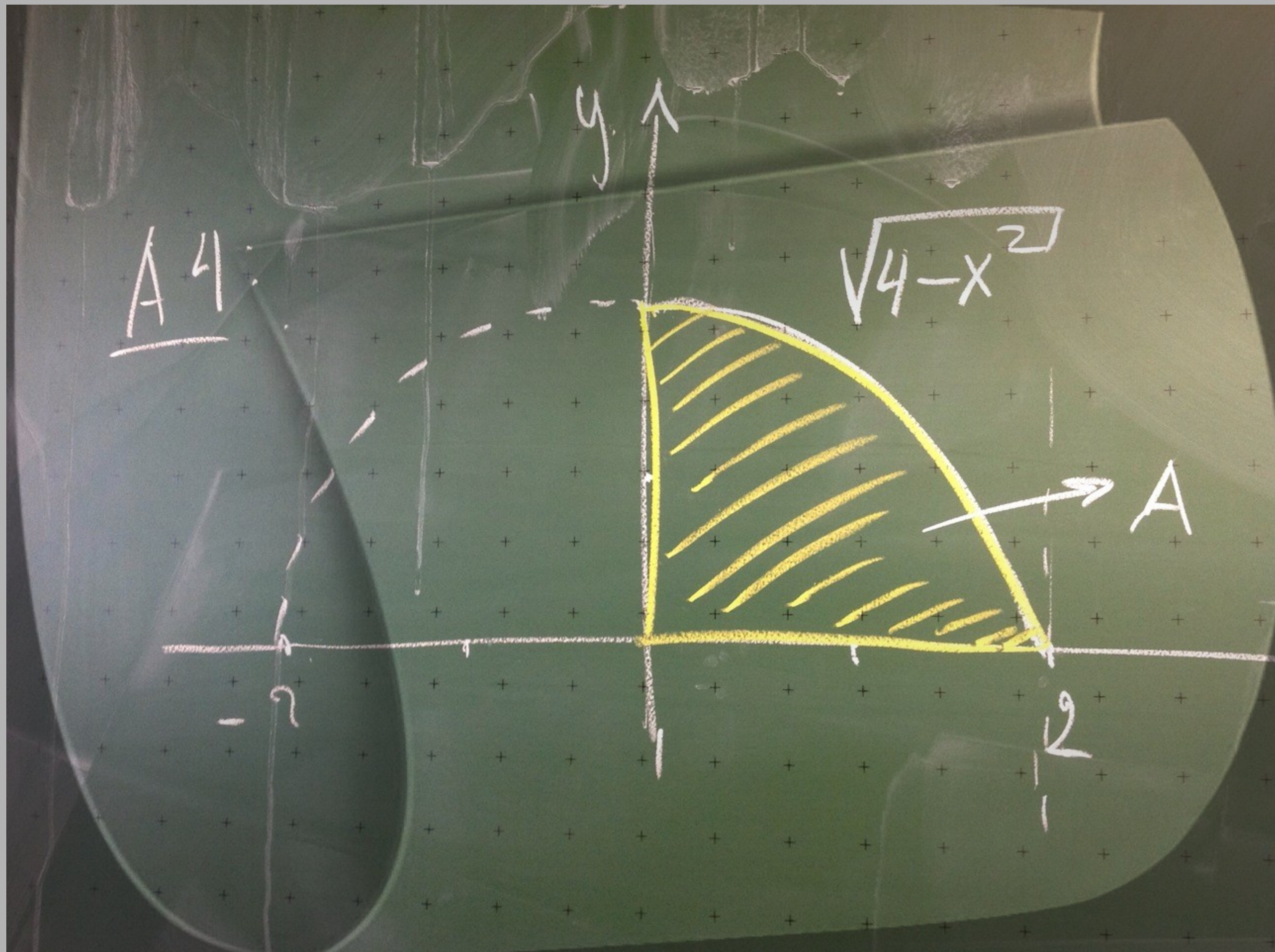

Doppelintegral in kartesischen Koordinaten

Aufgaben, Teil 1: Beliebige Integrationsgrenzen

Abb. 1: Die Darstellung des Integrationsbereiches A, Typ 1

$$a \leqslant x \leqslant b \,, \qquad g\,(x) \leqslant y \leqslant f\,(x)$$

Berechnen Sie die folgenden Doppelintegrale und zeichnen die entsprechenden Integrationsbereiche:

Aufgabe 1: $I = \int_{x=0}^{1} \int_{y=0}^{x} (x^2 + y^2)\, dy\, dx$

Aufgabe 2: $I = \int_{x=0}^{\pi/2} \int_{y=0}^{x} (1 + \sin y)\, dy\, dx$

Aufgabe 3: $I = \int_{y=0}^{1} \int_{x=-2}^{y} x\, y\, dx\, dy$

Aufgabe 4: $I = \int_{x=0}^{2} \int_{y=0}^{\sqrt{4-x^2}} x\, y\, dy\, dx$

Aufgabe 5: $I = \int_{x=0}^{3} \int_{y=0}^{\frac{1}{2}\sqrt{9-x^2}} x\, y\, dy\, dx$

Aufgabe 6: $$I = \int_{x=0}^{1} \int_{y=x^3}^{\sqrt{x}} (4\,x\,y - y^3)\; dy\, dx$$

Aufgabe 7: $$I = \int_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{y=-\sqrt{3+\cos(2\,x)}}^{\sqrt{\cos x}} f(x, y)\; dy\, dx$$

$a)\; f(x, y) = y, \quad b)\; f(x, y) = x\,y$

Aufgabe 8: $$I = \int_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{y=|x|-2}^{\cos x} x\; dy\, dx$$

Aufgabe 9: $$I = \int_{x=-2}^{2} \int_{y=\sqrt{1-\frac{x^2}{4}}}^{\sqrt{4-x^2}} f(x, y)\; dy\, dx$$

$a)\; f(x, y) = y, \quad b)\; f(x, y) = x\,y$

$c)\; f(x, y) = x, \quad d)\; f(x, y) = x^2$

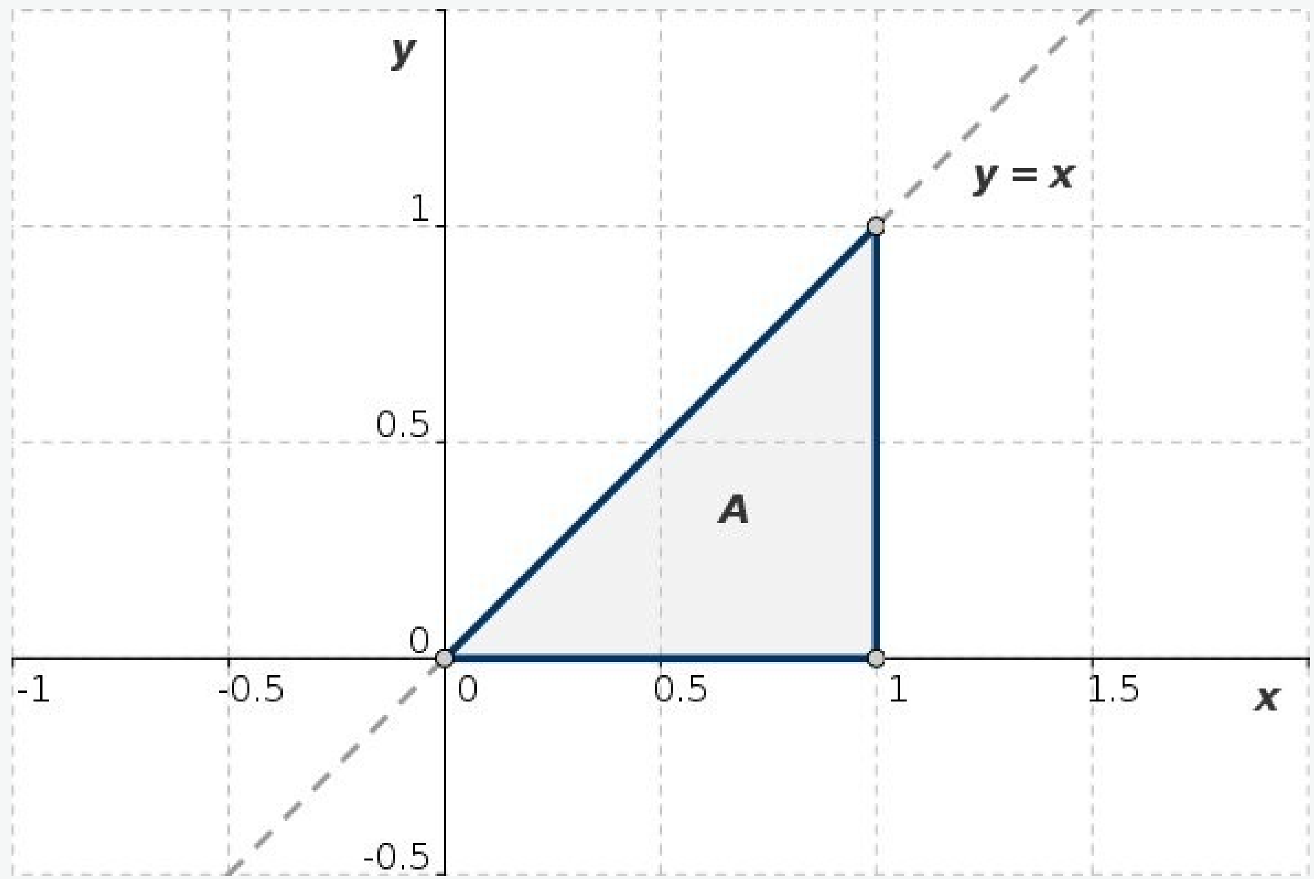

Abb. L1: Integrationsbereich der Aufgabe

$$I = \int_{x=0}^{1} \int_{y=0}^{x} (x^2 + y^2)\ dy\, dx = \int_{x=0}^{1} \left(\int_0^x (x^2 + y^2)\ dy \right) dx = \frac{4}{3} \int_0^1 x^3\ dx = \frac{1}{3}$$

$$A = \left\{ (x\,,\ y) \in \mathbb{R}^2 \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant x \right\}$$

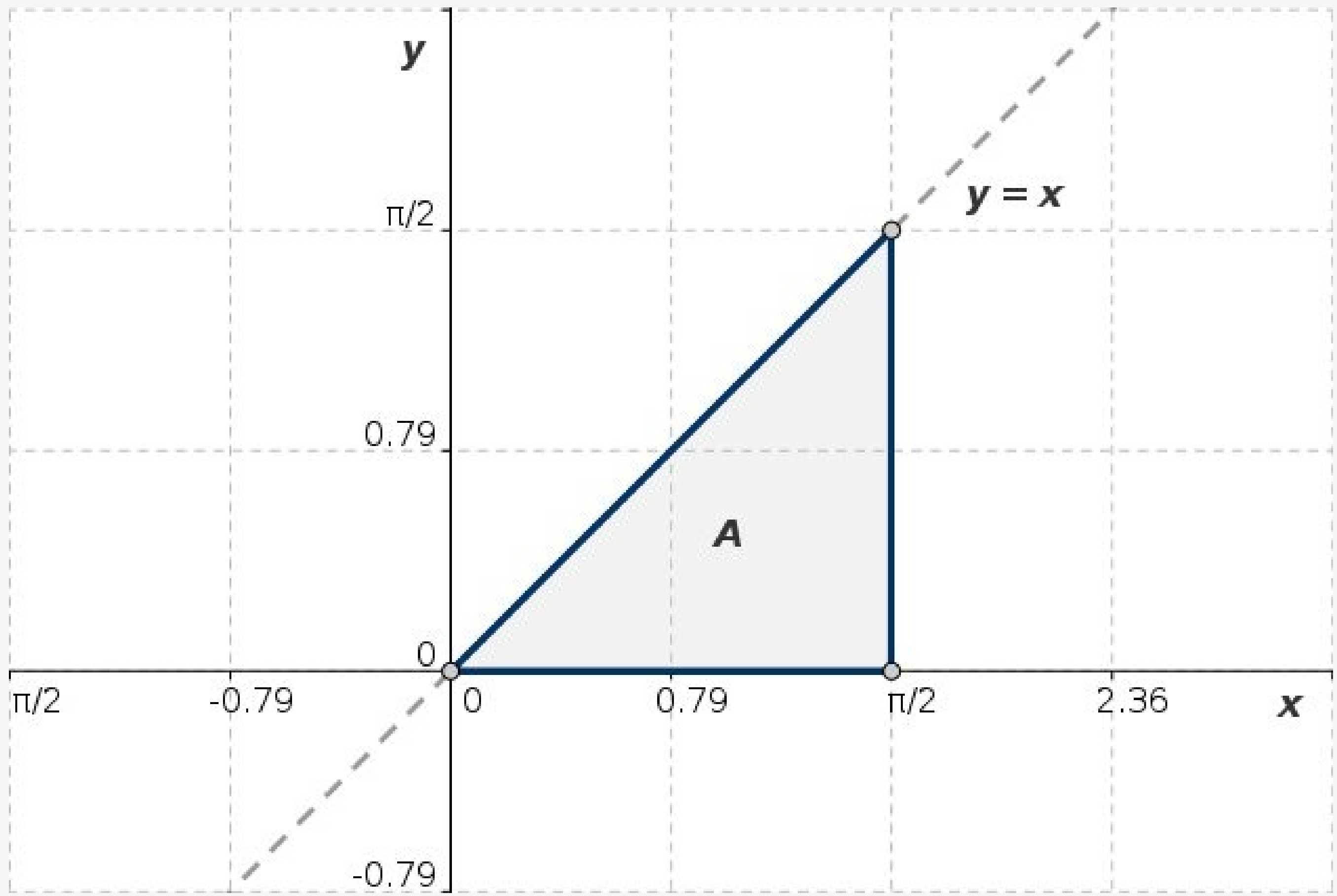

Abb. L2-a: Integrationsbereich der Aufgabe

$$I = \int_{x=0}^{\pi/2} \int_{y=0}^{x} (1 + \sin y)\, dy\, dx = \int_{0}^{\pi/2} (1 + x - \cos x)\, dx = \frac{\pi^2}{8} + \frac{\pi}{2} - 1 \simeq 1.80$$

$$A = \left\{ (x, y) \in \mathbb{R}^2 \quad 0 \leqslant x \leqslant \frac{\pi}{2}, \quad 0 \leqslant y \leqslant x \right\}$$

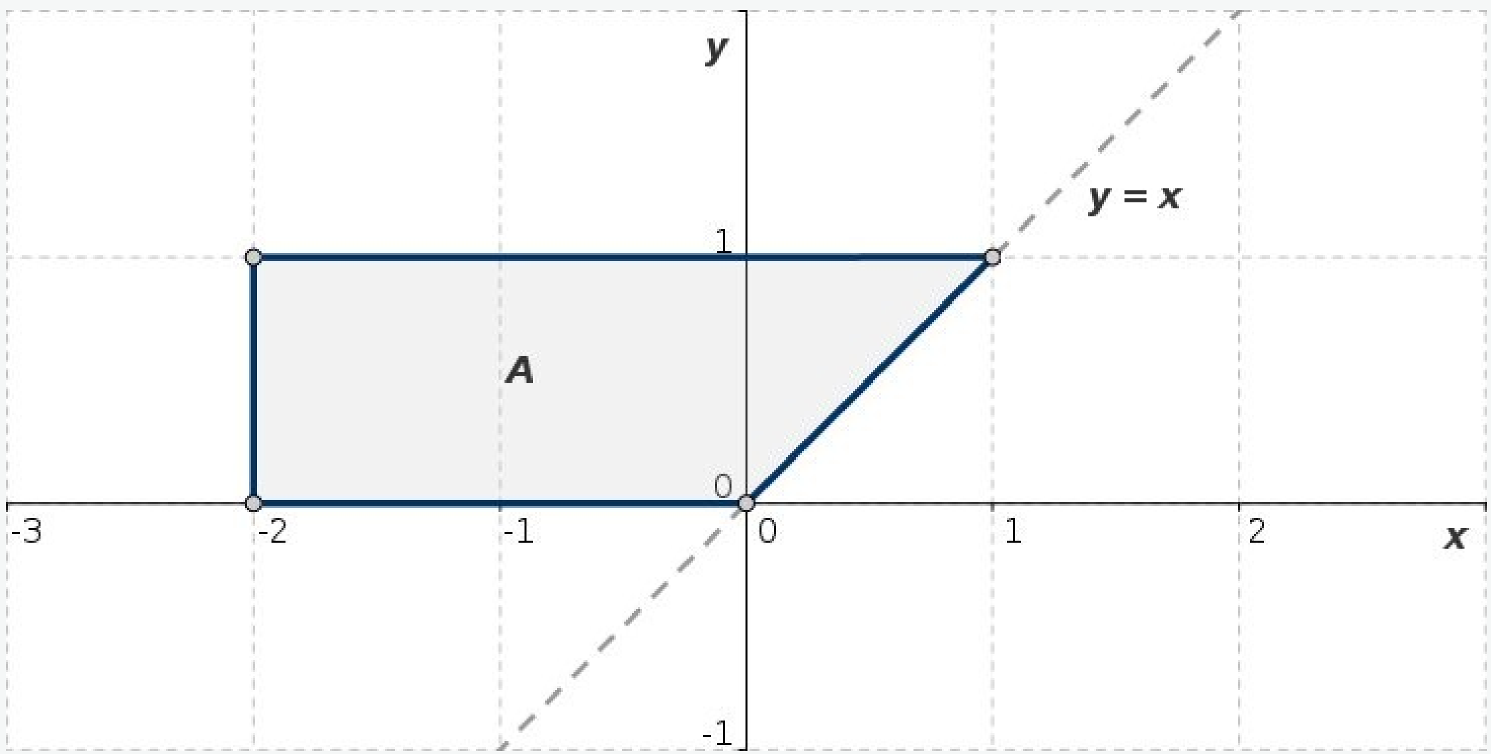

Abb. L3: Integrationsbereich der Aufgabe

$$I = \int_{y=0}^{1} \int_{x=-2}^{y} x \; y \; dx \, dy = \int_{y=0}^{1} y \left[\int_{x=-2}^{y} x \, dx \right] dy =$$

$$= \int_{0}^{1} y \left(\frac{y^2}{2} - 2 \right) dy = \int_{0}^{1} \left(\frac{y^3}{2} - 2\, y \right) dy = \left[\frac{y^4}{8} - y^2 \right]_0^1 = -\frac{7}{8}$$

$$A = \left\{ (x, \; y) \in \mathbb{R}^2 \quad -2 \leqslant x \leqslant y, \quad 0 \leqslant y \leqslant 1 \right\}$$

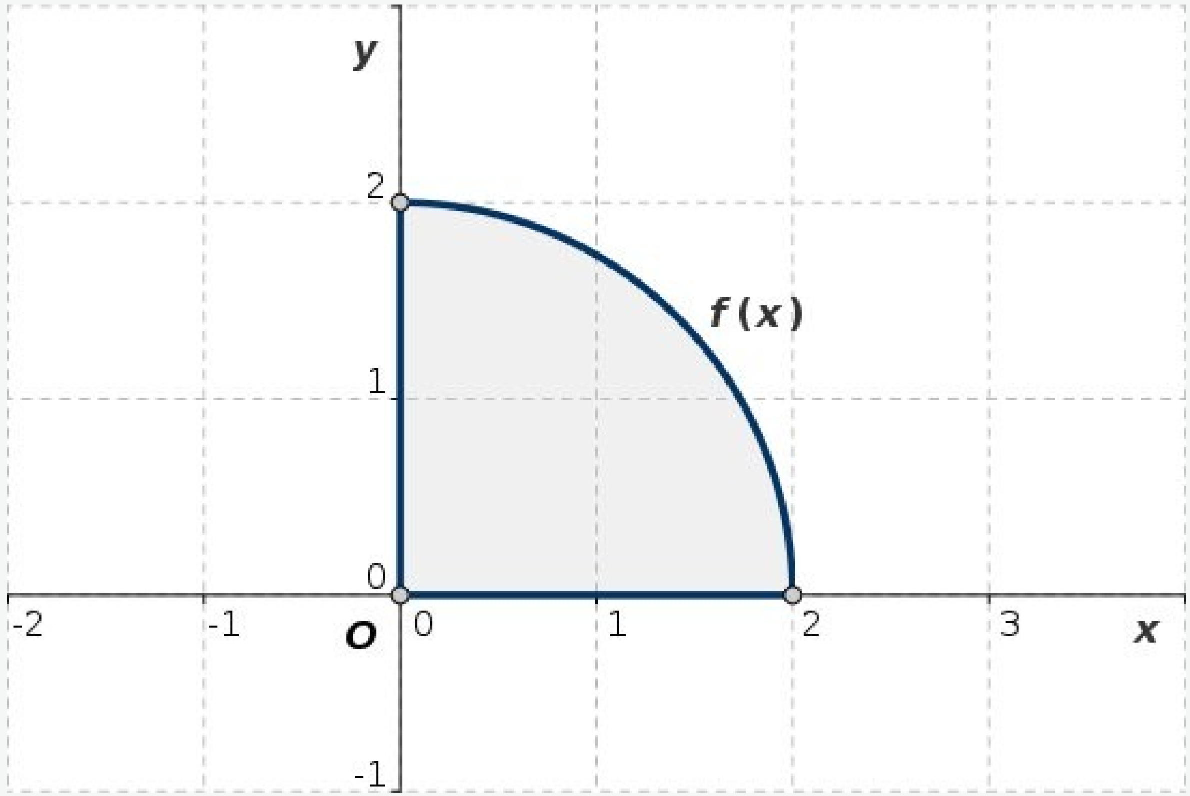

Abb. L4: Integrationsbereich der Aufgabe

$$I = \int_{x=0}^{2} \int_{y=0}^{\sqrt{4-x^2}} x \ y \ dy \, dx = \frac{1}{2} \int_{0}^{2} x \ (4 - x^2) \ dx = 2$$

$$A = \left\{ (x, \ y) \in \mathbb{R}^2, \quad 0 \leqslant x \leqslant 2, \quad 0 \leqslant y \leqslant \sqrt{4 - x^2} \right\}$$

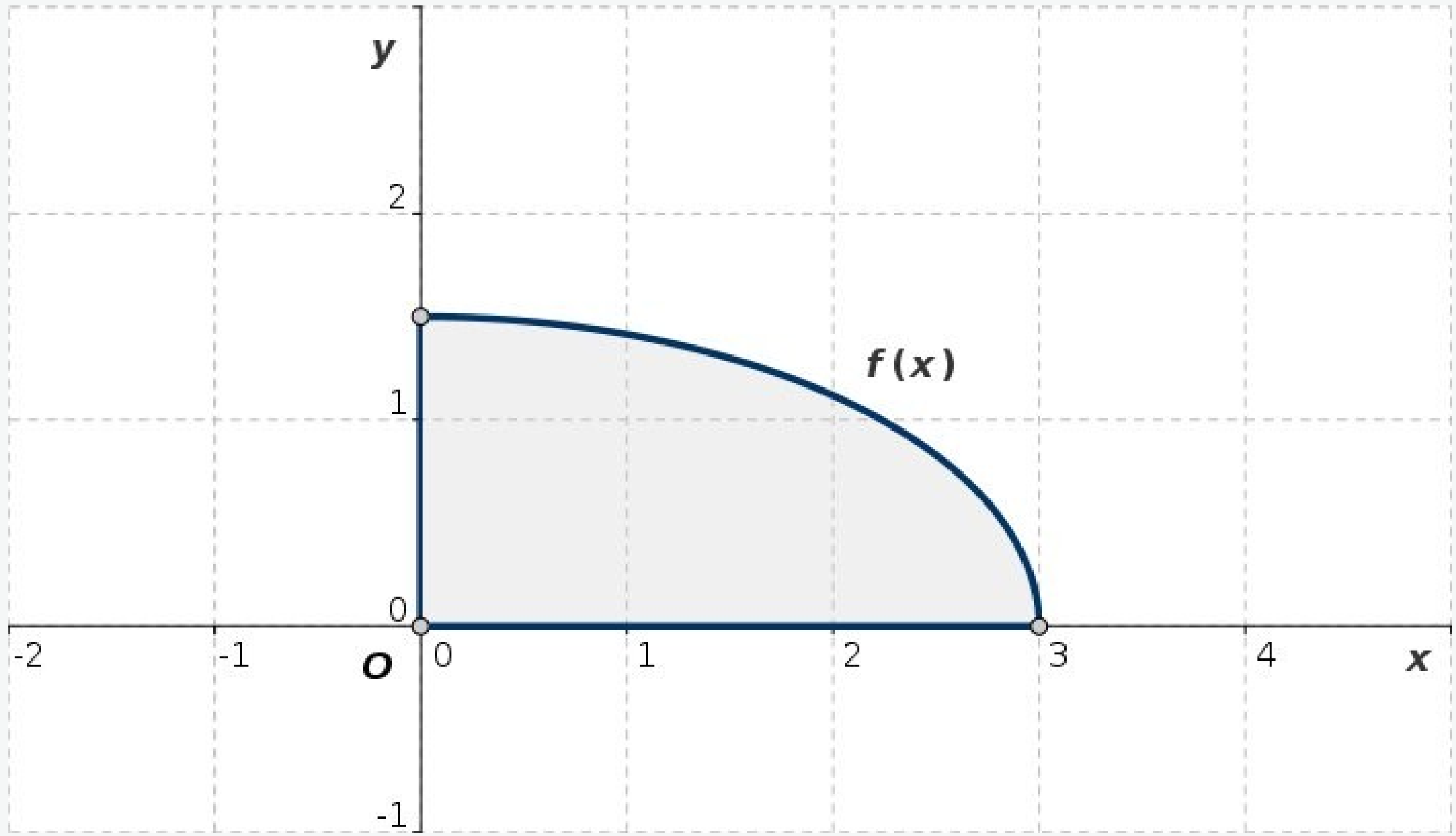

Abb. L5: Integrationsbereich der Aufgabe

$$I = \int_0^3 x \left(\int_0^{\frac{1}{2}\sqrt{9-x^2}} y \; dy \right) dx = \int_0^3 x \left[\frac{y^2}{2} \right]_0^{\frac{1}{2}\sqrt{9-x^2}} dx = \frac{1}{8} \int_0^3 (9x - x^3) \; dx = 2.53$$

$$A = \left\{ (x, \; y) \in \mathbb{R}^2 \quad 0 \leqslant x \leqslant 3, \quad 0 \leqslant y \leqslant \frac{1}{2}\sqrt{9 - x^2} \right\}$$

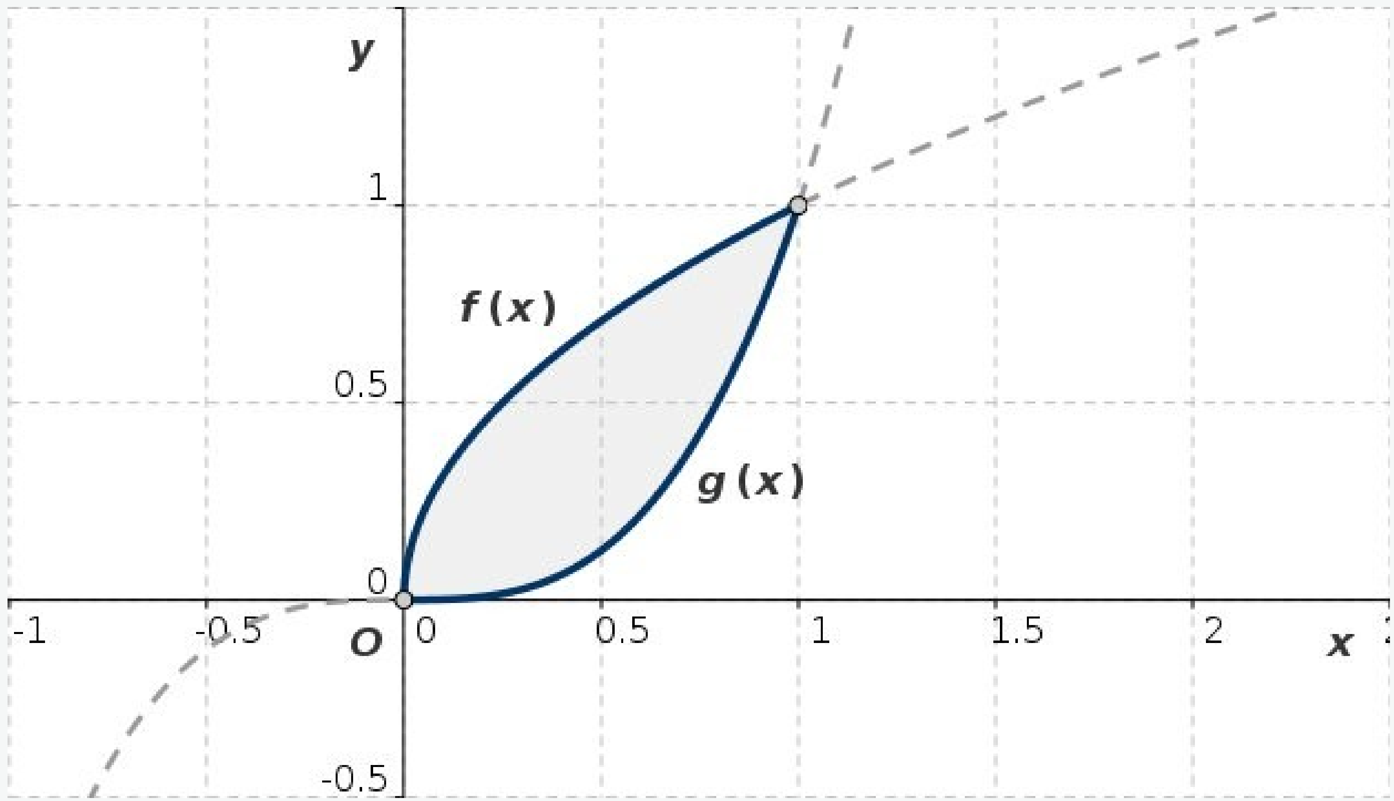

Abb. L6: Integrationsbereich der Aufgabe

$$f(x) = \sqrt{x}\,, \qquad g(x) = x^3$$

$$I = \int_{x=0}^{1} \left(\int_{x^3}^{\sqrt{x}} (4\,x\,y - y^3)\;dy \right) dx = \int_0^1 \left| 2\,xy^2 - \frac{y^4}{4} \right|_{x^3}^{\sqrt{x}} dx = \int_0^1 \left(\frac{7\,x^2}{4} - 2\,x^7 + \frac{x^{12}}{4} \right) dx = \frac{55}{156}$$

$$A = \left\{ (x\,,\; y) \in \mathbb{R}^2 \quad 0 \leqslant x \leqslant 1, \quad x^3 \leqslant y \leqslant \sqrt{x} \right\}$$

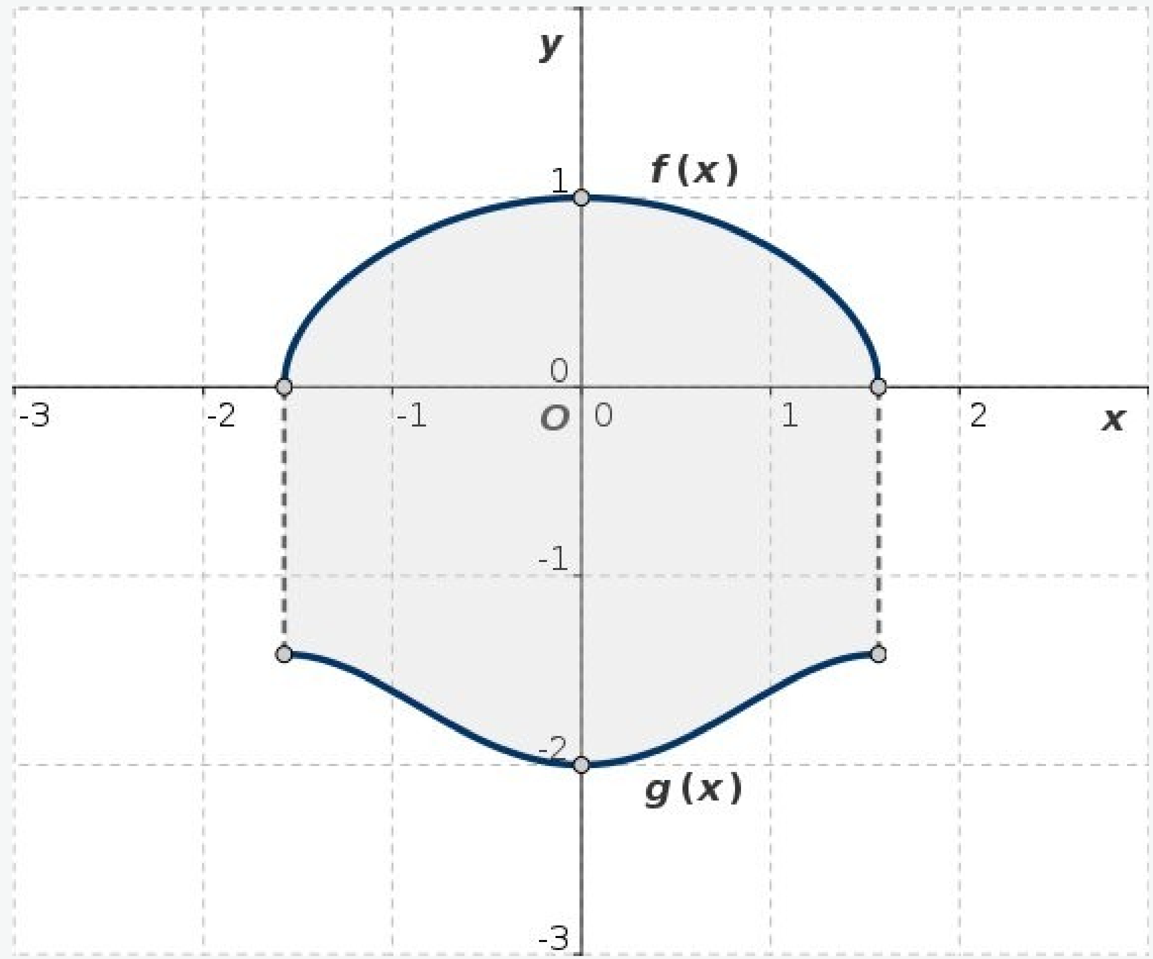

Abb. L7: Integrationsbereich der Aufgabe

$$A = \left\{ (x, \; y) \in \mathbb{R}^2, \quad -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}, \quad -\sqrt{3 + \cos(2x)} \leqslant y \leqslant \sqrt{\cos x} \right\}$$

$$a)\quad I = \int_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \int_{y=-\sqrt{3+\cos(2x)}}^{\sqrt{\cos x}} y\ dy = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x - 3 - \cos(2x))\, dx =$$

$$= \frac{1}{2}\left[\sin x - 3x - \frac{1}{2}\sin(2x)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 1 - \frac{3}{2}\pi$$

$$b)\quad I = \int_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}} x\, dx \int_{y=-\sqrt{3+\cos(2x)}}^{\sqrt{\cos x}} y\ dy = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x(\cos x - 3 - \cos(2x))\, dx =$$

$$= \frac{1}{2}\left[\cos x + x\sin x - \frac{x^2}{2} - x(\cos x \sin x + x) - \frac{1}{2}\cos^2 x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0$$

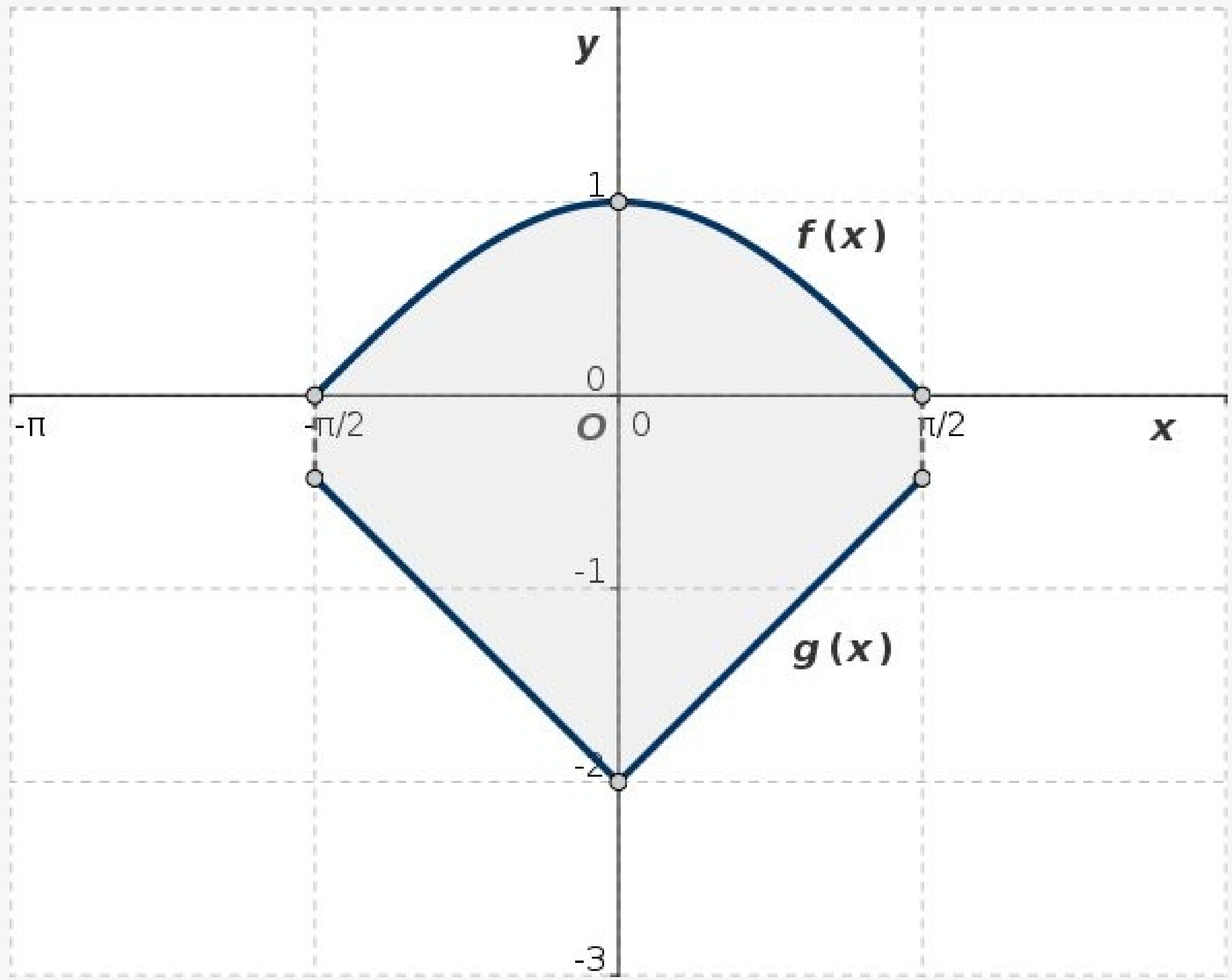

Abb. L8: Integrationsbereich der Aufgabe

$$A = \left\{ (x\,,\ y) \in \mathbb{R}^2 \quad -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}, \quad |\,x\,| - 2 \leqslant y \leqslant \cos x \right\}$$

$$I = \int_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}} x\,dx \int_{y=|x|-2}^{\cos x} dy =$$

$$= \int_{x=-\frac{\pi}{2}}^{0} x\,dx \int_{y=-x-2}^{\cos x} dy + \int_{x=0}^{\frac{\pi}{2}} x\,dx \int_{y=x-2}^{\cos x} dy =$$

$$= \int_{-\frac{\pi}{2}}^{0} x\,(\cos x + x + 2)\,dx + \int_{0}^{\frac{\pi}{2}} x\,(\cos x - x + 2)\,dx =$$

$$= \left[\cos x + x\sin x + \frac{x^3}{3} + x^2\right]_{-\frac{\pi}{2}}^{0} + \left[\cos x + x\sin x - \frac{x^3}{3} + x^2\right]_{0}^{\frac{\pi}{2}} =$$

$$= \left[-\frac{\pi}{2} + \frac{\pi^3}{24} - \frac{\pi^2}{4} + 1\right] + \left[\frac{\pi}{2} - \frac{\pi^3}{24} + \frac{\pi^2}{4} - 1\right] = 0$$

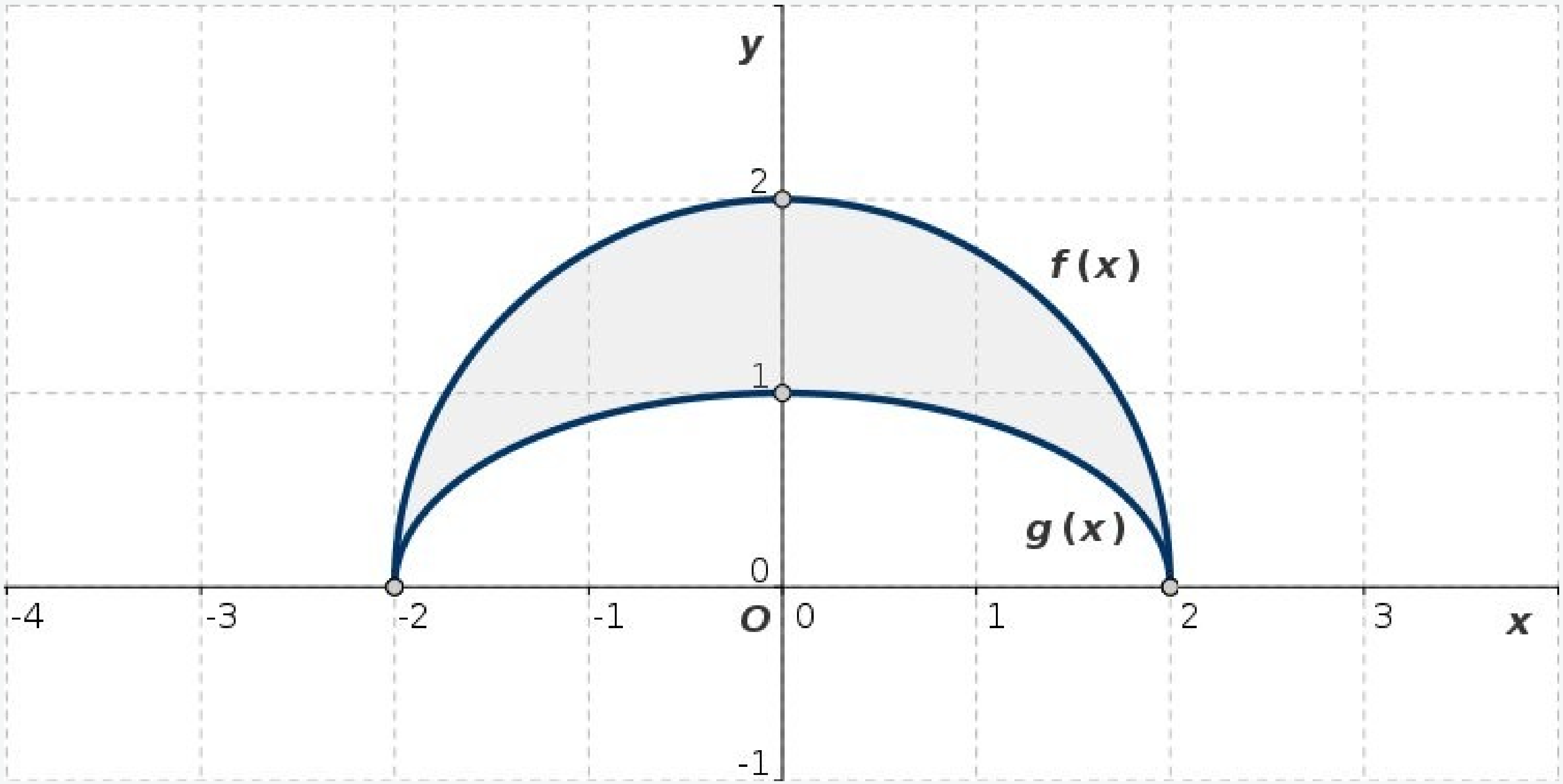

Abb. L9: Integrationsbereich der Aufgabe

$$A = \left\{ (x\,,\ y) \in \mathbb{R}^2, \quad -2 \leqslant x \leqslant 2, \quad \sqrt{1 - \frac{x^2}{4}} \leqslant y \leqslant \sqrt{4 - x^2} \right\}$$

$$a)\quad I = \int_{x=-2}^{2} dx \int_{y=\sqrt{1-\frac{x^2}{4}}}^{\sqrt{4-x^2}} y\ dy = \int_{-2}^{2} dx \left[\frac{y^2}{2}\right]_{\sqrt{1-\frac{x^2}{4}}}^{\sqrt{4-x^2}} =$$

$$= \frac{1}{2}\int_{-2}^{2}\left(4 - x^2 - 1 + \frac{x^2}{4}\right) dx = \frac{3}{2}\int_{-2}^{2}\left(1 - \frac{x^2}{4}\right) dx =$$

$$= \frac{3}{2}\left[x - \frac{x^3}{12}\right]_{-2}^{2} = 4$$

$$b)\quad I = \int_{x=-2}^{2} x\, dx \int_{y=\sqrt{1-\frac{x^2}{4}}}^{\sqrt{4-x^2}} y\ dy = \frac{3}{2}\int_{-2}^{2} x\left(1 - \frac{x^2}{4}\right) dx = 0$$

$$c)\quad I = \int_{x=-2}^{2} x\,dx \int_{y=\sqrt{1-\frac{x^2}{4}}}^{\sqrt{4-x^2}} dy = \frac{1}{2}\int_{-2}^{2} x\sqrt{4-x^2}\,dx = 0$$

$$d)\quad I = \int_{x=-2}^{2} x^2\,dx \int_{y=\sqrt{1-\frac{x^2}{4}}}^{\sqrt{4-x^2}} dy = \frac{1}{2}\int_{-2}^{2} x^2\sqrt{4-x^2}\,dx = \pi$$

Berechnen Sie die folgenden Doppelintegrale und zeichnen die entsprechenden Integrationsbereiche:

Aufgabe 10: $$I_{10} = \int_{y=0}^{1} \int_{x=-2}^{y} y\, e^{-x}\, dx\, dy$$

Aufgabe 11: $$I_{11} = \int_{y=0}^{\pi/2} \int_{x=0}^{y} \left(2 + \cos(2y)\right) dx\, dy$$

Aufgabe 12: $$I_{12} = \int_{x=0}^{\pi/6} \int_{y=0}^{2x} \left(4y + \cos(3y)\right) dy\, dx$$

Aufgabe 13: $$I_{13} = \int_{x=0}^{1} \int_{y=x^2-2}^{\sqrt{x}} \left(8xy^2 - 4y\right) dy\, dx$$

Aufgabe 14: $$I_{14} = \int_{x=-\frac{\pi}{2}}^{0} \int_{y=-\sqrt{2+\sin(2x)}}^{\sqrt{\cos x}} y\, dy\, dx$$

$$I_{10} = \int_{y=0}^{1} \int_{x=-2}^{y} y\, e^{-x}\, dx\, dy = -1 + \frac{2}{e} + \frac{e^2}{2}$$

$$I_{11} = \int_{y=0}^{\pi/2} \int_{x=0}^{y} (2 + \cos(2\, y))\, dx\, dy = \frac{1}{4} (\pi^2 - 2)$$

$$I_{12} = \int_{x=0}^{\pi/6} \int_{y=0}^{2\, x} (4\, y + \cos(3\, y))\, dy\, dx = \frac{1}{81} (\pi^3 + 9)$$

$$I_{13} = \int_{x=0}^{1} \int_{y=x^2-2}^{\sqrt{x}} (8\, x\, y^2 - 4\, y)\, dy\, dx = \frac{1102}{105} \simeq 10.5$$

$$I_{14} = \int_{x=-\frac{\pi}{2}}^{0} \int_{y=-\sqrt{2+\sin(2\, x)}}^{\sqrt{\cos x}} y\, dy\, dx = 1 - \frac{\pi}{2}$$