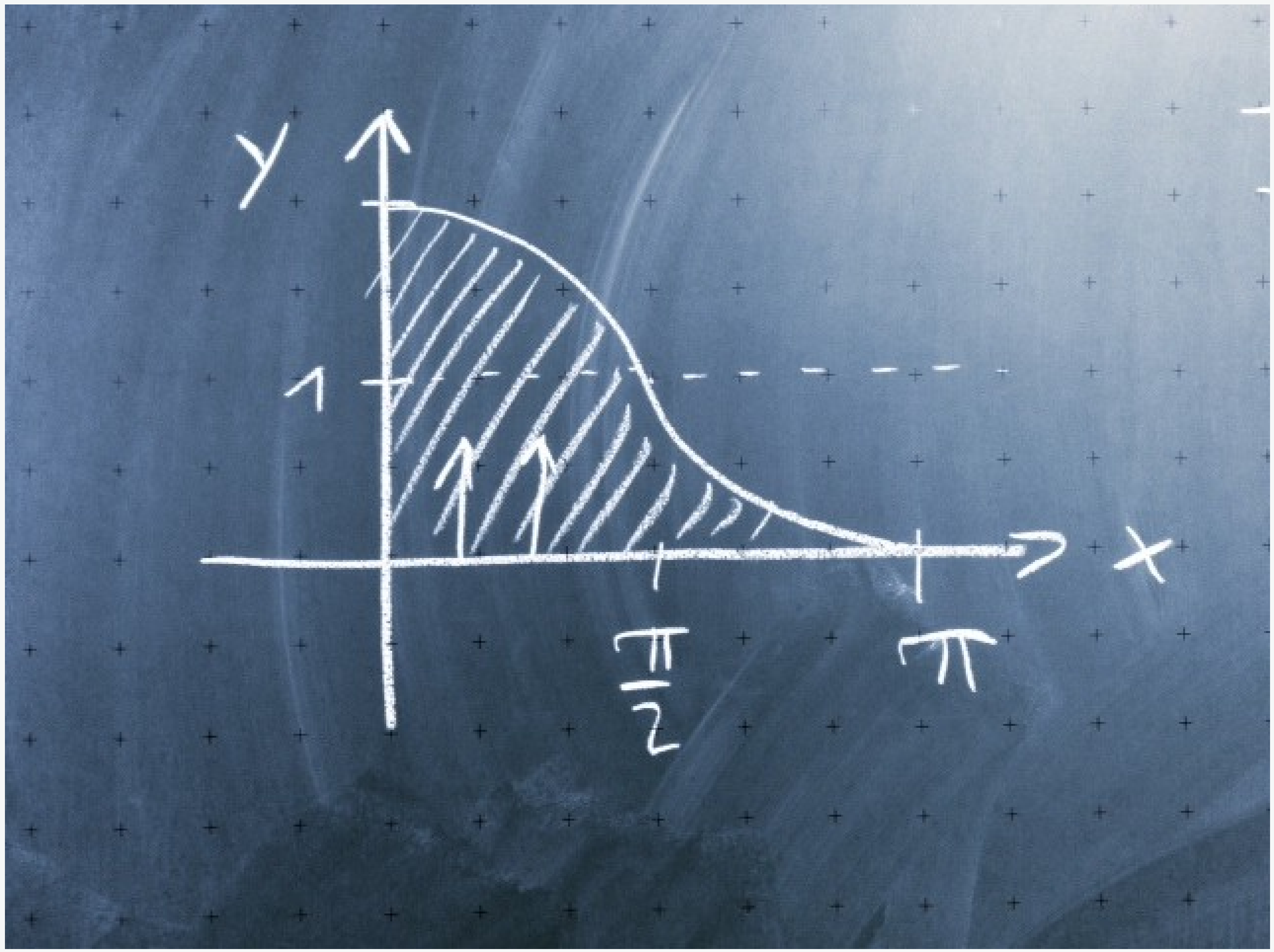
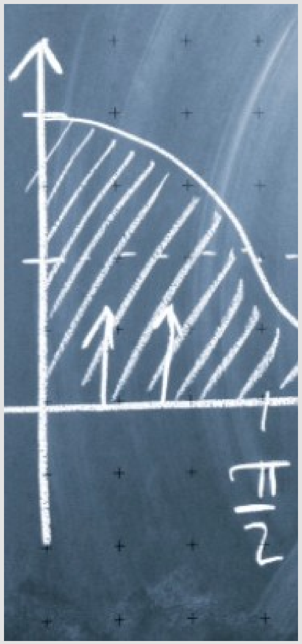
Doppelintegral in kartesischen Koordinaten

Aufgaben, Teil 3: Beliebige Integrationsgrenzen


y
1
x
π/2
π



Berechnen Sie die folgenden Doppelintegrale und zeichnen Sie den Integrationsbereich

Aufgabe 1:

$$I = \iint\limits_A f(x, y)\, dA, \qquad A: \quad y = 2x^2, \qquad y = 1 + x^2$$

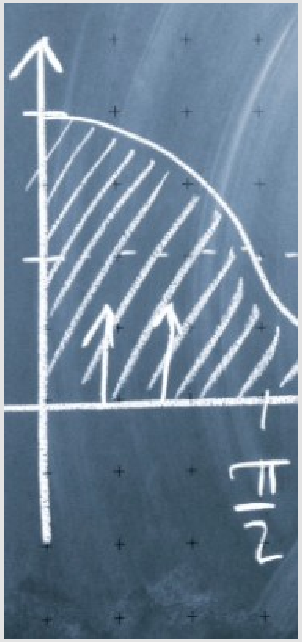
$a)\ f(x, y) = x + y, \qquad b)\ f(x, y) = x^2 + y$

$c)\ f(x, y) = x + x^2 + y$

Kann man, ohne Rechnungen durchzuführen, das Doppelintegral von folgenden Integranden zu bestimmen?

$d)\ f(x, y) = x, \qquad e)\ f(x, y) = y$

$$f)\ I = \int_{x=a}^{b} \int_{y=2x^2}^{1+x^2} x\, dy\, dx$$



Aufgabe 2: $I = \iint\limits_A (x + 2y)\, dA$

$A: \quad y = (3 - x)^2, \quad y = 4$

Aufgabe 3: $I = \iint\limits_A x\, y\; dx\, dy$

a) $A: \quad y = x^2 + 1, \quad y = 4x, \quad x = 0$

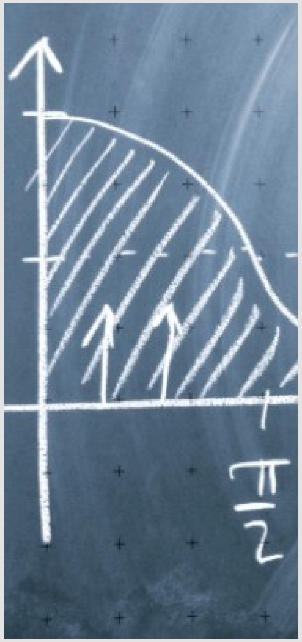
b) $A: \quad y = x^2 + 1, \quad y = 4x$

Aufgabe 4: $I = \iint\limits_A x\, dx\, dy$

$A: \quad y = x^3, \quad x + y = 2, \quad x = 0$

Aufgabe 5: $I = \iint\limits_A \sqrt{1 + x^4}\; dx\, dy$

$A: \quad y = x^3, \quad x = 2, \quad y = 0$



Aufgabe 6: $I = \iint_A \sin x \, dy \, dx$

$A: \quad 0 \leqslant y \leqslant \sin x, \qquad 0 \leqslant x \leqslant \pi$

Aufgabe 7: $I = \iint_A y^2 \sin x \, dy \, dx$

$A: \quad 0 \leqslant y \leqslant 1 + \cos x, \qquad 0 \leqslant x \leqslant \pi$

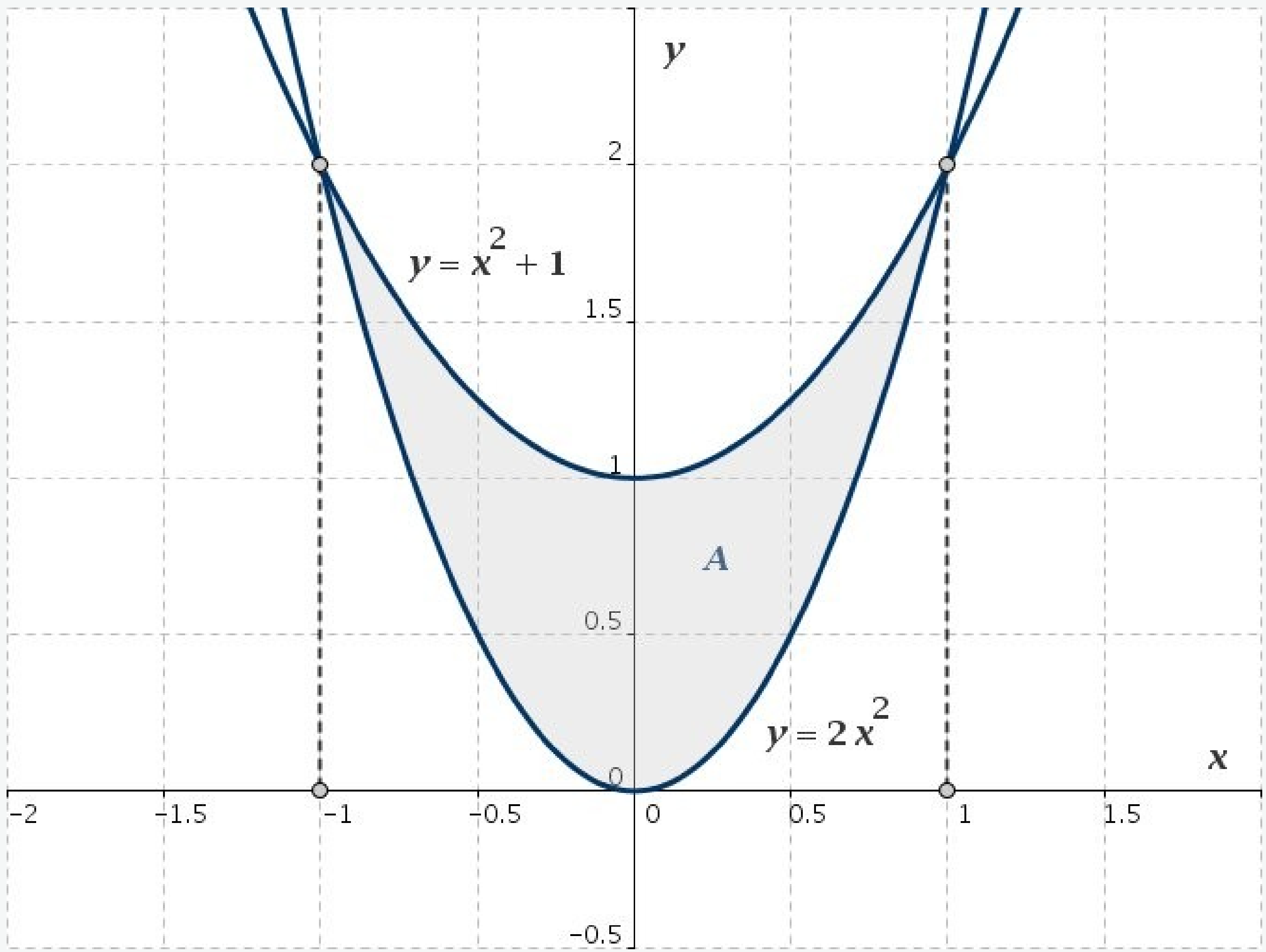


Abb. L1: Darstellung des Integrationsbereiches

$$A: \quad y = 2\,x^2, \quad y = 1 + x^2$$

$$a)\quad I = \iint_A (x + y)\, dA, \qquad A \;: y = 2x^2, \quad y = 1 + x^2$$

Die Integrationsfläche ist durch die beiden Parabeln $y = 2x^2$ und $y = 1 + x^2$ begrenzt. Die Schnittpunkte sind

$$S_1 = (-1,\ 2), \qquad S_2 = (1,\ 2)$$

$$-1 \leqslant x \leqslant 1, \qquad 2x^2 \leqslant y \leqslant 1 + x^2$$

$$I = \int_{x=-1}^{1} dx \int_{y=2x^2}^{1+x^2} (x + y)\, dy = \int_{-1}^{1} \left[x\,y + \frac{y^2}{2} \right]_{2x^2}^{1+x^2} dx =$$

- $$= \int_{-1}^{1} \left(\frac{1}{2} + x + x^2 - x^3 - \frac{3}{2}\, x^4 \right) dx = \left[\frac{x}{2} + \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \frac{3}{10}\, x^5 \right]_{-1}^{1} = \frac{16}{15}$$
- $$= 2 \int_{0}^{1} \left(\frac{1}{2} + x^2 - \frac{3}{2}\, x^4 \right) dx = \left[x + \frac{2}{3}\, x^3 - \frac{3}{5}\, x^5 \right]_{0}^{1} = \frac{16}{15}$$

$$b)\quad I = \int_{x=-1}^{1} dx \int_{y=2x^2}^{1+x^2} (x^2 + y)\, dy = \int_{-1}^{1} \left(\frac{1}{2} + 2x^2 - \frac{5}{2} x^4 \right) dx =$$

$$= \left[\frac{x}{2} + \frac{2}{3} x^3 - \frac{x^5}{2} \right]_{-1}^{1} = \frac{4}{3}$$

$$c)\quad I = \int_{x=-1}^{1} dx \int_{y=2x^2}^{1+x^2} (x + x^2 + y)\, dy = \frac{4}{3}$$

$$d)\quad I = \int_{x=-1}^{1} x\, dx \int_{y=2x^2}^{1+x^2} dy = 0$$

$$e)\quad I = \int_{x=-1}^{1} dx \int_{y=2x^2}^{1+x^2} y\, dy = \frac{16}{15}$$

$$f)\quad I = \int_{x=a}^{b} \int_{y=2x^2}^{1+x^2} x\, dy\, dx = \int_{x=a}^{b} (x - x^3)\, dx = \frac{1}{4}(a^4 - 2a^2 - b^4 + 2b^2)$$

Integrationsgrenzen: Zusammenfassung

$$a)\quad f(x, y) = x + y \quad \rightarrow \quad \frac{16}{15}$$

$$b)\quad f(x, y) = x^2 + y \quad \rightarrow \quad \frac{4}{3}$$

$$c)\quad f(x, y) = x + x^2 + y \quad \rightarrow \quad \frac{4}{3}$$

$$d)\quad f(x, y) = x \quad \rightarrow \quad 0$$

$$e)\quad f(x, y) = y \quad \rightarrow \quad \frac{16}{15}$$

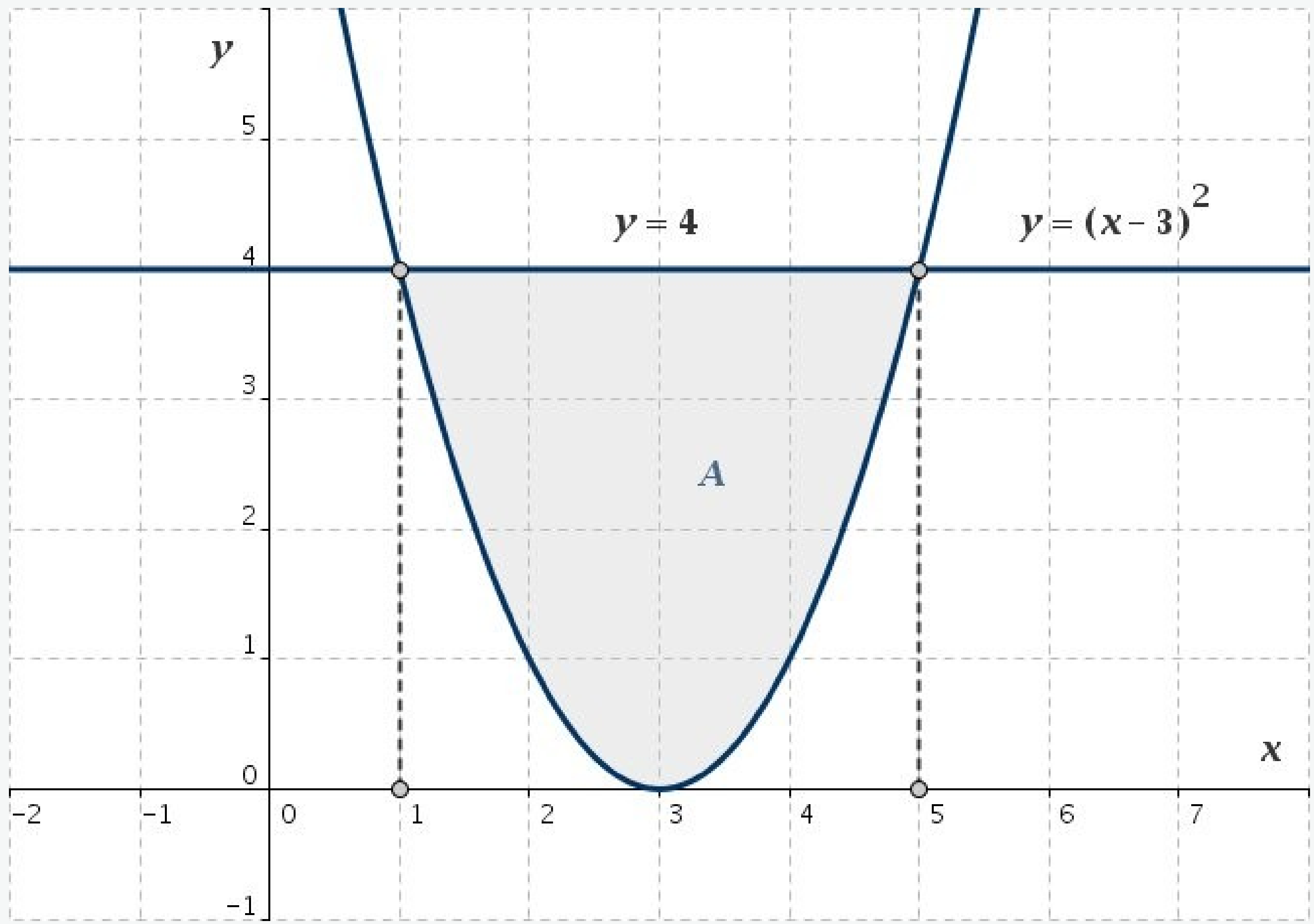


Abb. L2: Darstellung des Integrationsbereiches

1. Möglichkeit:

$$I = \int_{y=0}^{4} \int_{x=3-\sqrt{y}}^{3+\sqrt{y}} (x + 2\,y)\; dx\;\, dy = \int_0^4 (6\sqrt{y} + 4\,y\sqrt{y})\; dy = 83.2$$

2. Möglichkeit:

$$I = \int_{x=1}^{5} \int_{y=(x-3)^2}^{4} (x + 2\,y)\; dy\, dx = \int_{1}^{5} dx \left[x\,y + y^2\right]_{(x-3)^2}^{4} =$$

$$= \int_{1}^{5} \left(16 - (x - 3)^4 + x\,(4 - (x - 3)^2)\,\right) dx = 83.2$$

$$(a + b)^4 = a^4 + 4\,a^3\,b + 6\,a^2\,b^2 + 4\,a\,b^3 + b^4$$

Der Rechenaufwand ist geringer, wenn zuerst nach x und dann nach y integriert wird.

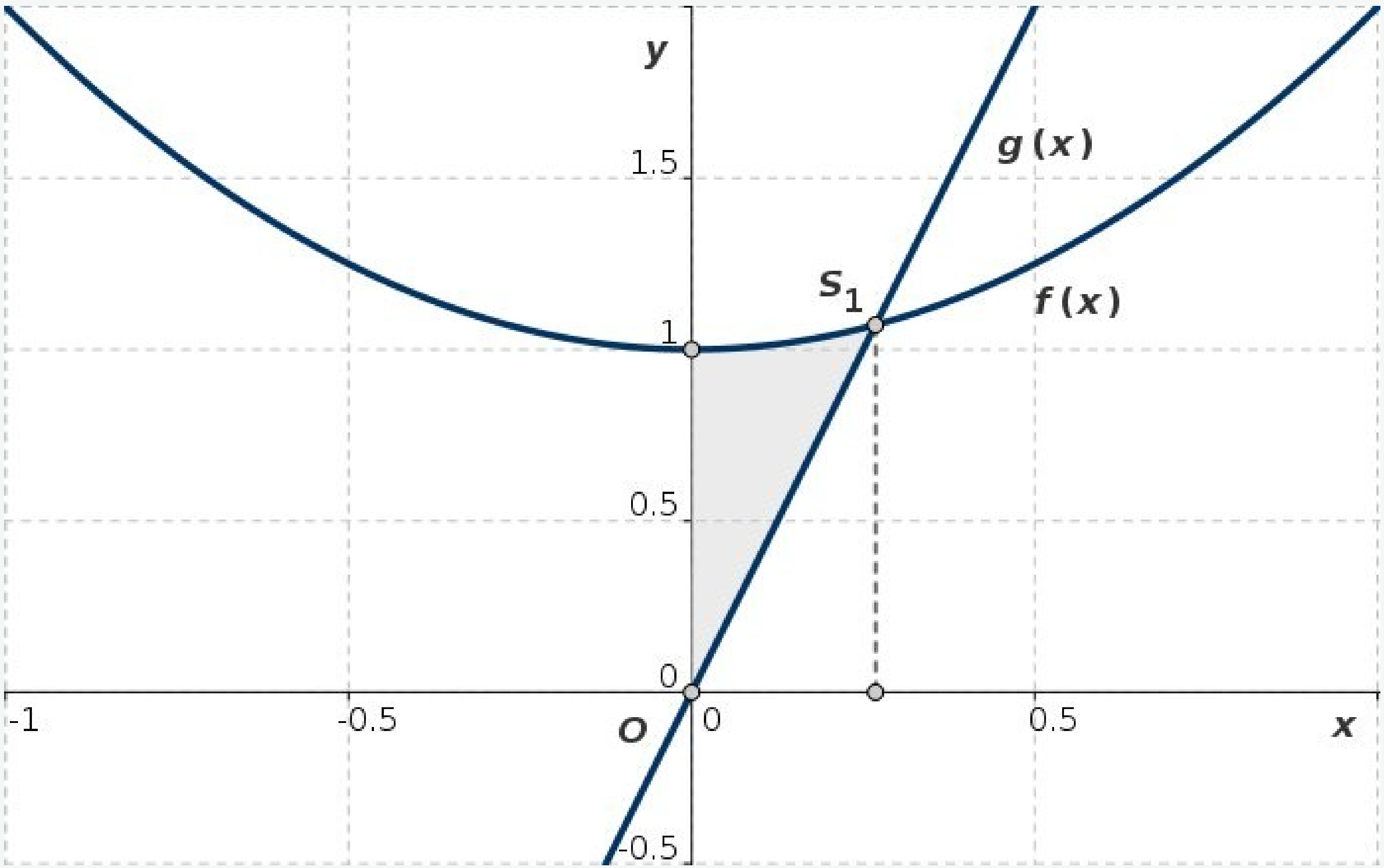


Abb. L3a: Darstellung des Integrationsbereiches

$$A: \quad y = x^2 + 1, \quad y = 4x, \quad x = 0$$

$$f(x) = x^2 + 1, \qquad g(x) = 4x$$

Der Integrationsbereich ist durch die Parabel $y = x^2 + 1$, die Gerade $y = 4x$ und die y-Achse begrenzt. Die Schnittpunkte bestimmt man durch die Lösung der quadratischen Gleichung

$$f(x) = g(x) \quad \Leftrightarrow \quad x^2 - 4x + 1 = 0 \quad \Rightarrow$$

$$x_1 = 2 - \sqrt{3}, \qquad x_2 = 2 + \sqrt{3}$$

$$S_1 = (2 - \sqrt{3},\ f(2 - \sqrt{3})) \simeq (0.27,\ 1.07)$$

$$I = \iint_A x\, y\, dx\, dy = \int_0^{2-\sqrt{3}} x\, dx \int_{4x}^{x^2+1} y\, dy = \frac{1}{2} \int_0^{2-\sqrt{3}} x\,(x^4 - 14x^2 + 1)\, dx =$$

$$= \frac{1}{12}(2 - \sqrt{3})^6 - \frac{7}{4}(2 - \sqrt{3})^4 + \frac{1}{4}(2 - \sqrt{3})^2 \simeq 0.01$$

Integrationsgrenzen: Lösung 3b

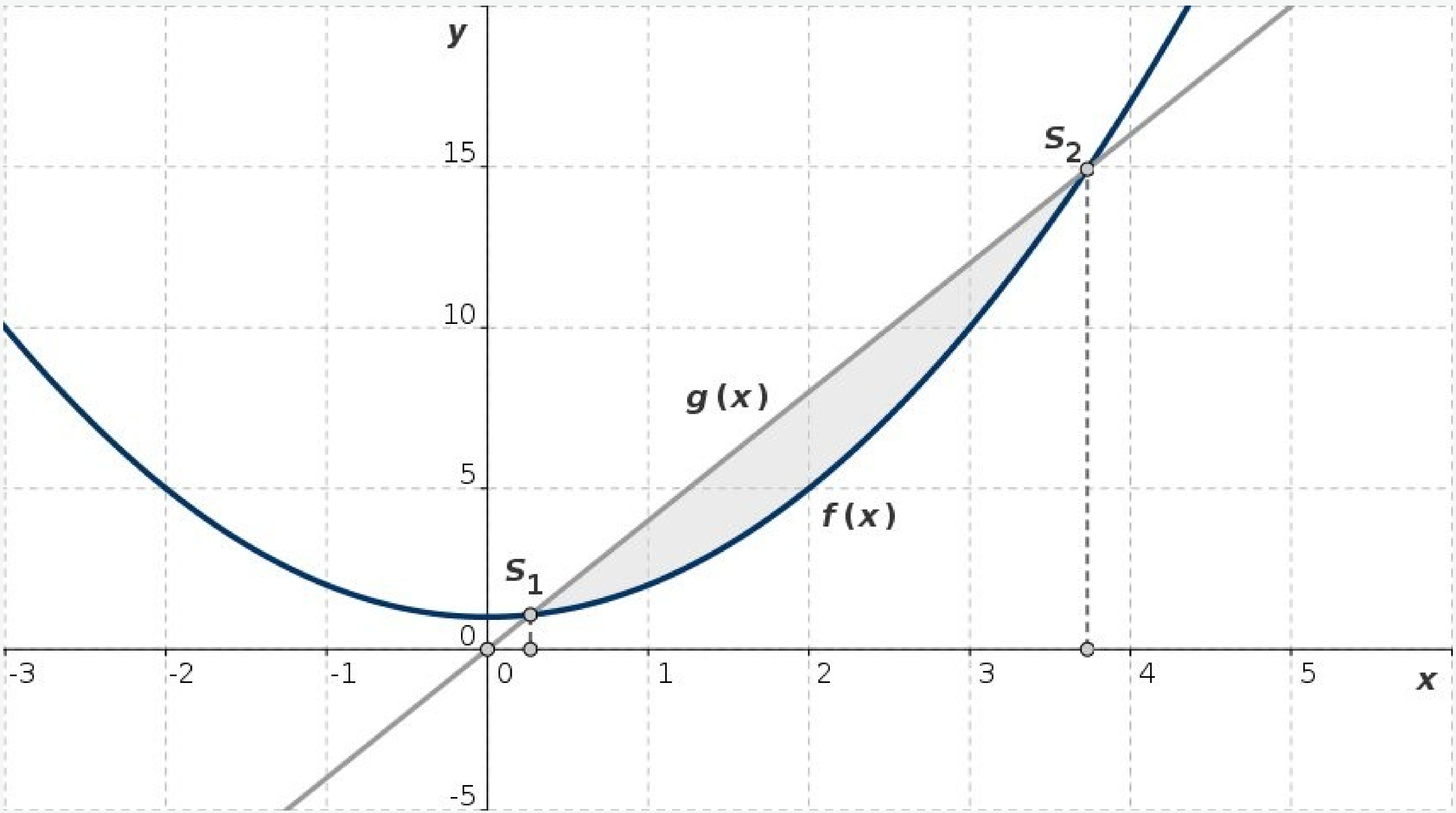


Abb. L3b: Darstellung des Integrationsbereiches

$$A: \quad y = x^2 + 1, \quad y = 4x, \qquad f(x) = x^2 + 1, \qquad g(x) = 4x$$

$$S_1 = (2 - \sqrt{3},\ f(2 - \sqrt{3})), \qquad S_2 = (2 + \sqrt{3},\ f(2 + \sqrt{3}))$$

$$I = \iint\limits_A x\, y\, dx\, dy = \int\limits_{2-\sqrt{3}}^{2+\sqrt{3}} x\, dx \int\limits_{x^2+1}^{4x} y\, dy =$$

$$= -\frac{1}{2} \int\limits_{2-\sqrt{3}}^{2+\sqrt{3}} x\left(x^4 - 14\,x^2 + 1\right) dx =$$

$$= \left[-\frac{1}{12}\, x^2 \left(x^4 - 21\,x^2 + 3\right)\right]_{2-\sqrt{3}}^{2+\sqrt{3}}$$

$$= 64\sqrt{3} = 110.85$$

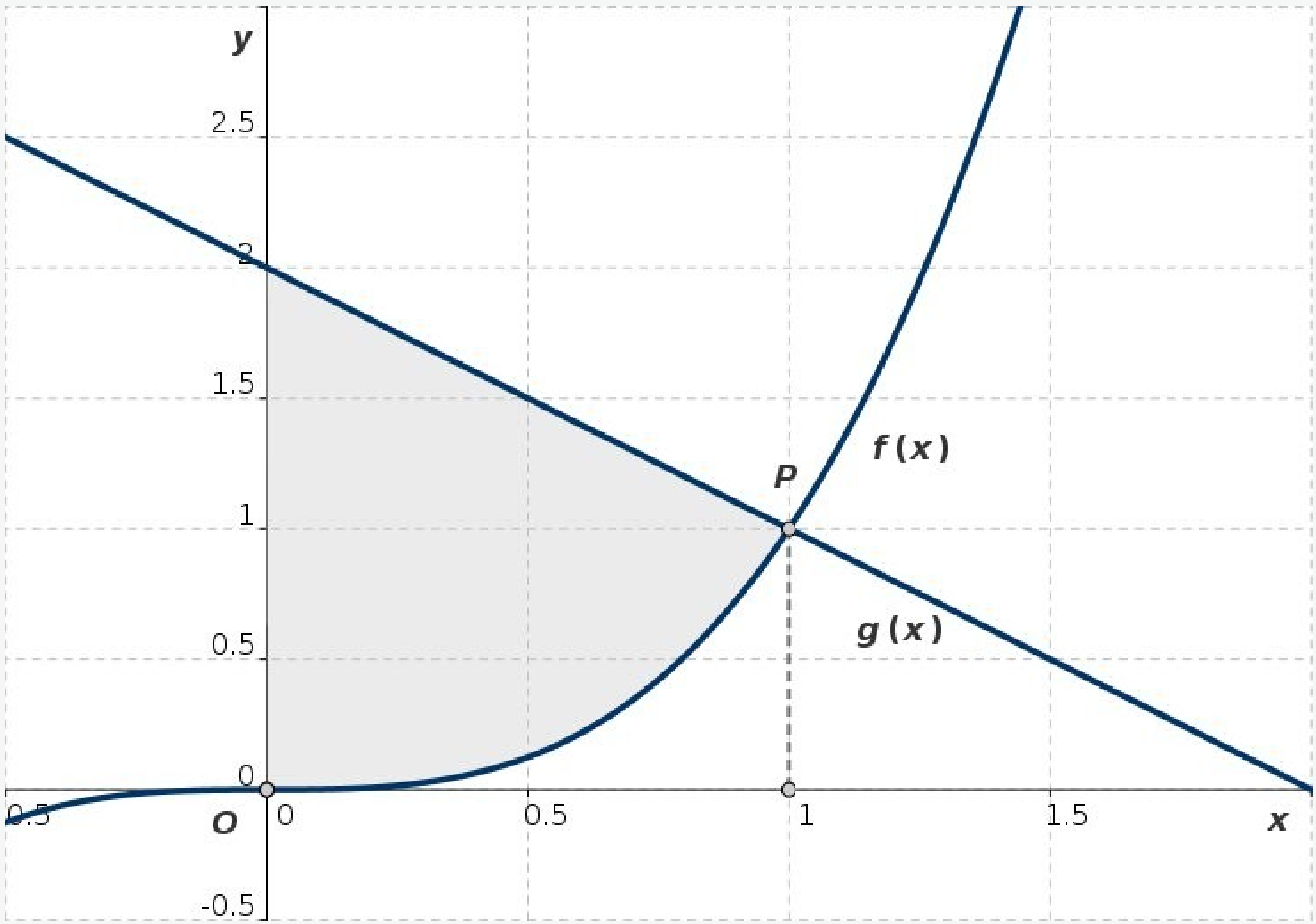


Abb. L4: Darstellung des Integrationsbereiches

$$A: \quad y = x^3, \quad x + y = 2, \quad x = 0; \quad f(x) = x^3, \quad g(x) = 2 - x$$

Integrationsgrenzen: Lösung 4

$$f(x) = g(x) \quad \Leftrightarrow \quad x^3 = 2 - x \quad \Rightarrow \quad x_P = 1$$

$$I = \iint_A x\,dx\,dy = \int_0^1 x\,dx \int_{x^3}^{2-x} dy = \int_0^1 x\,(2 - x - x^3)\,dx =$$

$$= \left[x^2 - \frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = \frac{7}{15}$$

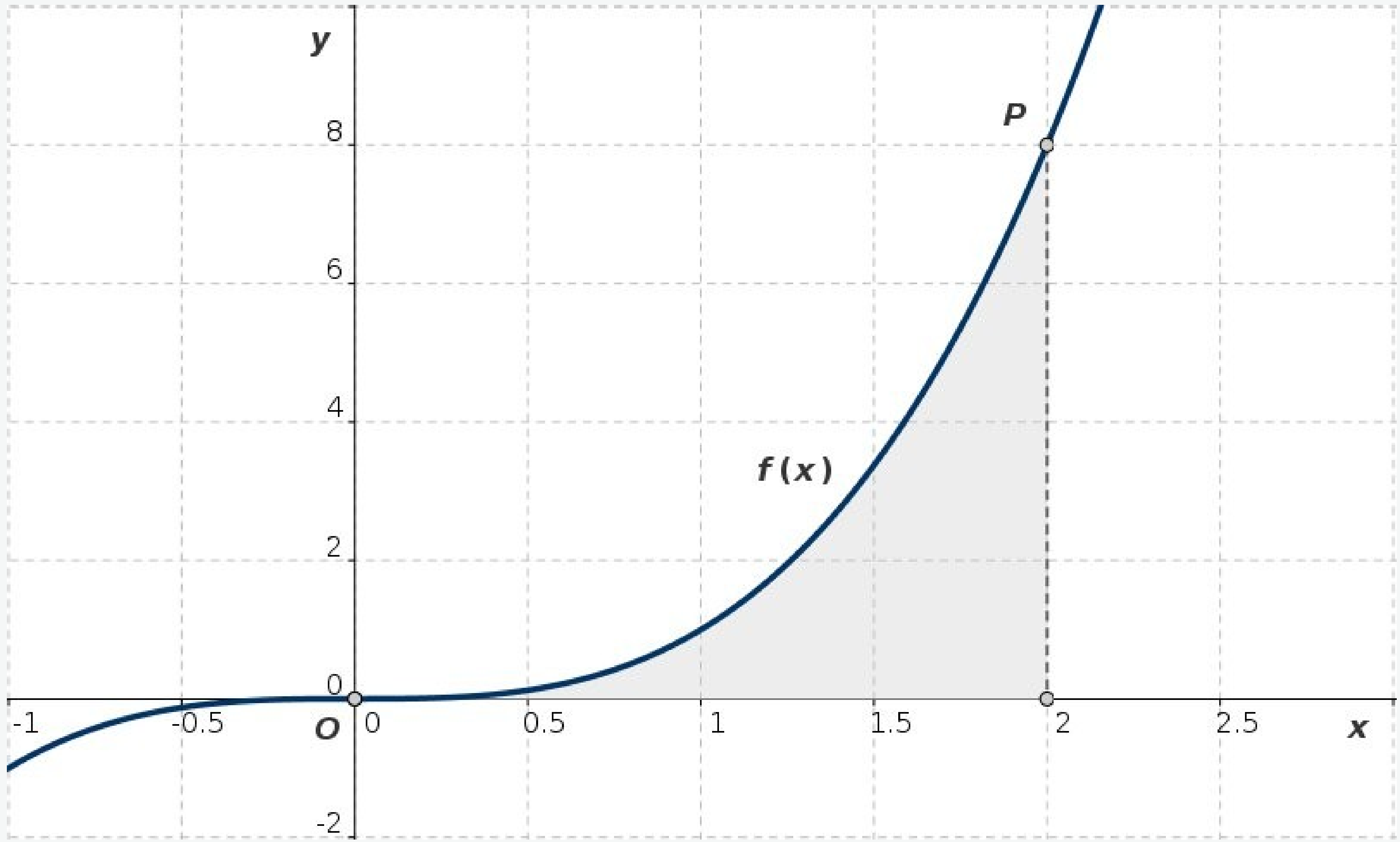


Abb. L5: Darstellung des Integrationsbereiches

$$A: \quad y = x^3, \quad x = 2, \quad y = 0$$

$$I = \iint\limits_A \sqrt{1 + x^4}\, dx\, dy = \int_0^2 \sqrt{1 + x^4}\, dx \int_0^{x^3} dy = \int_0^2 x^3 \sqrt{1 + x^4}\, dx =$$

$$= \frac{1}{6} \left(1 + x^4\right)^{\frac{3}{2}} = \frac{17}{6} \sqrt{17} - \frac{1}{6} \simeq 11.52$$

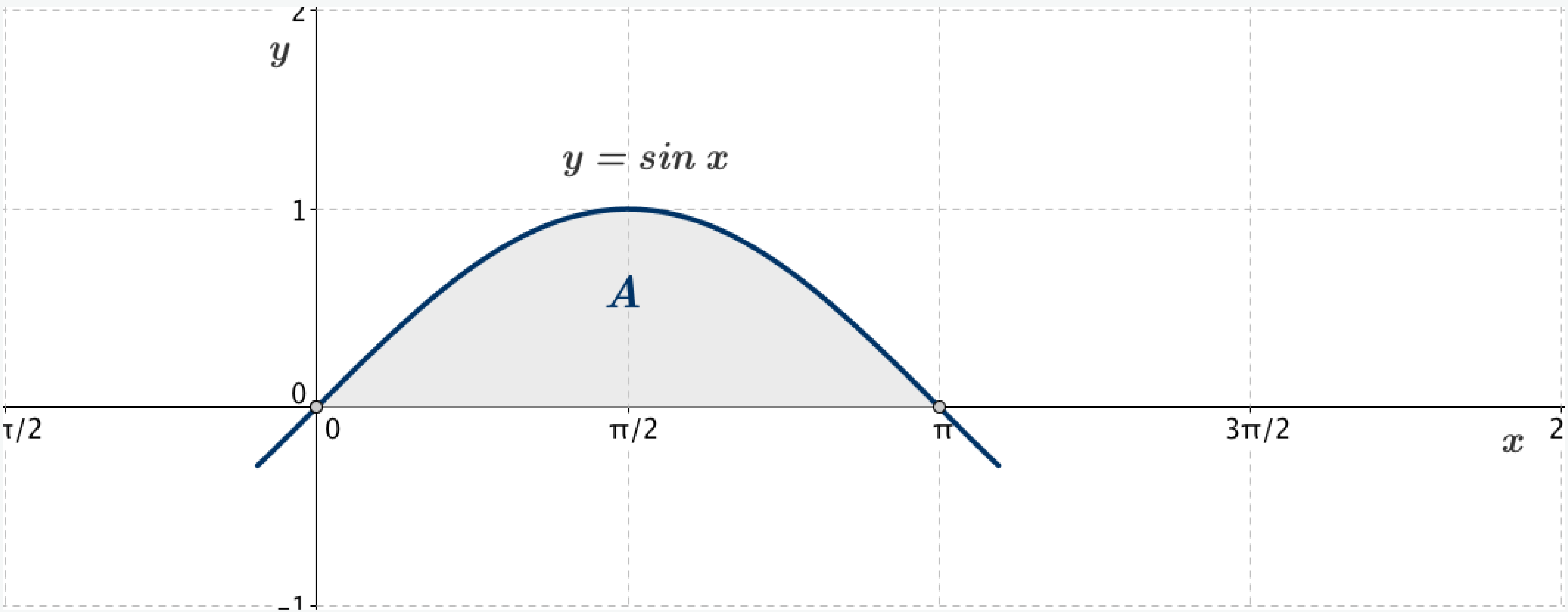


Abb. L6: Darstellung des Integrationsbereiches

$$A: \quad 0 \leqslant y \leqslant \sin x\,, \quad 0 \leqslant x \leqslant \pi$$

$$I = \int_{x=0}^{\pi} \sin x \, dx \int_{y=0}^{\sin x} dy = \int_{x=0}^{\pi} \sin^2 x \, dx = \left[\frac{x}{2} - \frac{\sin(2x)}{4} \right]_0^{\pi} = \frac{\pi}{2}$$

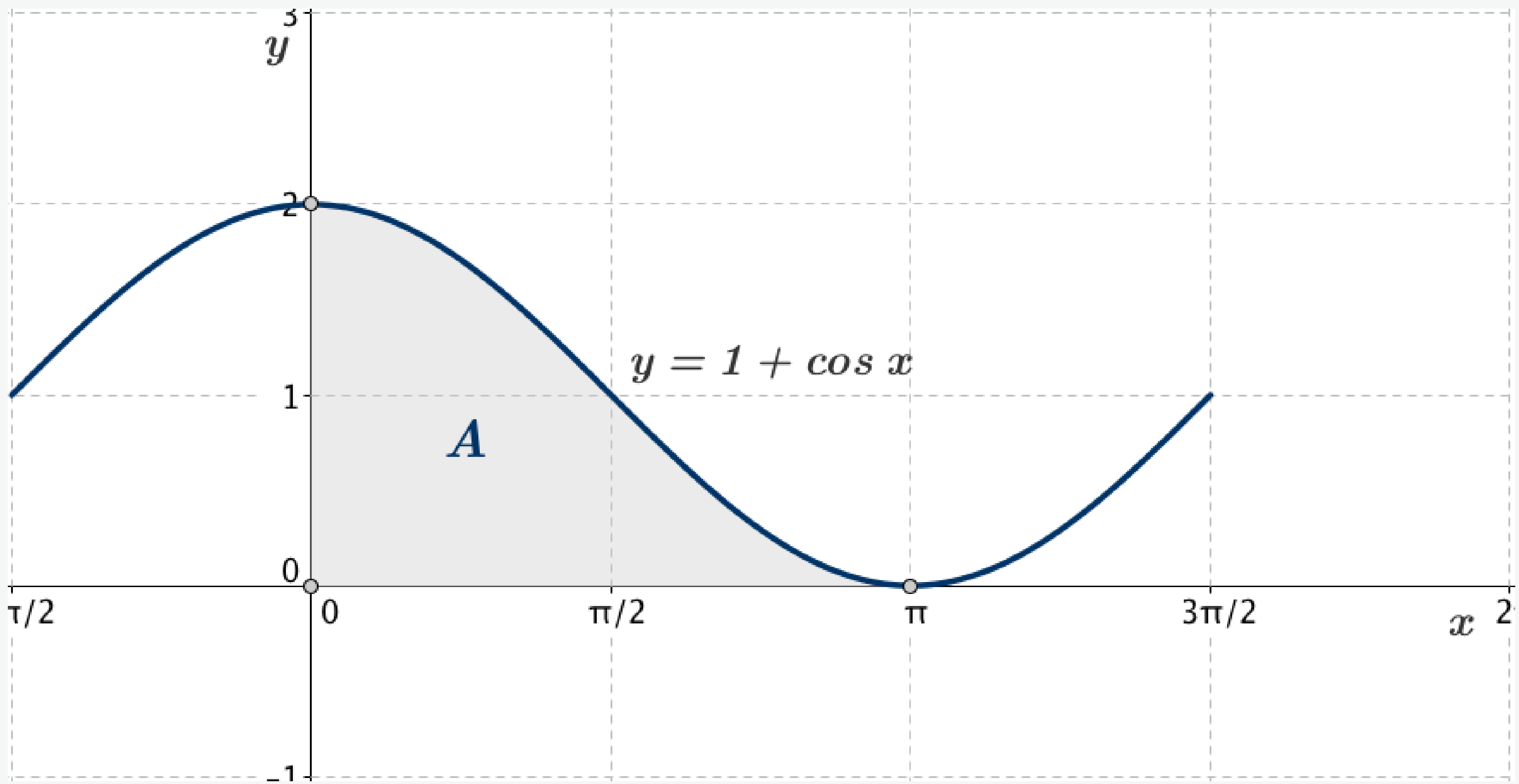


Abb. L7: Darstellung des Integrationsbereiches

$$A: \quad 0 \leqslant y \leqslant 1 + \cos x\,, \qquad 0 \leqslant x \leqslant \pi$$

$$I = \int_{x=0}^{\pi} \sin x \, dx \int_{y=0}^{1+\cos x} y^2 \, dy = \int_0^{\pi} \sin x \, dx \left[\frac{y^3}{3} \right]_0^{1+\cos x} =$$

$$= \frac{1}{3} \int_0^{\pi} (1 + \cos x)^3 \sin x \, dx =$$

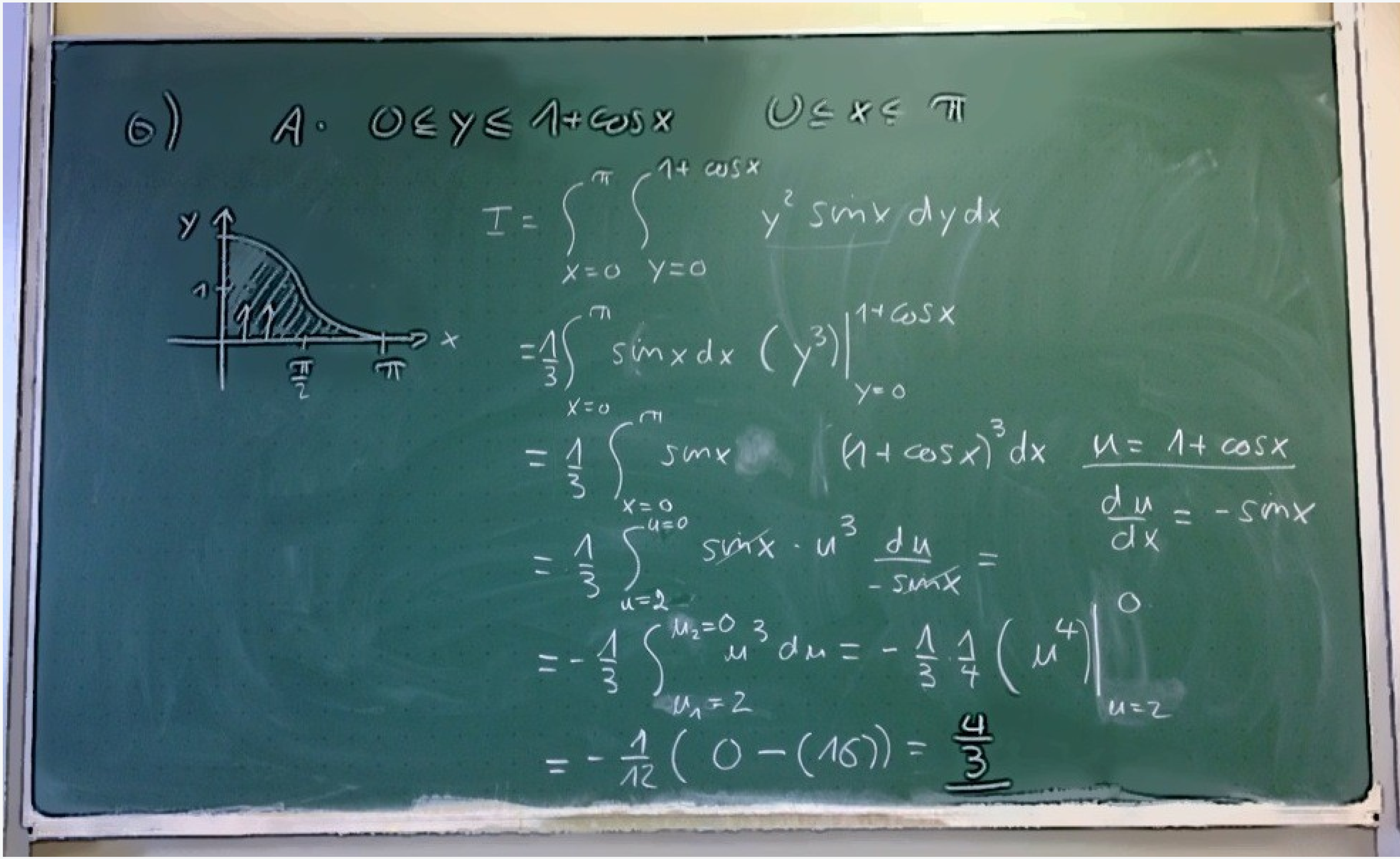
$$= \frac{1}{3} \int_0^{\pi} (1 + 3\cos x + 3\cos^2 x + \cos^3 x) \sin x \, dx =$$

$$= \frac{1}{3} \int_0^{\pi} \left(\sin x + \frac{3}{2} \sin(2x) + 3\cos^2 x \sin x + \cos^3 x \sin x \right) dx =$$

$$= \frac{1}{3} \left[-\cos x - \frac{3}{4} \cos(2x) - \cos^3 x - \frac{1}{4} \cos^4 x \right]_0^{\pi} = \frac{4}{3}$$

$$\int \cos^n (a x) \sin(a x) \, dx = -\frac{1}{a(n+1)} \cos^{n+1}(a x) \qquad (n \neq -1)$$

$$\int \cos^2 x \sin x \, dx = -\frac{1}{3} \cos^3 x, \qquad \int \cos^3 x \sin x \, dx = -\frac{1}{4} \cos^4 x$$


6) A: $0 \le y \le 1+\cos x \quad 0 \le x \le \pi$
$I = \int_{x=0}^{\pi} \int_{y=0}^{1+\cos x} y^2 \sin x \, dy \, dx$
$= \frac{1}{3} \int_{x=0}^{\pi} \sin x \, dx \, (y^3) \Big|_{y=0}^{1+\cos x}$
$= \frac{1}{3} \int_{x=0}^{\pi} \sin x \, (1+\cos x)^3 dx$
$u = 1+\cos x$
$\frac{du}{dx} = -\sin x$
$= \frac{1}{3} \int_{u=2}^{u=0} \sin x \cdot u^3 \frac{du}{-\sin x} =$
$= -\frac{1}{3} \int_{u_1=2}^{u_2=0} u^3 du = -\frac{1}{3} \frac{1}{4} (u^4) \Big|_{u=2}^{0}$
$= -\frac{1}{12} (0-(16)) = \frac{4}{3}$
y
1
x
$\frac{\pi}{2}$
π