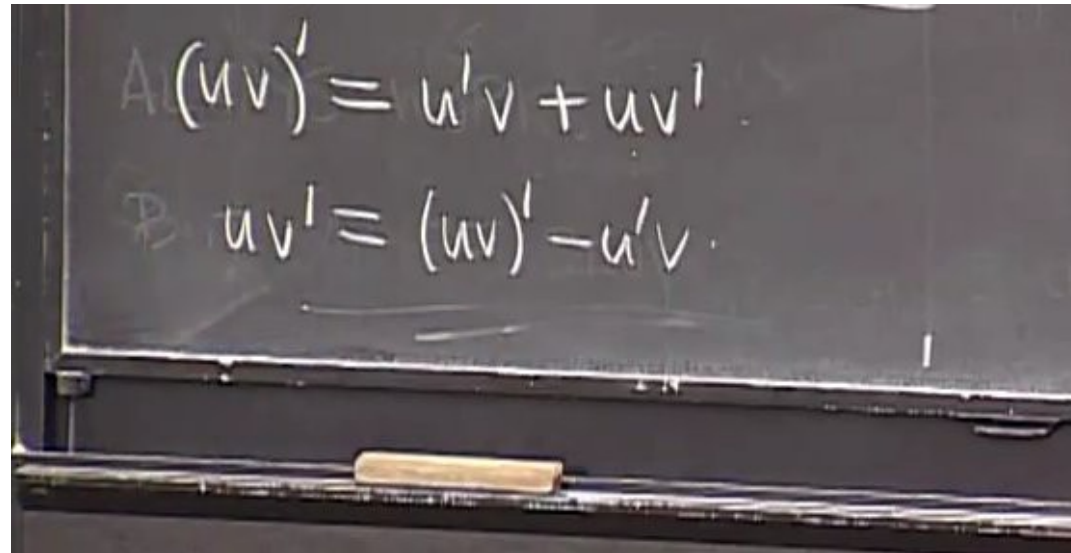
Partielle Integration



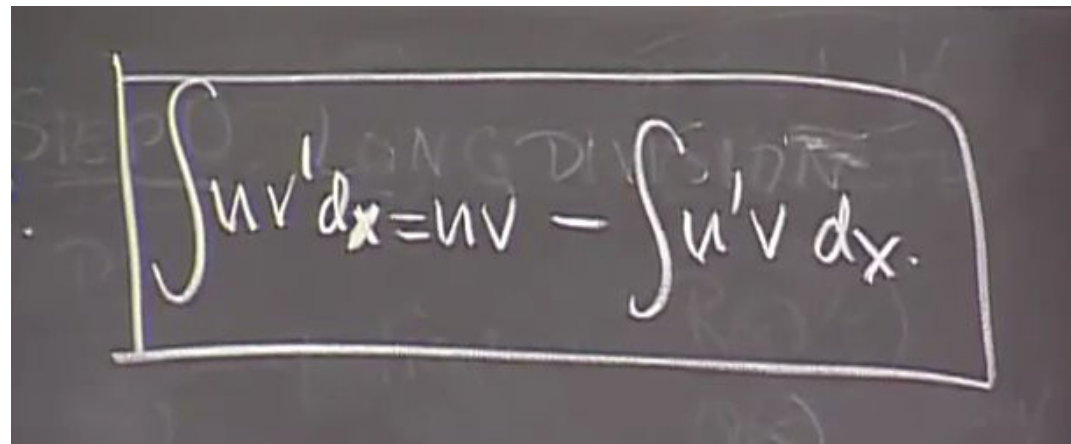
$$\int x \cdot \cos x \, dx$$

Wie kann man ein solches Integral bestimmen?

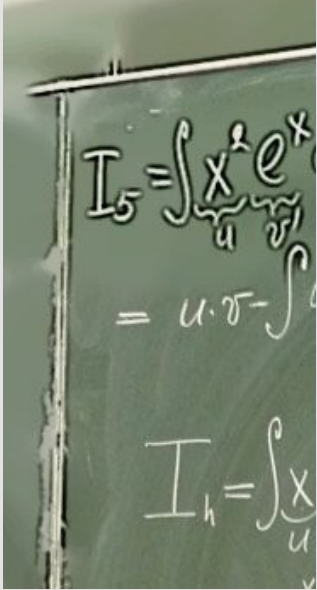
Während eine Summe integrierbarer Funktionen leicht zu integrieren ist (Summenregel), gibt es keine einfache Regel für die Integration von Produkten von Funktionen. Nur in speziellen Fällen gelingt die Integration durch eine geeignete Substitution. Wir haben zwar eine Regel für die Differentiation von Produkten, aber daraus folgt keine Regel für die Integration beliebiger Produkte. Und dennoch ergibt sich aus der Produktregel der Differentiation ein sehr nützliches Integrationsverfahren.



A) $(uv)' = u'v + uv'$
B) $uv' = (uv)' - u'v$



$\int uv' dx = uv - \int u'v dx$



$$\int u v' dx = u v - \int u' v dx$$

Ausgangsintegral

Hilfsintegral

Unter welchen Voraussetzungen verwendet man diese Formel?

Für eine geeignete, richtige Zerlegung des Integranden in zwei Faktorfunktionen ist Folgendes wichtig:

- die Stammfunktion $v(x)$ zur Faktorfunktion $v'(x)$ muss problemlos bestimmt werden können;
- das Hilfsintegral muss elementar lösbar sein.

$$\int u v' dx = u v - \int u' v dx$$

$$\int x \cdot e^x dx \Rightarrow u v' = x \cdot e^x$$

Zerlegung 1: $u = x, \quad v' = e^x$

Zerlegung 2: $u = e^x, \quad v' = x$

Wir bestimmen das Integral für beide Zerlegungen.

Partielle Integration: Beispiel

$$\int x \cdot e^x dx \Rightarrow uv' = x \cdot e^x$$

Zerlegung 1: $u = x, \quad v' = e^x$

$$u = x, \quad u' = 1$$

$$v' = e^x, \quad v = \int v' dx = \int e^x dx = e^x$$

$$\int x \cdot e^x dx = x \cdot e^x - \int 1 \cdot e^x dx = x \cdot e^x - e^x + C = (x - 1) \cdot e^x + C$$

Zerlegung 2: $u = e^x, \quad v' = x$

$$u = e^x, \quad u' = e^x$$

$$v' = x, \quad v = \int v' dx = \int x dx = \frac{x^2}{2}$$

$$\int x \cdot e^x dx = e^x \cdot \frac{x^2}{2} - \int e^x \cdot \frac{x^2}{2} dx = \frac{x^2}{2} \cdot e^x - \frac{1}{2} \int x^2 \cdot e^x dx$$

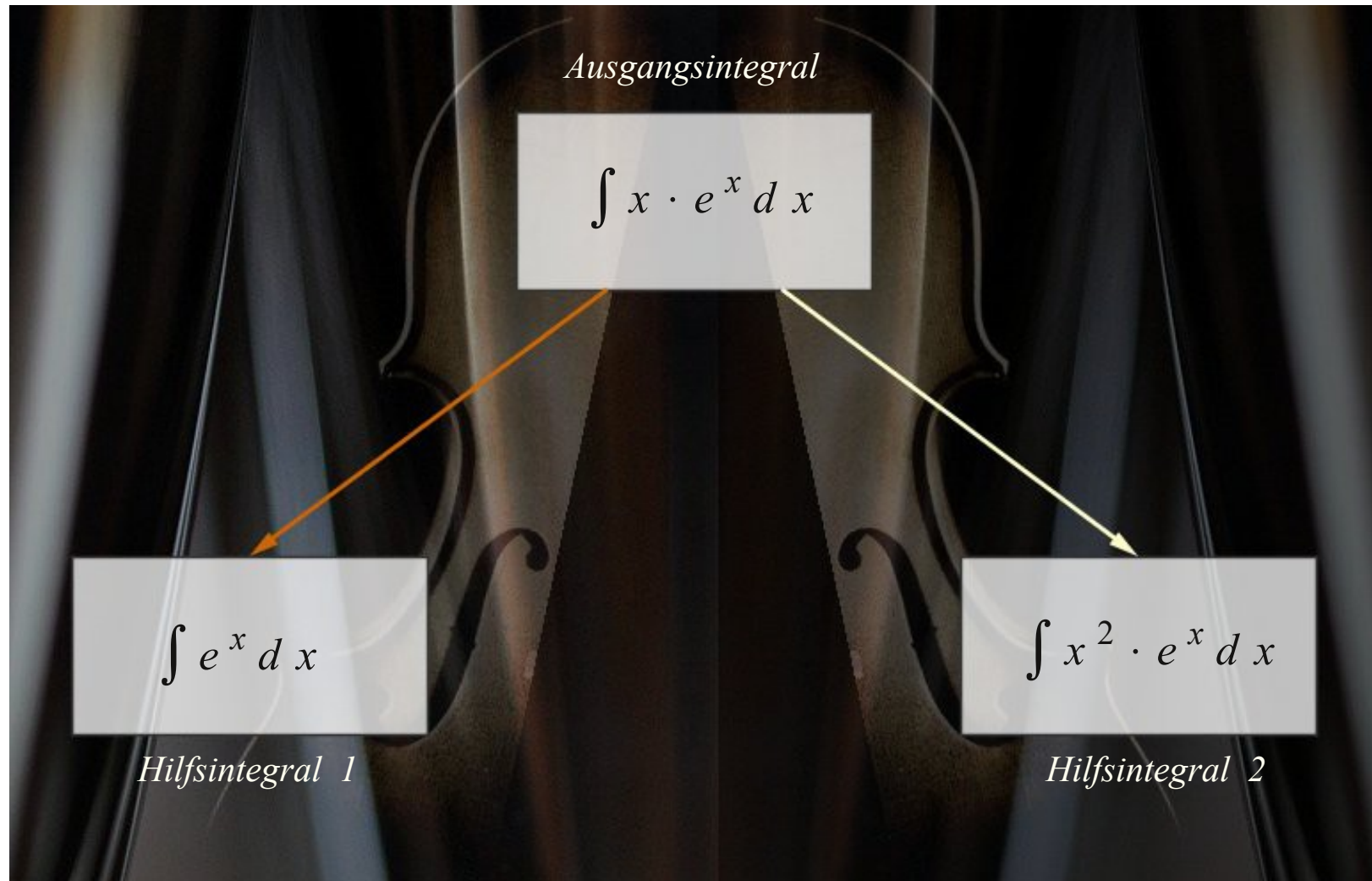
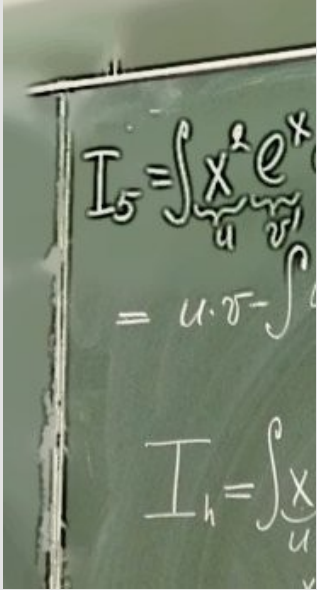


Abb. 1: Partielle Integration – Vergleich von zwei unterschiedlichen Zerlegungen

Nur die erste Zerlegung führt zu einer einfachen Lösung.



Berechnen Sie folgende Integrale:

Aufgabe 1: $I_1 = \int \ln x \, dx$

Aufgabe 2: $I_2 = \int (\ln x)^2 \, dx$

Aufgabe 3: $I_3 = \int e^x \sin x \, dx$

Lösung 1: $I_1 = \int \ln x \, dx$

$$u \cdot v' = \ln x : \quad u = \ln x, \quad u' = \frac{1}{x}, \quad v' = 1, \quad v = \int 1 \, dx = x$$

$$\int \ln x \, dx = \ln x \cdot x - \int dx = x (\ln x - 1) + C$$

Lösung 2: $I_2 = \int (\ln x)^2 \, dx$

$$u \cdot v' = (\ln x)^2 : \quad u = (\ln x)^2, \quad u' = \frac{2}{x} \cdot \ln x, \quad v' = 1, \quad v = x$$

$$\begin{aligned} \int (\ln x)^2 \, dx &= x \cdot (\ln x)^2 - 2 \int \ln x \, dx = x \cdot (\ln x)^2 - 2 I_1 = \\ &= x \cdot (\ln x)^2 - 2 x (\ln x - 1) + C \end{aligned}$$

$$I_3 = \int e^x \cdot \sin x \, dx$$

$$u = \sin x, \quad v' = e^x$$

$$u' = \cos x, \quad v = e^x$$

$$\int e^x \cdot \sin x \, dx = e^x \sin x - \int e^x \cdot \cos x \, dx + C$$

- War die partielle Integration sinnvoll?
- Ja, wenn man das entstehende Integral nochmals partiell integriert:

$$u = \cos x, \quad v' = e^x$$

$$u' = -\sin x, \quad v = e^x$$

$$\begin{aligned} \int e^x \cdot \sin x \, dx &= e^x \sin x - \left(e^x \cos x + \int e^x \sin x \, dx \right) + C = \\ &= e^x (\sin x - \cos x) - \int e^x \sin x \, dx + C \end{aligned}$$

$$\int e^x \cdot \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

Berechnen Sie folgende Integrale:

$$I_1 = \int x \cdot \sin x \, dx$$

$$I_2 = \int x \ln x \, dx$$

$$I_3 = \int \frac{\ln x}{x} \, dx$$

$$I_4 = \int x \cdot \cos(2x) \, dx$$

$$I_5 = \int x^2 \cdot e^x \, dx$$

$$I_6 = \int x^2 \cdot \cos x \, dx$$

$$I_1 = \int x \cdot \sin x \, dx$$

Zerlegung:

$$u = x, \quad v' = \sin x$$

Resultat:

$$I_1 = -x \cos x + \sin x + C$$

Partielle Integration: Lösung 4-2

$$I_2 = \int x \ln x \, dx$$

Zerlegung:

$$u = \ln x, \quad v' = x$$

The image shows a chalkboard with handwritten mathematical work. The integral $I_2 = \int x \ln x \, dx$ is written on the left. To its right, the substitution $u = \ln x$ and $u' = \frac{1}{x}$ are noted. Below that, $v' = x$ and $v = \int x \, dx = \frac{x^2}{2}$ are written. The main calculation follows: $\int x \ln x \, dx = \ln(x) \frac{x^2}{2} - \int \frac{x^2}{2x} \, dx$. This is then simplified to $= \ln(x) \frac{x^2}{2} - \frac{1}{2} \int x \, dx$, then to $= \ln(x) \frac{x^2}{2} - \frac{1}{2} \frac{x^2}{2} + C$, and finally to $= \frac{1}{2} x^2 \left(\ln x - \frac{1}{2} \right) + C$.

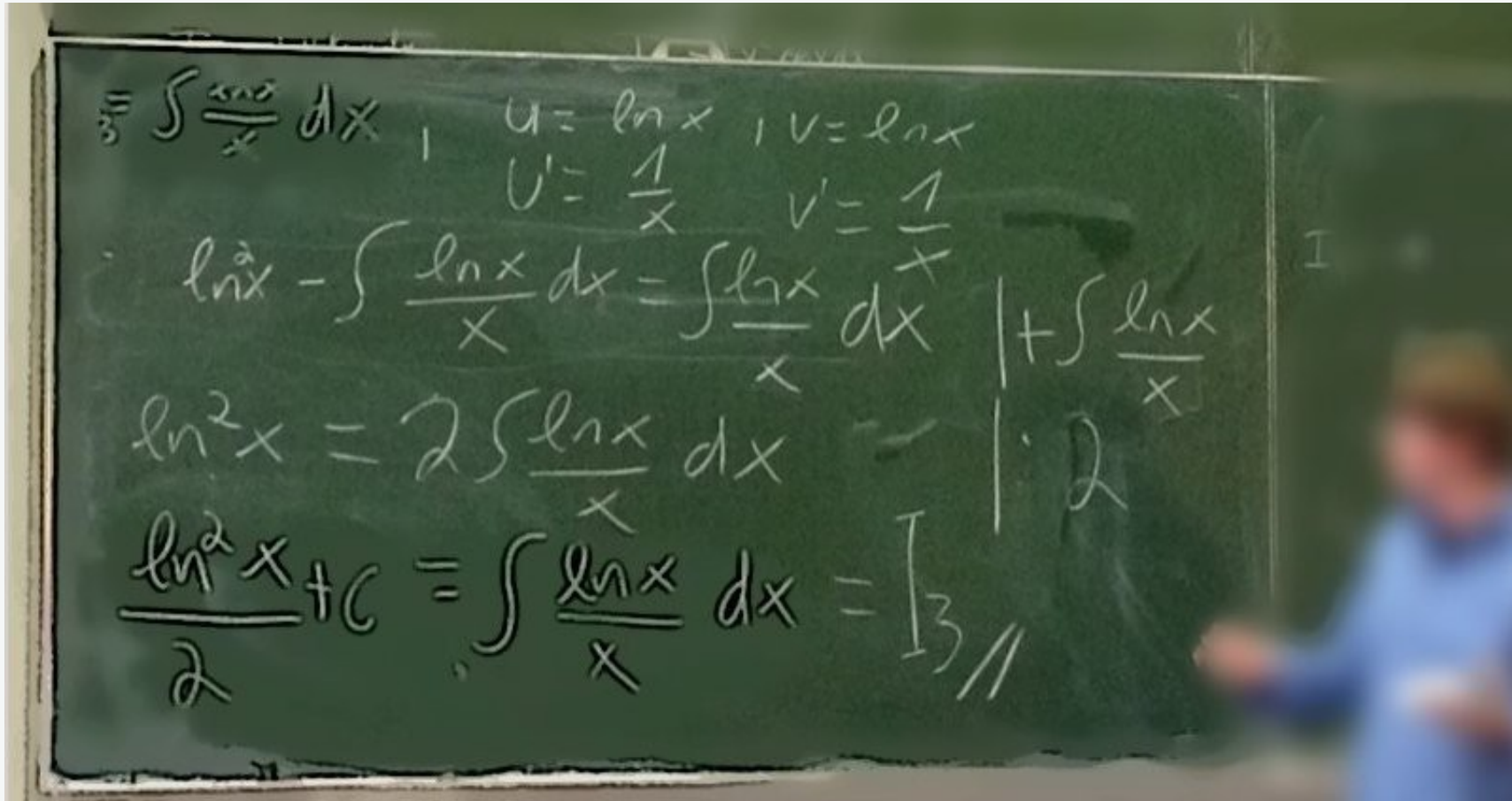
$$I_2 = \frac{x^2}{2} \left(\ln x - \frac{1}{2} \right) + C$$

Partielle Integration: Lösung 4-3

$$I_3 = \int \frac{\ln x}{x} dx$$

Zerlegung:

$$u = \ln x, \quad v' = \frac{1}{x}$$



$$I_3 = \frac{1}{2} (\ln x)^2 + C$$

The image shows a chalkboard with handwritten mathematical work. At the top, the integral $I_4 = x \cdot \cos(2x)$ is written. To its right, the choice of $u = x$ and $v' = \cos(2x)$ is noted, along with their derivatives $u' = 1$ and $v = \sin(2x)/2$. The main calculation follows: $I_4 = \frac{x}{2} \sin(2x) - \frac{1}{2} \int \sin(2x) dx$, which is then simplified to $= \frac{x}{2} \sin(2x) + \frac{1}{4} \cos(2x) + C$. A second part of the work shows the substitution $u = 2x$ for the integral $\int \cos(2x) dx$, leading to $\int \cos u \cdot \frac{du}{2} = \frac{1}{2} \int \cos u \cdot du = \frac{1}{2} \sin u = \frac{1}{2} \sin(2x)$.

$$I_4 = \int x \cdot \cos(2x) dx = \frac{x}{2} \sin(2x) + \frac{1}{4} \cos(2x) + C$$

$$I_5 = \int \underbrace{x^2}_u \cdot \underbrace{e^x}_v dx =$$

$$= u \cdot v - \int u' \cdot v dx = x^2 e^x - 2 \int x e^x dx =$$

$$= x^2 e^x - 2 I_h$$

$$I_h = \int \underbrace{x}_u \cdot \underbrace{e^x}_v dx = x e^x - \int e^x dx =$$

$$= x e^x - e^x = I_h$$

$$I_5 = x^2 e^x - 2(x e^x - e^x) + C = \underline{e^x (x^2 - 2x + 2) + C}$$

(1) $\left| \begin{array}{l} u = x^2, \quad u' = 2x, \\ v' = e^x, \quad v = e^x \end{array} \right.$

(2) $\left| \begin{array}{l} u = x, \quad u' = 1 \\ v' = e^x, \quad v = e^x \end{array} \right.$

$$I_5 = \int x^2 \cdot e^x dx = e^x (x^2 - 2x + 2) + C$$

The chalkboard shows the following steps for solving the integral $I_6 = \int x^2 \cos x dx$ using integration by parts:

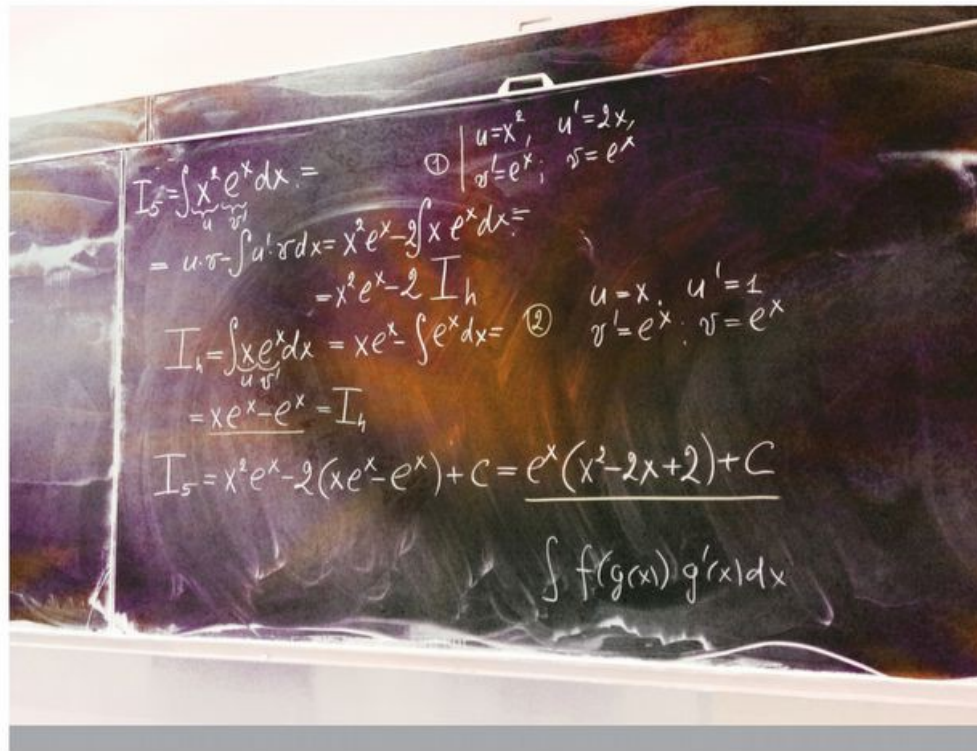
- Initial integral: $I_6 = \int x^2 \cos x dx$
- First step: $I_6 = x^2 \sin x - \int 2x \sin x dx$
- Second step: $I_6 = x^2 \sin x + 2x \cos x + \int 2 \cos x dx$
- Third step: $I_6 = x^2 \sin x + 2x \cos x + 2 \sin x + C$
- Final simplified form: $I_6 = \sin x (x^2 + 2) + 2x \cos x + C$

Handwritten notes on the right side of the board specify the choices for u and v in each step:

- Step 1: $u = x^2, v = \sin x$
 $u' = 2x, v' = \cos x$
- Step 2: $u = 2x, v = -\cos x$
 $u' = 2, v' = \sin x$

$$I_6 = \int x^2 \cdot \cos x dx = x^2 \sin x + 2 x \cos x - 2 \sin x + C$$

2012



JANUARY

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FEBRUARY

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AUGUST

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SEPTEMBER

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OCTOBER

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NOVEMBER

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DECEMBER

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