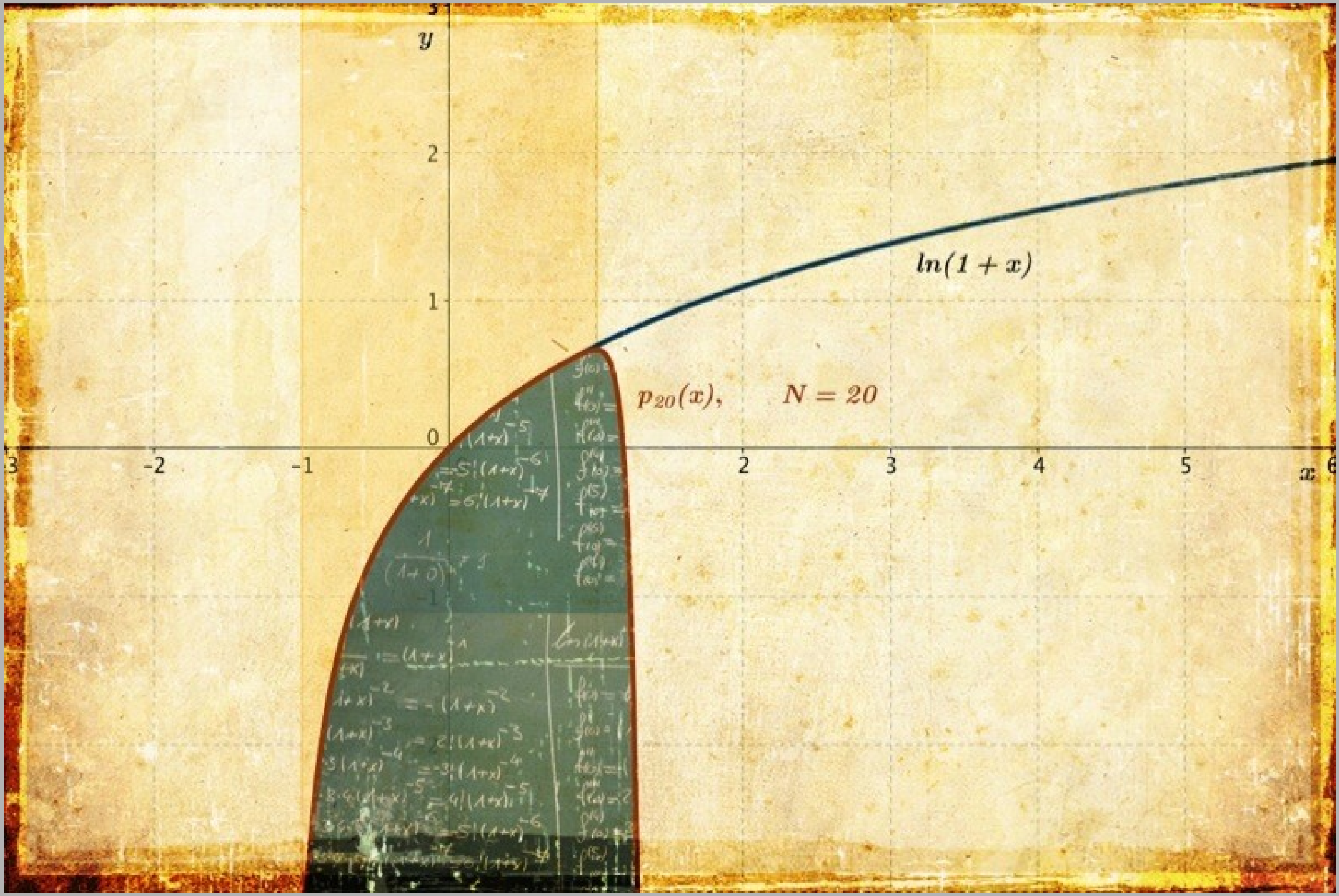
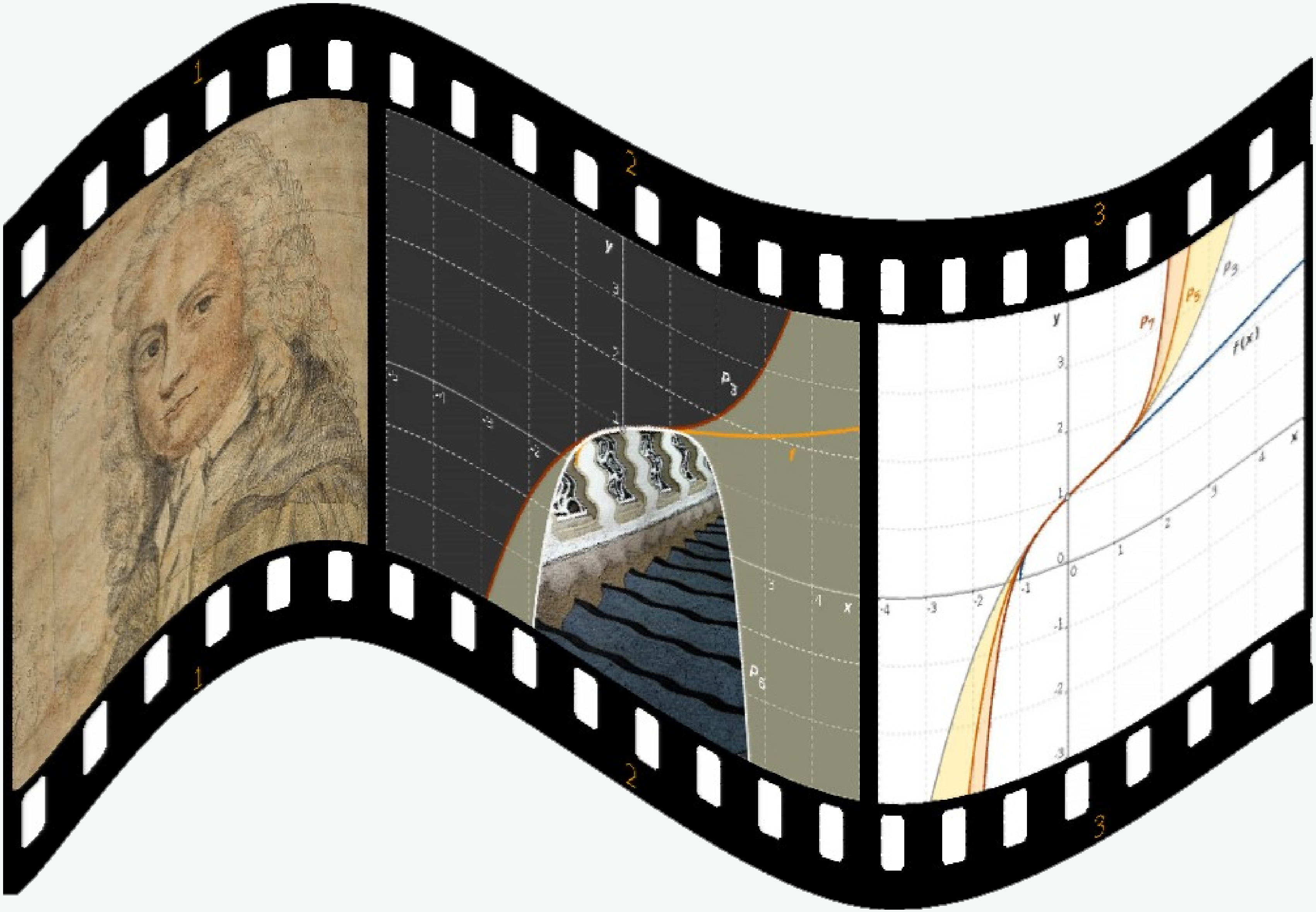
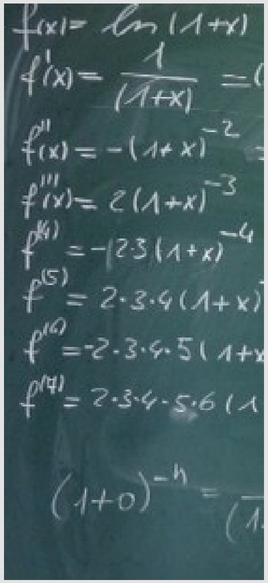


Maclaurinsche Reihe: Aufgaben





Entwickeln Sie die folgenden Funktionen in eine Maclaurinsche Reihe und berechnen Sie den Konvergenzradius

Aufgabe 9: $f(x) = (1+x)^m$

$a)\ \sqrt{1+x}\,, \qquad b)\ \sqrt{1-x}$

$c)\ (1+x)^{-1}\,, \qquad d)\ (1-x)^{-1}$

Aufgabe 10: $f(x) = \ln(1+x)$

Aufgabe 11: $f(x) = \ln(1-x)$

Aufgabe 12: $f(x) = \ln\left(\dfrac{1+x}{1-x}\right)$

Aufgabe 13: $f(x) = \arcsin x$

$$f(x) = (1+x)^m, \qquad f(0) = 1$$

$$f'(x) = m(1+x)^{m-1}, \qquad f'(0) = m$$

$$f''(x) = m(m-1)(1+x)^{m-2}, \qquad f''(0) = m(m-1)$$

$$f'''(x) = m(m-1)(m-2)(1+x)^{m-3}$$

$$f'''(0) = m(m-1)(m-2)$$

$$f^{(4)}(0) = m(m-1)(m-2)(m-3)$$

$$f^{(k)}(0) = m(m-1)(m-2)\ \ldots\ (m-k+1)$$

$$f(x) = f(0) + \frac{f'(0)}{1!}\, x + \frac{f''(0)}{2!}\, x^2 + \frac{f'''(0)}{3!}\, x^3 + \ldots =$$

$$= 1 + \frac{m}{1!}\, x + \frac{m(m-1)}{2!}\, x^2 + \frac{m(m-1)(m-2)}{3!}\, x^3 +$$

$$+ \frac{m(m-1)(m-2)(m-3)}{4!}\, x^4 + \ldots + \frac{m\ldots(m-k+1)}{4!}\, x^k + \ldots =$$

$$(1+x)^m = 1 + \frac{m}{1!}\,x + \frac{m(m-1)}{2!}\,x^2 + \frac{m(m-1)(m-2)}{3!}\,x^3 + \dots$$

$$(1+x)^m = \sum_{n=0}^{\infty} a_n\, x^n, \qquad a_n = \frac{m(m-1)\dots(m-n+1)}{n!}$$

$$\frac{a_n}{a_{n+1}} = \frac{m(m-1)\dots(m-n+1)}{n!} \cdot \frac{(n+1)!}{m(m-1)\dots(m-n)} = \frac{n+1}{m-n}$$

$$r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{m-n} \right| = \lim_{n\to\infty} \left| \frac{1+\frac{1}{n}}{\frac{m}{n}-1} \right| = 1$$

Maclaurinsche Reihe: Lösung 9

$$a)\ (1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5}{128}\,x^4 + \ldots$$

$$b)\ (1-x)^{1/2} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5}{128}\,x^4 - \ldots$$

$$c)\ (1+x)^{-1} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \ldots$$

$$d)\ (1-x)^{-1} = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots$$

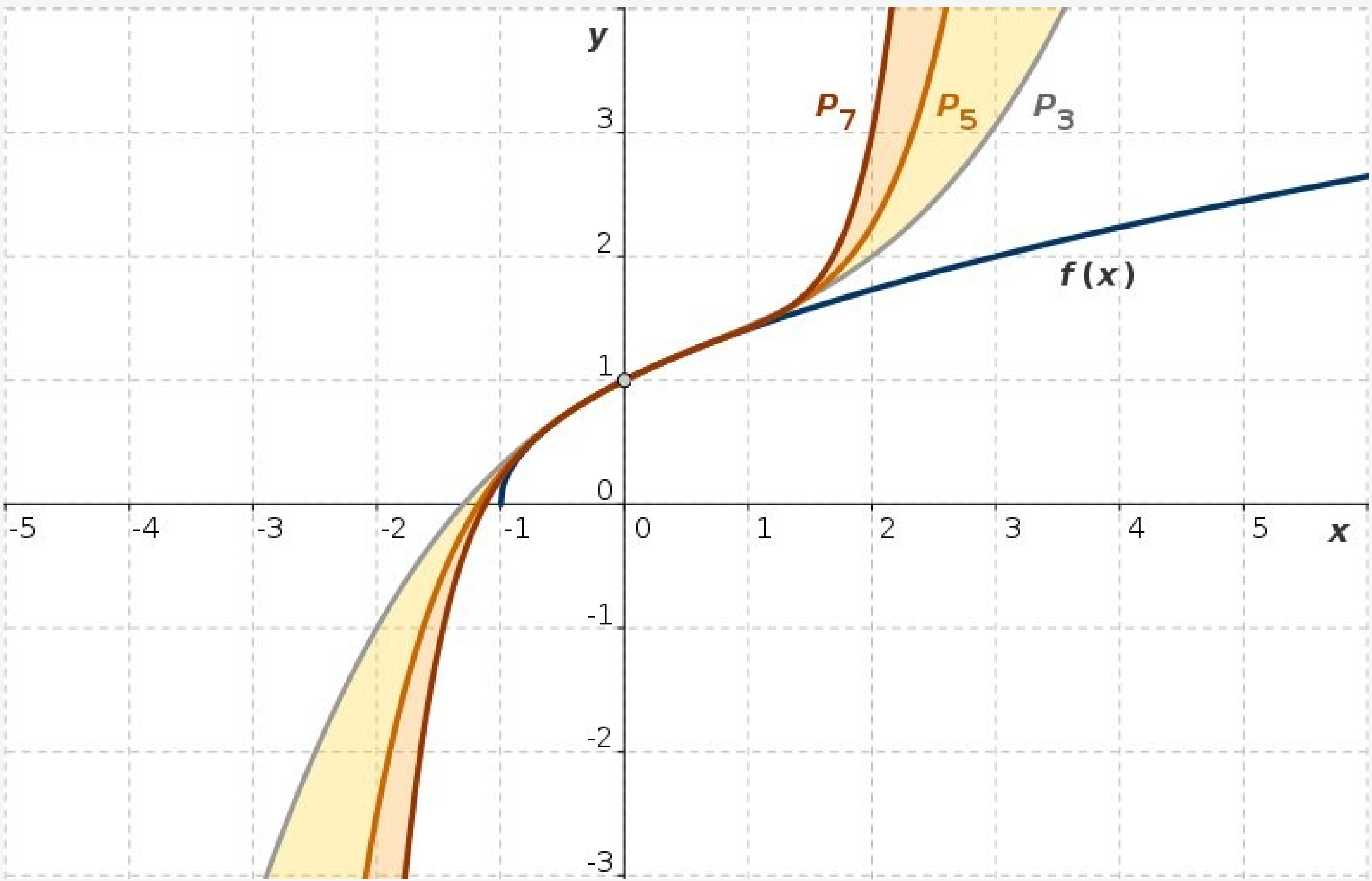


Abb. L9-1: Die Funktion f (x) = cos x und Näherungspolynome 3., 5. und 7. Grades

$$f(x) = (1 + x)^{1/2} = \frac{1}{\sqrt{1 + x}}$$

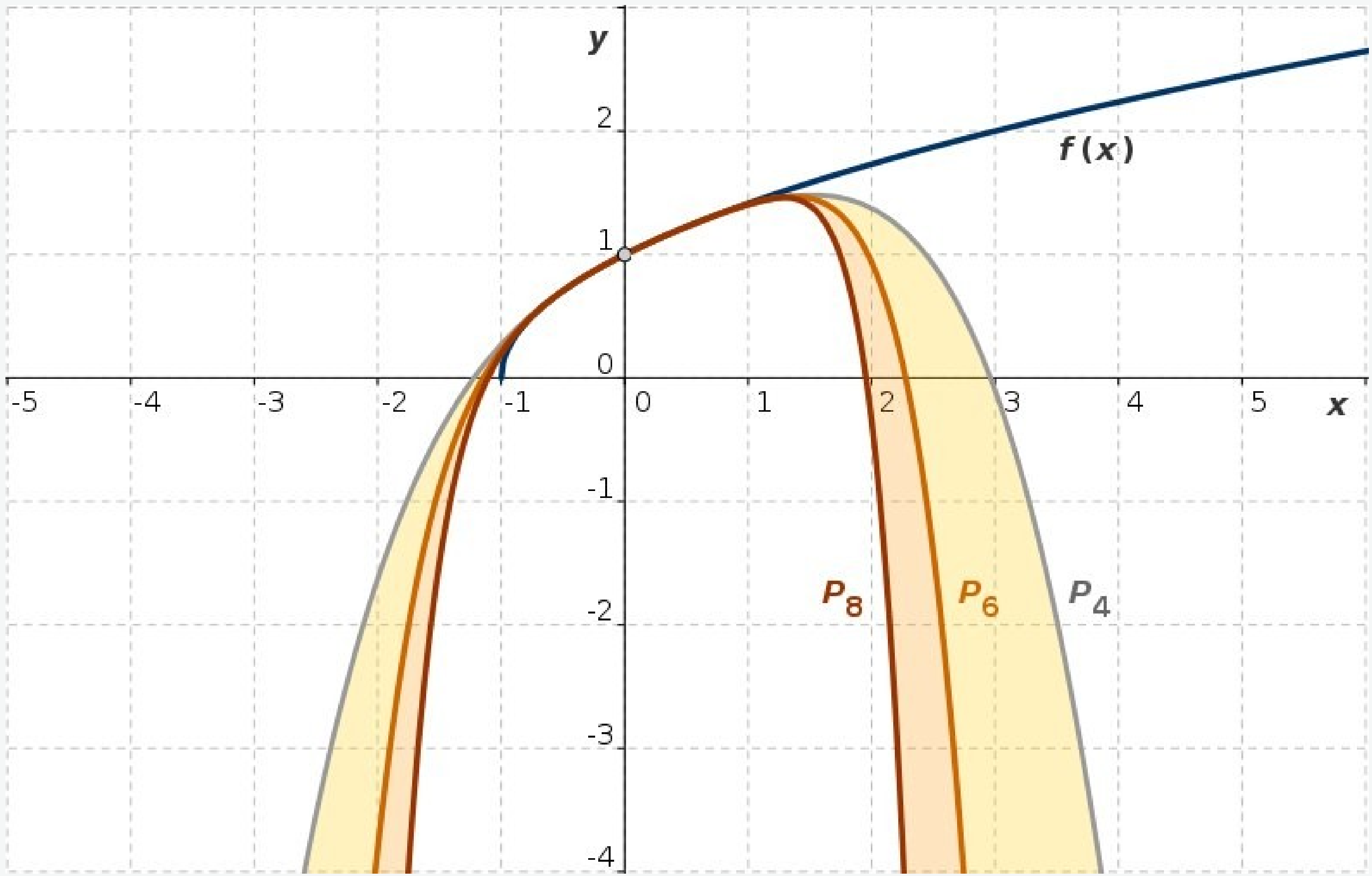


Abb. L9-2: Die Funktion f (x) = cos x und Näherungspolynome 4., 6. und 8. Grades

$$f(x) = (1 + x)^{1/2} = \frac{1}{\sqrt{1 + x}}$$

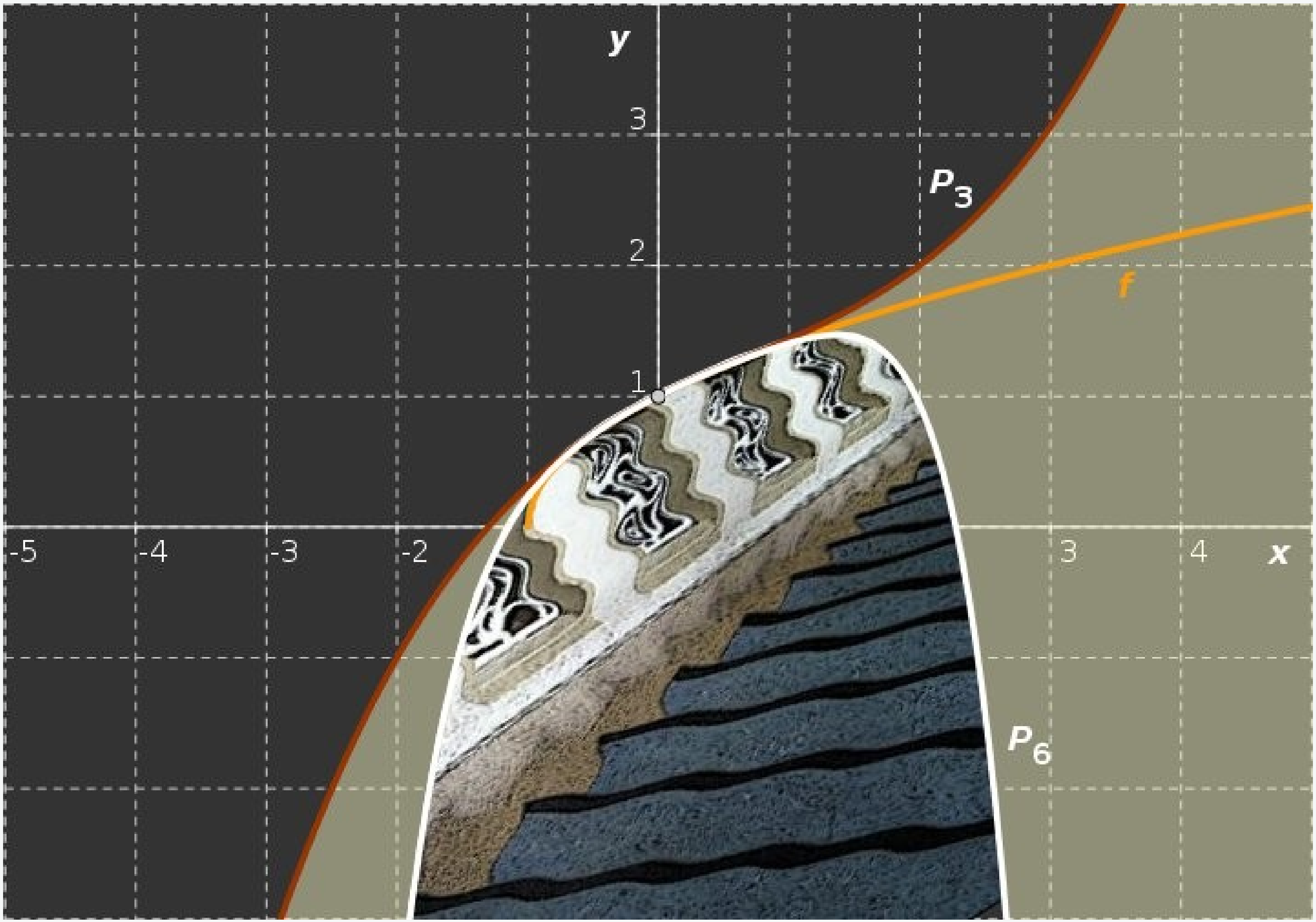


Abb. L9-3: Die Funktion f (x) = cos x und Näherungspolynome 3. und 6. Grades

$$f(x) = \ln(1+x), \qquad f(0) = \ln(1) = 0$$

$$\ln'(1+x) = \frac{1}{1+x} = (1+x)^{-1}, \qquad \ln'(1+x)\big|_{x=0} = 1$$

$$\ln''(1+x) = -(1+x)^{-2}, \qquad \ln''(1+x)\big|_{x=0} = -1$$

$$\ln'''(1+x) = 2\,(1+x)^{-3}, \qquad \ln'''(1+x)\big|_{x=0} = 2 = 2!$$

$$\ln^{(4)}(1+x) = -2\cdot 3\,(1+x)^{-4}, \qquad \ln^{(4)}(1+x)\big|_{x=0} = -2\cdot 3 = -3!$$

$$\ln^{(5)}(1+x) = 2\cdot 3\cdot 4\,(1+x)^{-5}, \qquad \ln^{(5)}(1+x)\big|_{x=0} = 2\cdot 3\cdot 4 = 4!$$

$$f(x) = f(0) + \frac{f'(0)}{1!}\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \dots$$

$$= \ln 1 + \frac{\ln'(1+x)\big|_{x=0}}{1!}\,x + \frac{\ln''(1+x)\big|_{x=0}}{2!}\,x^2 + \frac{\ln'''(1+x)\big|_{x=0}}{3!}\,x^3 + \dots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\,x^n$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots =$$

$$= \sum_{n=1}^{\infty} a_n \, x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \, x^n$$

$$a_n = \frac{(-1)^{n+1}}{n}, \qquad a_{n+1} = \frac{(-1)^n}{n+1}$$

$$r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \left| \frac{n+1}{n} \right| = 1, \qquad x \in (-1, \; 1]$$

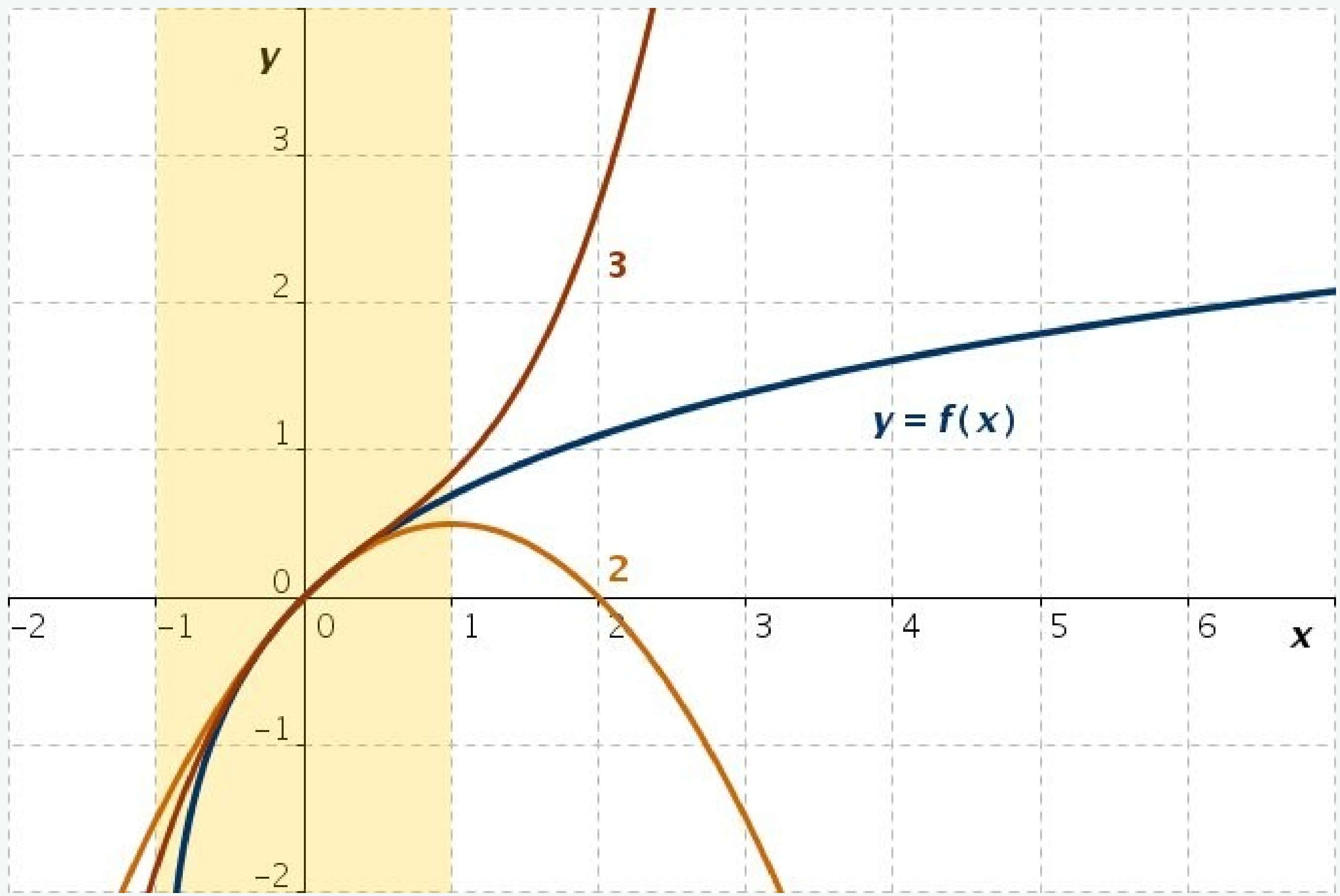


Abb. 1L0-1: Die Funktion f (x) = ln (1 + x) und Näherungspolynome 2. und 3. Grades

$$p_2 = x - \frac{x^2}{2}, \qquad p_3 = x - \frac{x^2}{2} + \frac{x^3}{3}$$

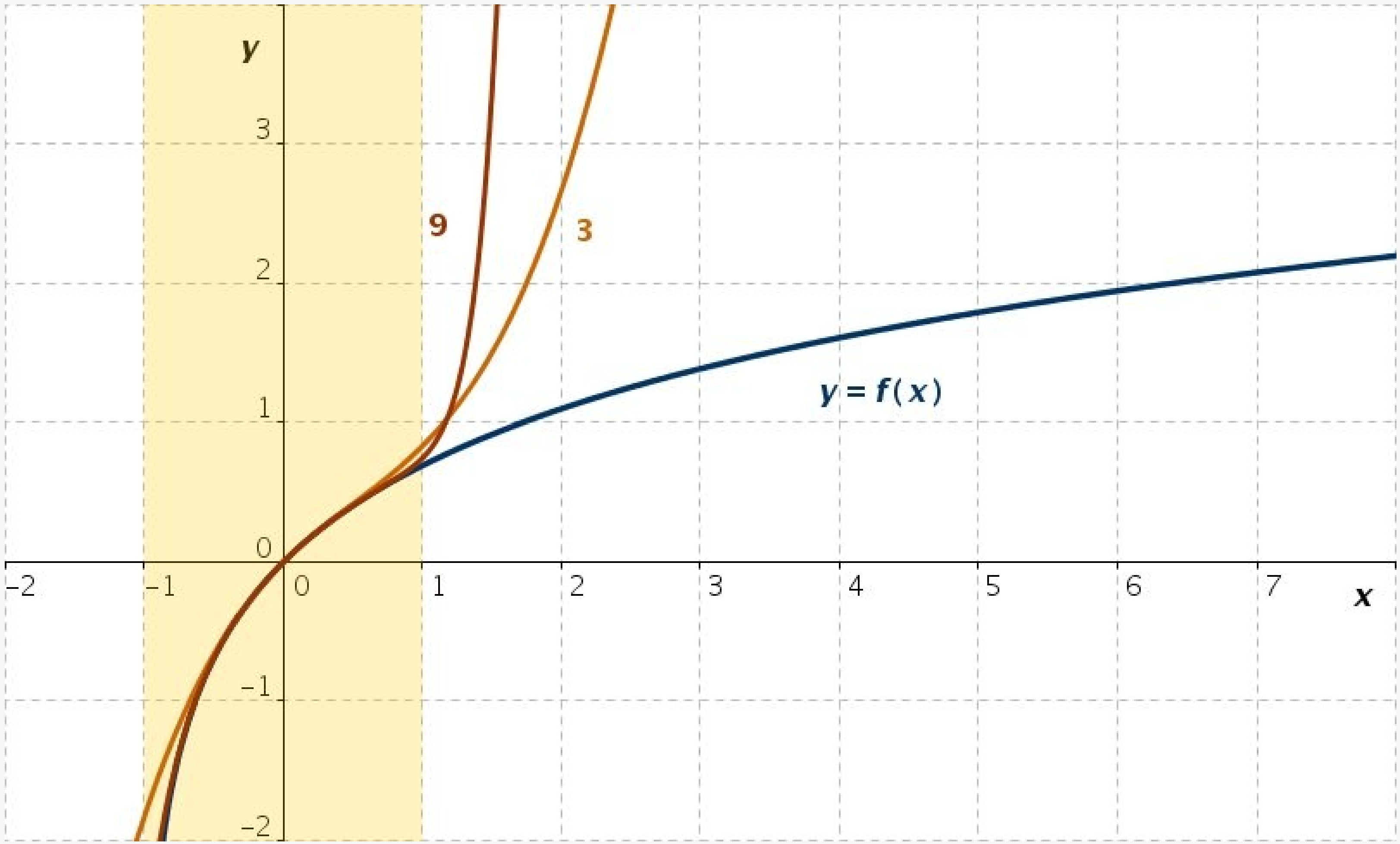


Abb. L10-2: Die Funktion f (x) = ln (1 + x) und Näherungspolynome 3. und 9. Grades

$$p_3 = x - \frac{x^2}{2} + \frac{x^3}{3}, \qquad p_9 = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{x^9}{9}$$

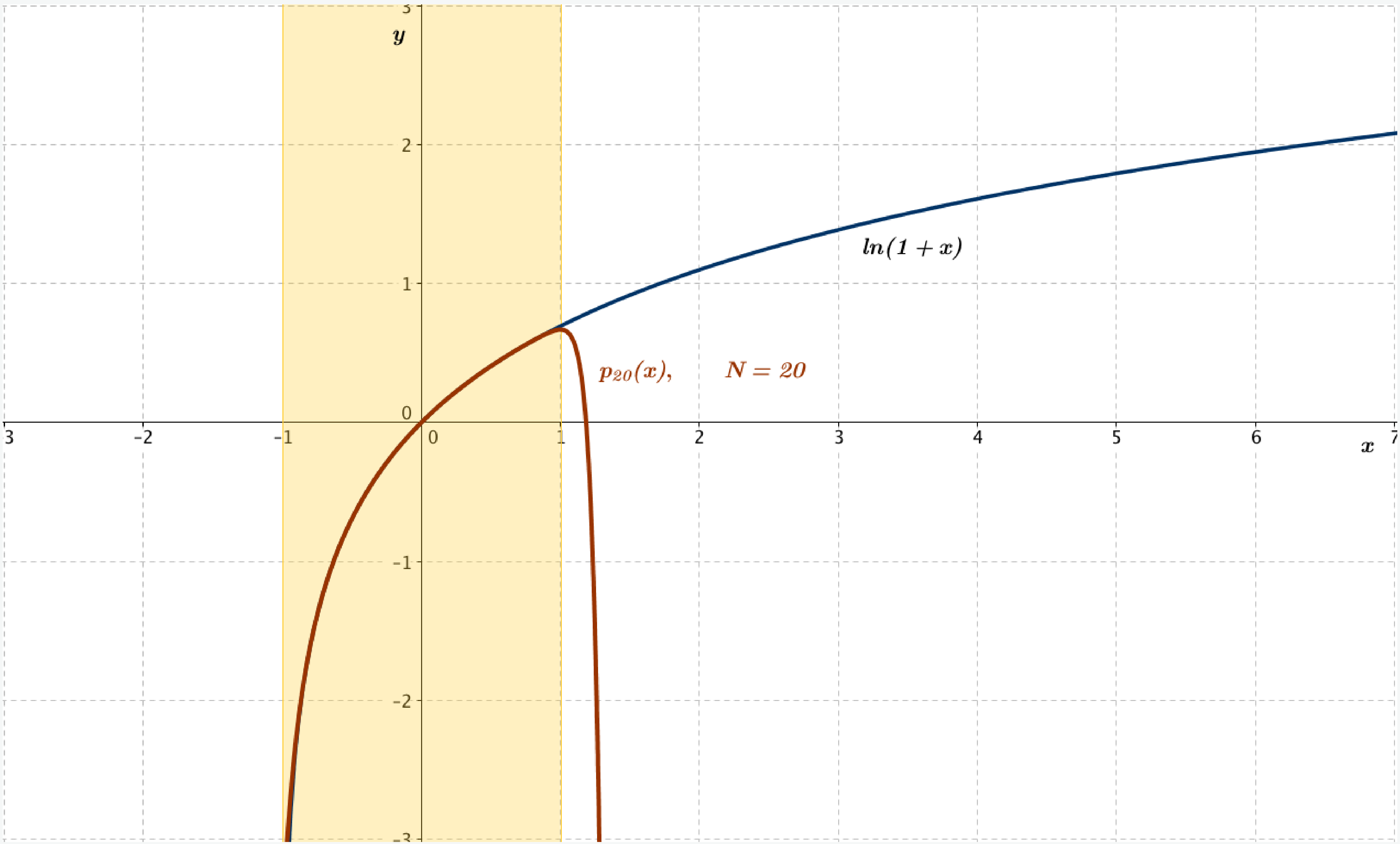


Abb. L10-3: Die Funktion f (x) = ln (1 + x) und Näherungspolynome 20. Grades

$$\ln'(1+x) = \frac{1}{1+x}$$

$$\int_0^x \frac{dt}{1+t} = \ln(1+t)\,\Big|_0^x = \ln(1+x) - \ln 1 = \ln(1+x)$$

$$\ln(1+x) = \int_0^x \frac{dt}{1+t}$$

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + t^4 + \ldots$$

$$\ln(1+x) = \int_0^x \frac{dt}{1+t} = \int_0^x (1 - t + t^2 - t^3 + t^4 - \ldots)\, dt =$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\, x^n$$

$$(|r| < 1)$$

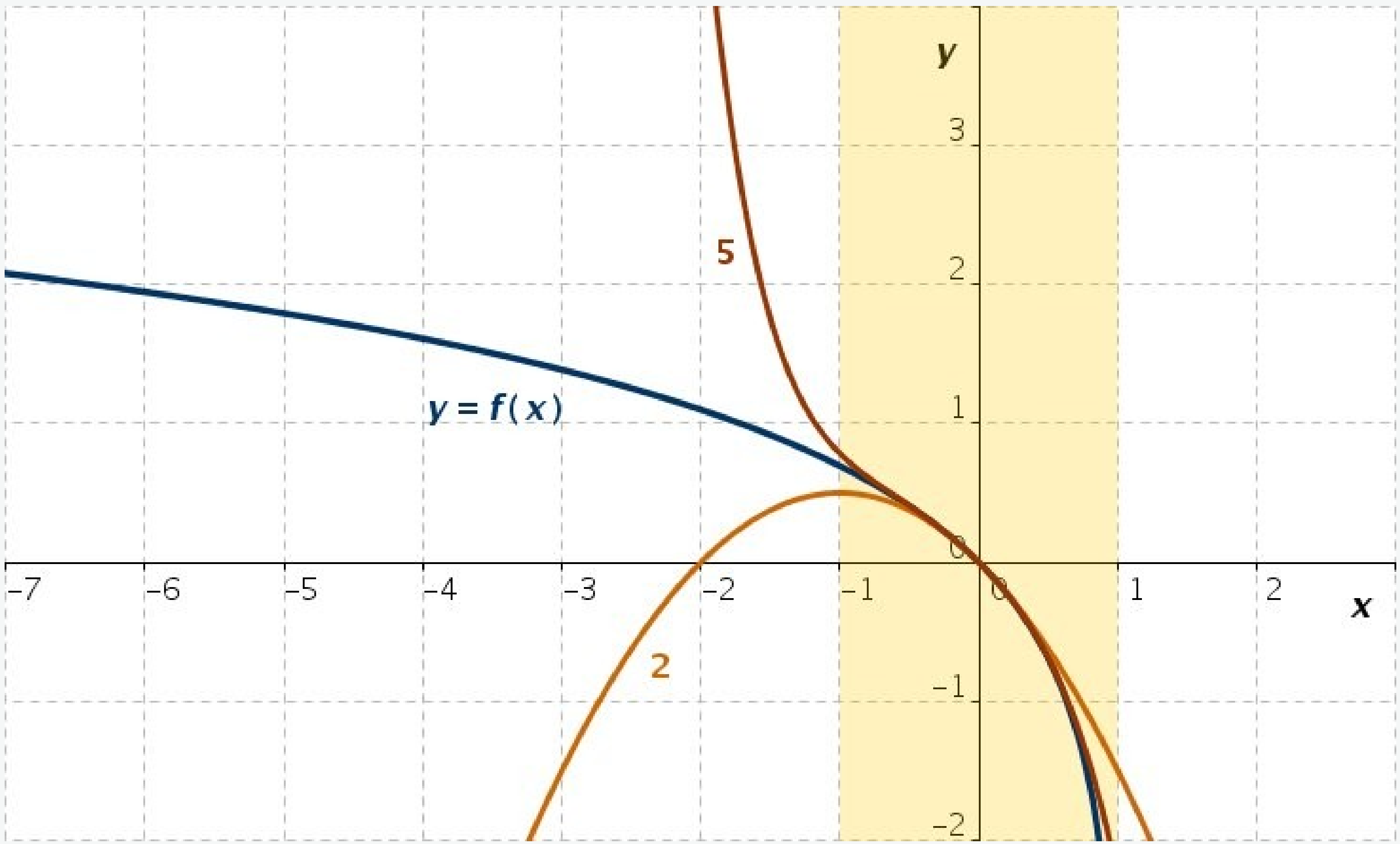


Abb. L11: Die Funktion f (x) = ln (1 - x) und Näherungspolynome 2. und 5. Grades

$$\ln(1 - x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \ldots\right) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$P_2 = -\left(x + \frac{x^2}{2}\right), \qquad P_5 = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5}\right)$$

Maclaurinsche Reihe: Lösung 12

Möglichkeit 1:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$$

$$\ln(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \ldots\right)$$

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) =$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right) = 2\sum_{n=1}^{\infty} \frac{x^{2n+1}}{2n+1}$$

$$(|r| < 1)$$

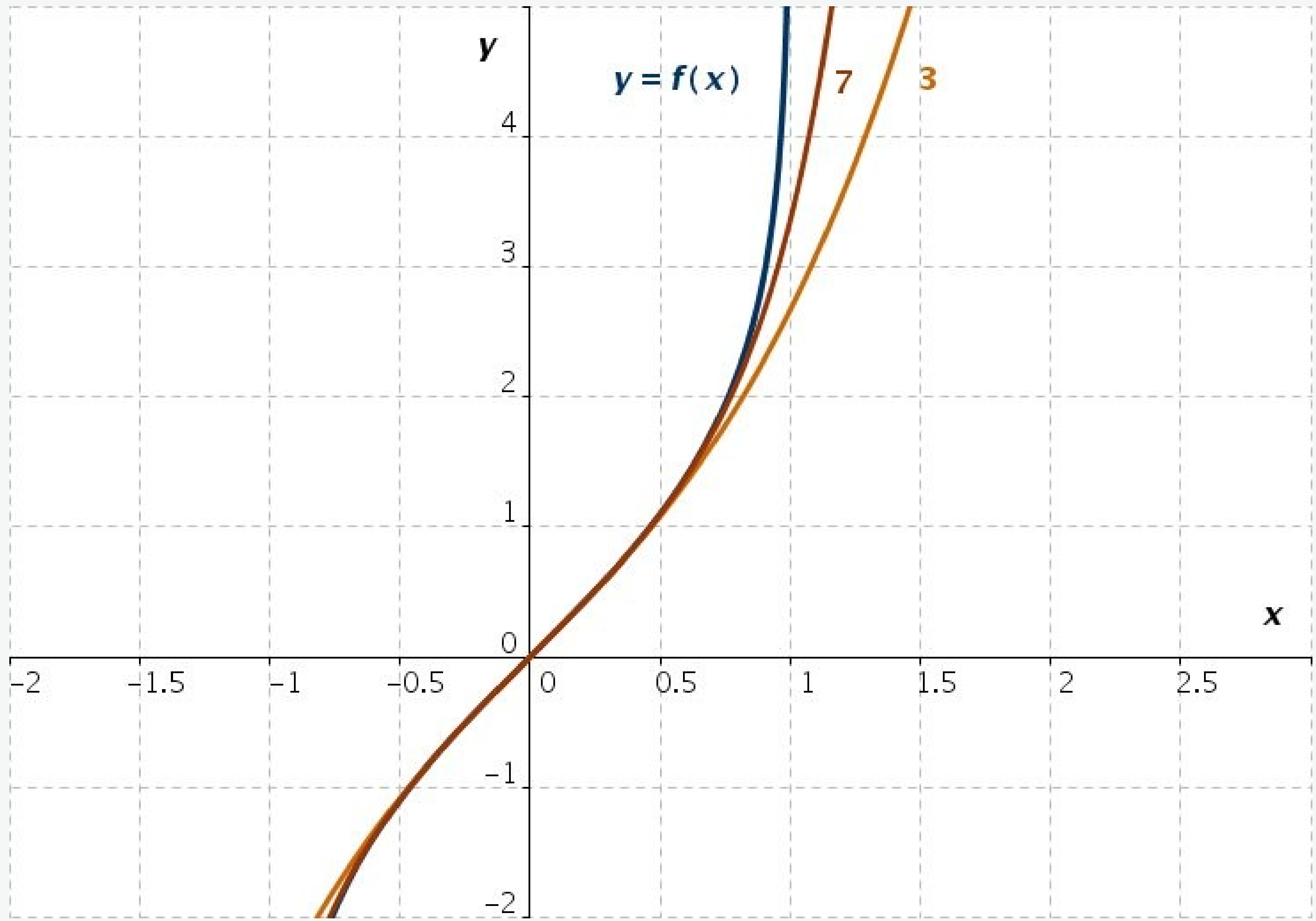


Abb. L12: Die Funktion y = f (x) und Näherungspolynome 3. und 7. Grades

$$f(x) = \ln\left(\frac{1+x}{1-x}\right), \qquad p_3 = 2\left(x + \frac{x^3}{3}\right), \qquad p_7 = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7}\right)$$

$$\ln'\left(\frac{1+x}{1-x}\right) = \ln'(1+x) - \ln'(1-x) = \frac{1}{1+x} + \frac{1}{1-x} = \frac{2}{1-x^2}$$

$$2\int_0^x \frac{dt}{1-t^2} = \ln\left|\frac{1+t}{1-t}\right| \Big|_0^x = \ln\frac{1+x}{1-x}, \qquad |x| < 1$$

$$(1+x)^m = 1 + \frac{m}{1!}x + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \dots$$

$$\frac{1}{1-t^2} = (1-t^2)^{-1}, \qquad x = -t^2, \qquad m = -1$$

$$(1-t^2)^{-1} = 1 + t^2 + t^4 + t^6 + \dots$$

$$2\int_0^x \frac{dt}{1-t^2} = 2\int_0^x (1 + t^2 + t^4 + t^6 + \dots)\, dt =$$

$$= 2\left[t + \frac{t^3}{3} + \frac{t^5}{5} + \frac{t^7}{7} + \dots\right]_0^x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots$$

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots\right) = 2\sum_{n=1}^{\infty} \frac{x^{2n+1}}{2n+1} \qquad (|r| < 1)$$

$$f(x) = \arcsin x, \qquad f'(x) = \frac{1}{\sqrt{1 - x^2}} = (1 - x^2)^{-\frac{1}{2}}$$

$$\int_0^x \frac{dt}{\sqrt{1 - t^2}} = \int_0^x (1 - t^2)^{-\frac{1}{2}}\, dt = \arcsin x$$

Wir vergleichen diese Ableitung mit der allgemeinen Formel:

$$(1 + t)^m = 1 + \frac{m}{1!}\, t + \frac{m(m-1)}{2!}\, t^2 + \frac{m(m-1)(m-2)}{3!}\, t^3 + \ldots$$

$$t = -x^2, \qquad m = -\frac{1}{2} \quad \Rightarrow$$

$$(1 - t^2)^m = 1 - \frac{m}{1!}\, t^2 + \frac{m(m-1)}{2!}\, t^4 - \frac{m(m-1)(m-2)}{3!}\, t^6 +$$

$$+ \frac{m(m-1)(m-2)(m-3)}{4!}\, t^8 + \ldots$$

$$\int_0^x (1 - t^2)^{-\frac{1}{2}}\, dt = x - \frac{m}{1!}\, \frac{x^3}{3} + \frac{m(m-1)}{2!}\, \frac{x^5}{5} - \frac{m(m-1)(m-2)}{3!}\, \frac{x^7}{7} + \ldots$$

$$\int_0^x \left(1 - t^2\right)^{-\frac{1}{2}} dt = x + \frac{1}{2 \cdot 3}\, x^3 + \frac{3}{2^2 \cdot 2!}\, \frac{x^5}{5} + \frac{3 \cdot 5}{2^3 \cdot 3!}\, \frac{x^7}{7} + \ldots =$$

$$= \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n - 1)}{2^n \cdot n!}\, \frac{x^{2n+1}}{2n+1}$$

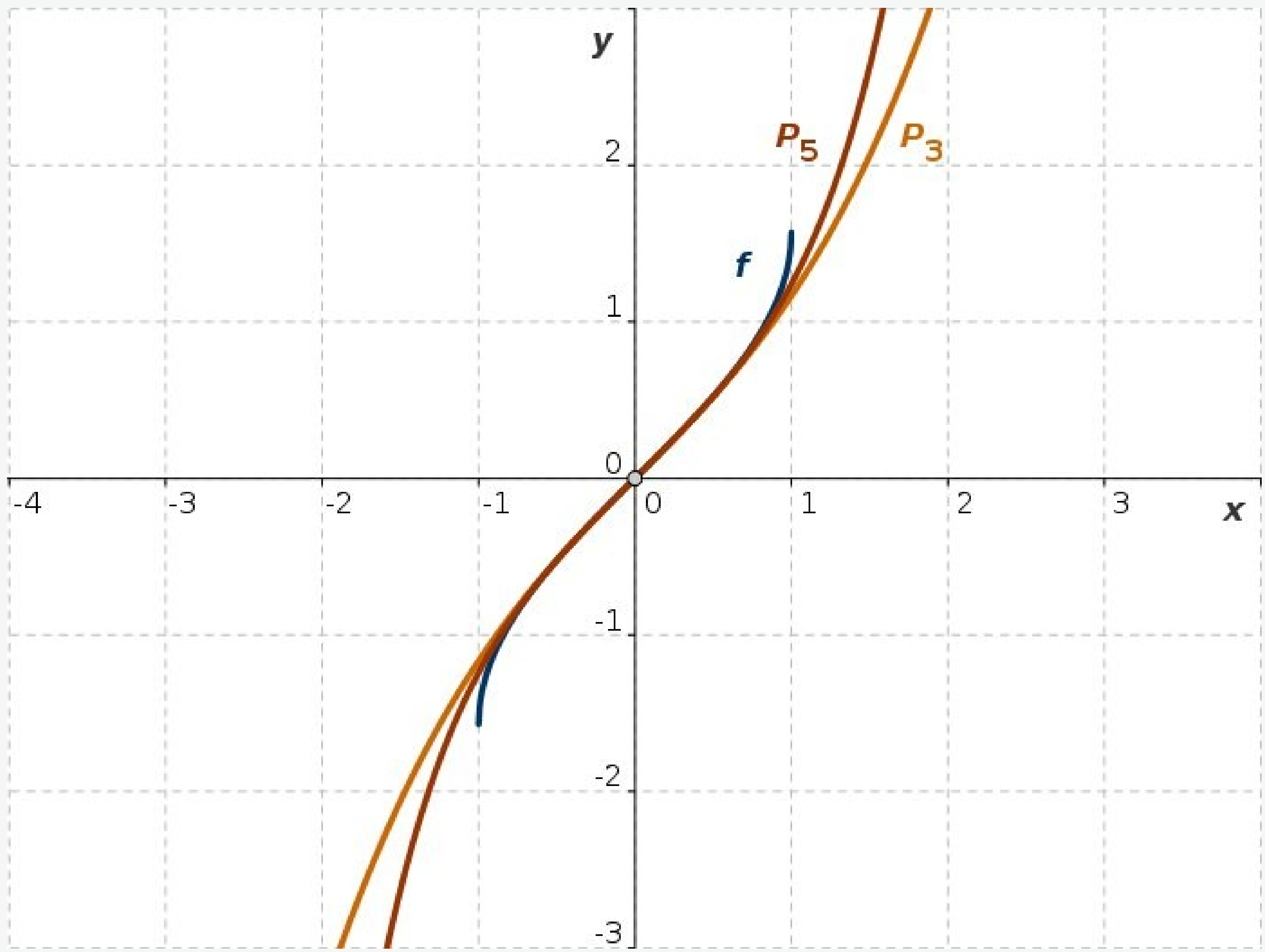


Abb. L13: Die Funktion f (x) = arcsin x und Näherungspolynome 3. und 5. Grades

$$f(x) = \arcsin x$$

Ma 2 – Lubov Vassilevskaya