Haus (Fragment), Mönckebergstraße, Hamburg

Doppelintegral in Polarkoordinaten

Wichtige Formeln für trigonometrische Funktionen

$$\cos^2\varphi + \sin^2\varphi = 1$$

$$\sin 2\varphi = 2\sin\varphi\cos\varphi, \qquad \cos 2\varphi = \cos^2\varphi - \sin^2\varphi$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin^2\varphi = \frac{1}{2}(1 - \cos 2\varphi)$$

$$\sin^3\varphi = \frac{1}{4}(3\sin\varphi - \sin 3\varphi)$$

$$\sin^4\varphi = \frac{1}{8}(\cos 4\varphi - 4\cos 2\varphi + 3)$$

$$\cos^2\varphi = \frac{1}{2}(1 + \cos 2\varphi)$$

$$\cos^3\varphi = \frac{1}{4}(3\cos\varphi + \cos 3\varphi)$$

$$\cos^4\varphi = \frac{1}{8}(\cos 4\varphi + 4\cos 2\varphi + 3)$$

$$\int \sin(a\,x)\,dx = -\frac{1}{a}\cos(a\,x)$$

$$\int \sin^2(a\,x)\,dx = \frac{x}{2} - \frac{1}{4\,a}\sin(2\,a\,x)$$

$$\int \sin^3(a\,x)\,dx = -\frac{1}{a}\cos(a\,x) + \frac{1}{3\,a}\cos^3(a\,x)$$

$$\int \cos(a\,x)\,dx = \frac{1}{a}\sin(a\,x)$$

$$\int \cos^2(a\,x)\,dx = \frac{x}{2} + \frac{1}{4\,a}\sin(2\,a\,x)$$

$$\int \cos^3(a\,x)\,dx = \frac{1}{a}\sin(a\,x) - \frac{1}{3\,a}\sin^3(a\,x)$$

$$\int e^{a\,x}\,dx = \frac{1}{a}\,e^{a\,x}$$

Doppelintegral in Polarkoordinaten: Aufgaben 1-6

Berechnen Sie folgende Integrale:

Aufgabe 1:

$$I_a = \int_{\varphi=0}^{\frac{\pi}{2}} \int_{r=0}^{\cos\varphi} r \, dr \, d\varphi, \qquad I_b = \int_{\varphi=0}^{\frac{\pi}{2}} \int_{r=0}^{\sin^2\varphi} r \, dr \, d\varphi$$

Aufgabe 2:

$$I = \int_A (x^2 + y^2) \, dx \, dy, \quad A: \; 1 \leqslant x^2 + y^2 \leqslant 4, \quad x, y \geqslant 0$$

Aufgabe 3:

$$I = \int_A x \, y \, dx \, dy, \quad A: \; 1 \leqslant r \leqslant 3, \quad 0 \leqslant \varphi \leqslant \frac{\pi}{2}$$

Aufgabe 4:

$$I = \int_A e^{-(x^2 + y^2)}\, dx\, dy$$

$a)\ A: \ x^2 + y^2 \leqslant a^2, \qquad b)\ A: \ x^2 + y^2 \leqslant 1$

Aufgabe 5:

$$I = \int_A e^{-(x^2 + y^2)} (x^2 + y^2)\, dx\, dy, \qquad A: \ x^2 + y^2 \leqslant 4$$

Aufgabe 6:

$$I = \int_A e^{-(x^2 + y^2)} \frac{x\, y}{x^2 + y^2}\, dx\, dy$$

$A: \quad x^2 + y^2 \leqslant 4, \qquad x, y \geqslant 0$

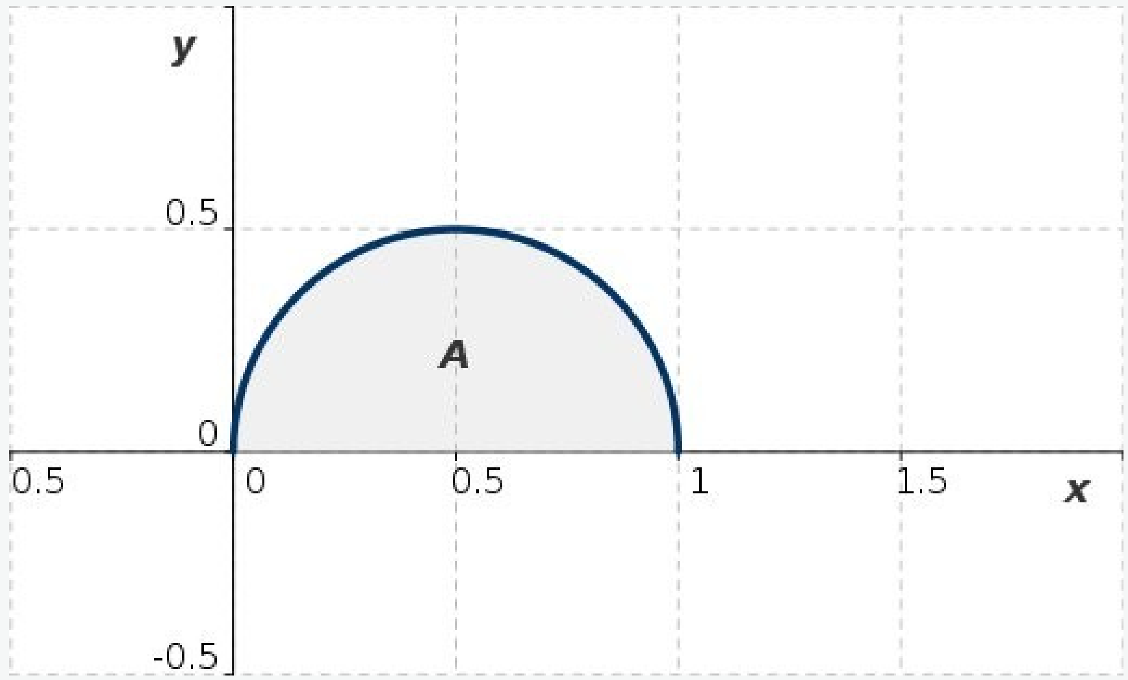

Abb. L1-a: Darstellung des Integrationsbereiches A

$$I_a = \int_{\varphi=0}^{\frac{\pi}{2}} \int_{r=0}^{\cos\varphi} r \; dr \; d\varphi = \int_{\varphi=0}^{\frac{\pi}{2}} \frac{\cos^2\varphi}{2} \; d\varphi = \frac{1}{4} \int_{\varphi=0}^{\frac{\pi}{2}} (1 + \cos 2\varphi) \; d\varphi =$$

$$= \frac{1}{4} \left[\varphi + \frac{\sin 2\varphi}{2} \right]_{\varphi=0}^{\frac{\pi}{2}} = \frac{\pi}{8} \simeq 0.393$$

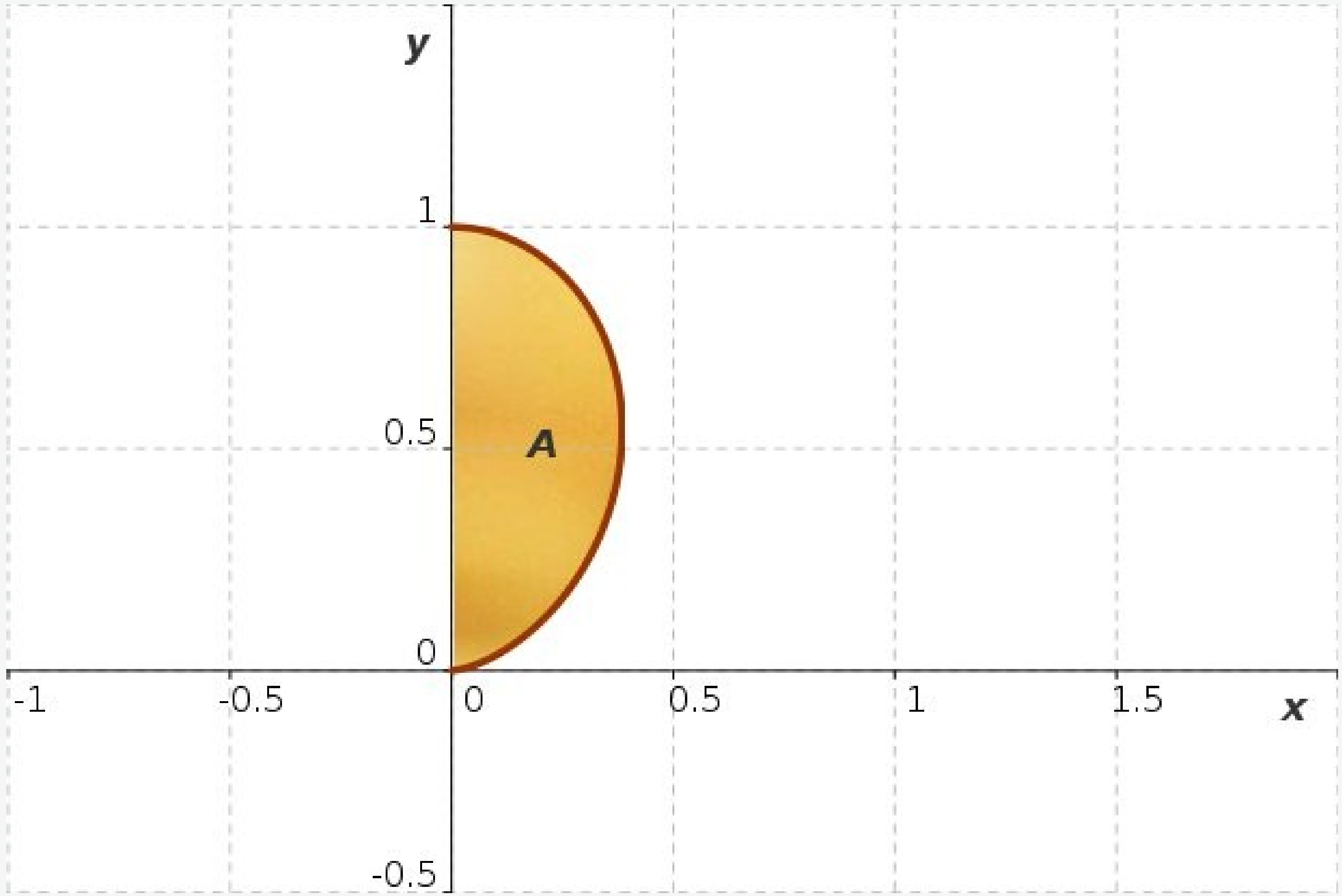

Abb. L1-b: Darstellung des Integrationsbereiches A

$$I_b = \int_{\varphi=0}^{\frac{\pi}{2}} \int_{r=0}^{\sin^2\varphi} r\ dr\ d\varphi = \frac{1}{2} \int_{\varphi=0}^{\frac{\pi}{2}} \sin^4\varphi\ d\varphi = \frac{1}{16} \int_{\varphi=0}^{\frac{\pi}{2}} (3 - 4\cos 2\varphi + \cos 4\varphi)\ d\varphi =$$

$$= \frac{3}{32}\,\pi \simeq 0.295$$

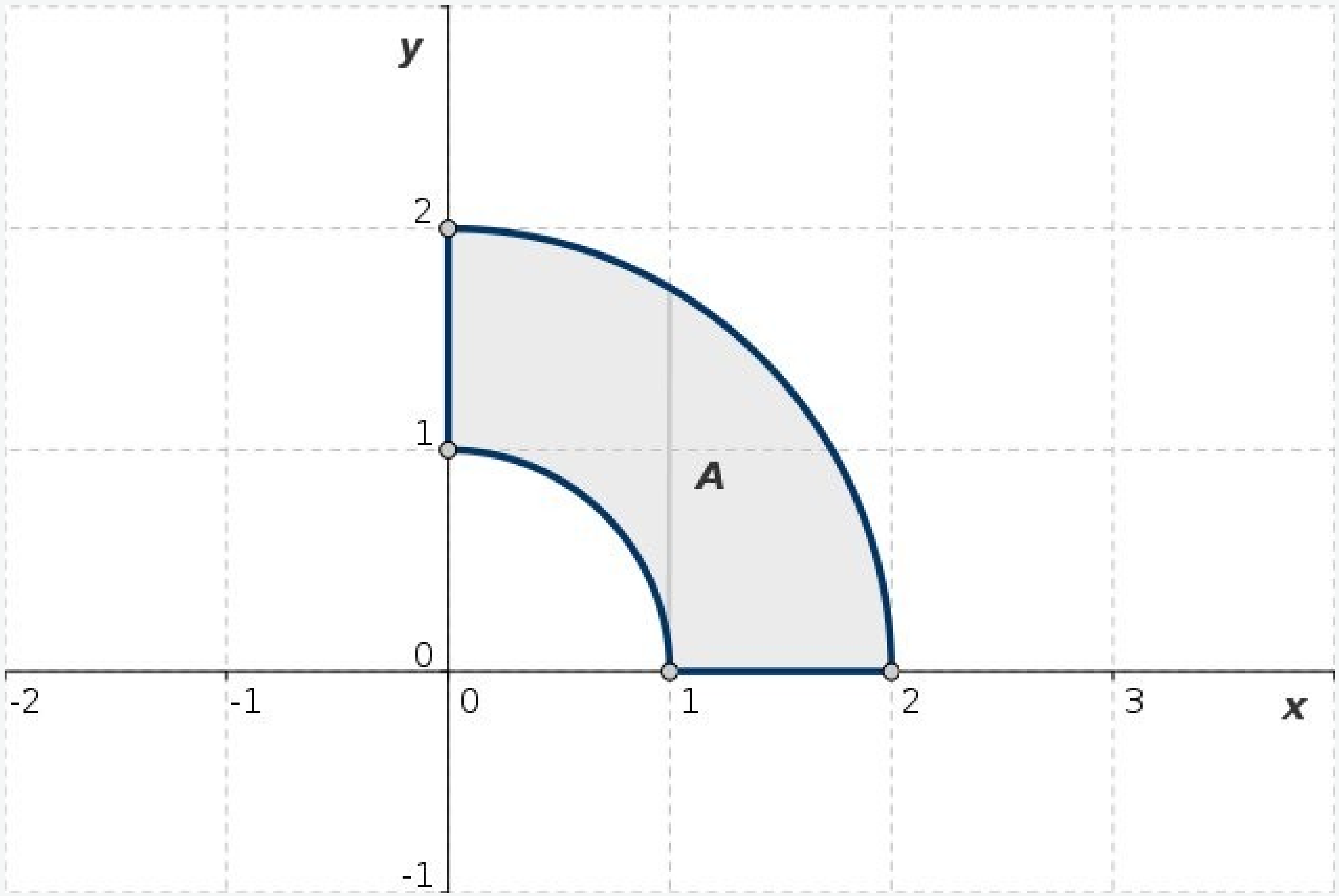

Abb. L2: Darstellung des Integrationsbereiches A

$$A: \quad 1 \leqslant x^2 + y^2 \leqslant 4, \quad x, y \geqslant 0 \quad \Leftrightarrow \quad 1 \leqslant r \leqslant 2, \quad 0 \leqslant \varphi \leqslant \frac{\pi}{2}$$

$$I = \int_A (x^2 + y^2)\ dy\, dx\,, \qquad A: \quad 1 \leqslant r \leqslant 2, \qquad 0 \leqslant \varphi \leqslant \frac{\pi}{2}$$

$$\int_A f(x, y)\ dA = \int_{\varphi_1}^{\varphi_2} \int_{r_1(\varphi)}^{r_2(\varphi)} f(r\cos\varphi,\ r\sin\varphi)\ r\, dr\ d\varphi$$

$$x^2 + y^2 = r^2 \cos^2\varphi + r^2 \sin^2\varphi = r^2$$

$$I = \int_A (x^2 + y^2)\ dx\ dy = \int_1^2 r^2\ r\, dr \int_0^{\frac{\pi}{2}} d\varphi = \frac{\pi}{2} \int_1^2 r^3\ dr = \frac{15}{8}\ \pi$$

Doppelintegral in Polarkoordinaten: Lösung 3

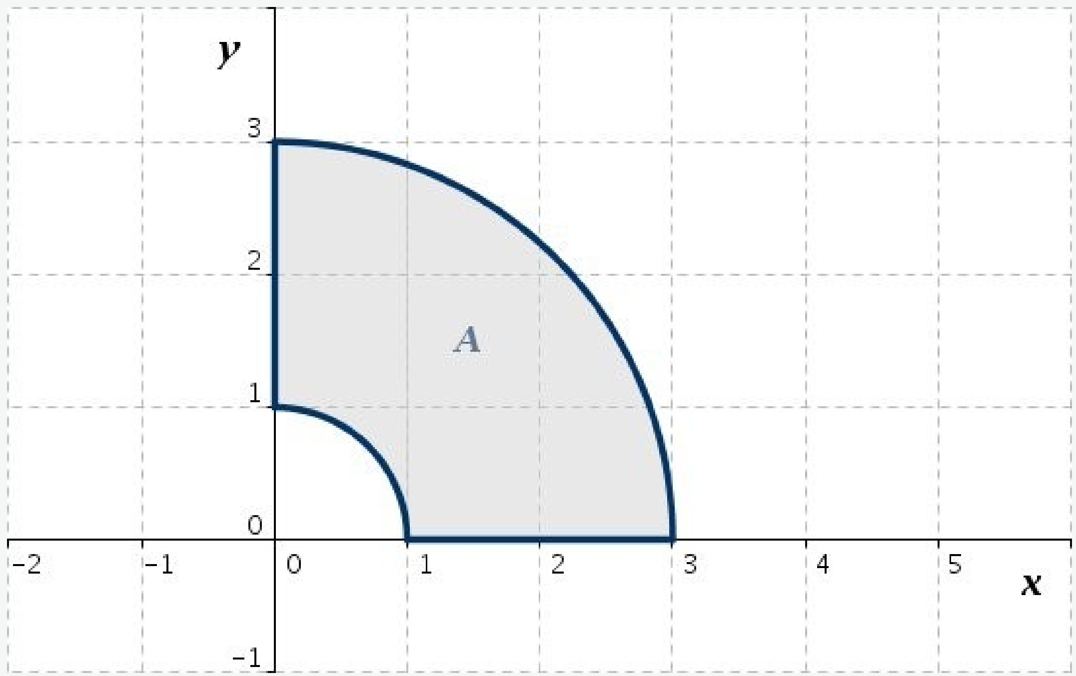

Abb. L3: Darstellung des Integrationsbereiches A

$$I = \int_A x\, y\; dy\, dx = \int_{\varphi=0}^{\frac{\pi}{2}} \int_{r=1}^{3} (r\cos\varphi)\,(r\sin\varphi)\; r\, dr\, d\varphi = \frac{1}{2}\int_{\varphi=0}^{\frac{\pi}{2}} \int_{r=1}^{3} r^3 \sin 2\varphi\; dr\, d\varphi =$$

$$= \frac{1}{2}\left(\int_{\varphi=0}^{\frac{\pi}{2}} \sin 2\varphi\; d\varphi\right)\cdot\left(\int_{r=1}^{3} r^3\; dr\right) = 10 \int_0^{\frac{\pi}{2}} \sin 2\varphi\; d\varphi = 10$$

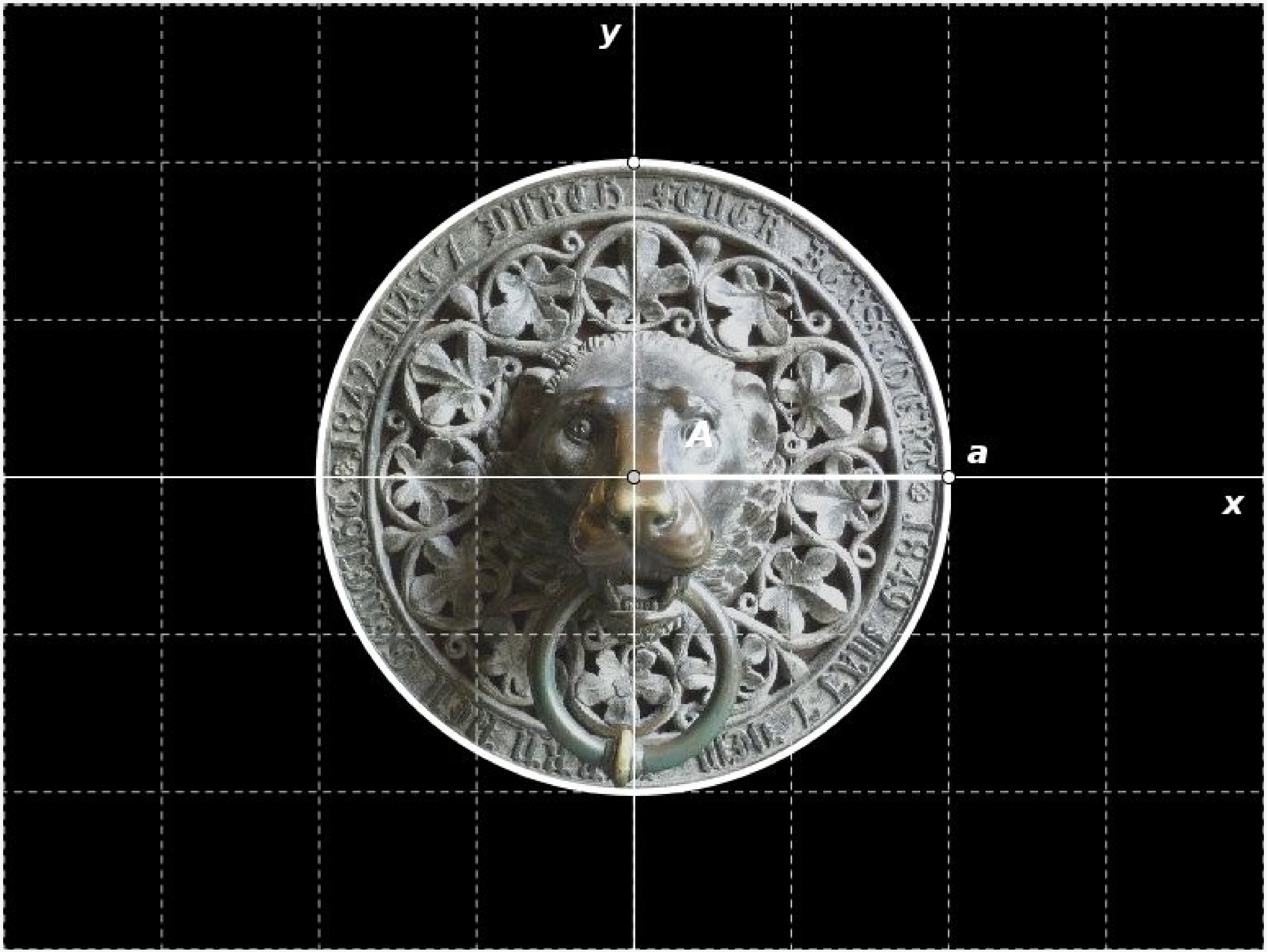

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Abb. L4: Darstellung des Integrationsbereiches A

$$I = \int_A e^{-(x^2 + y^2)}\, dx\, dy$$

a) $A_a: \ x^2 + y^2 \leqslant a^2 \qquad : 0 \leqslant r \leqslant a, \qquad 0 \leqslant \varphi \leqslant 2\pi$

b) $A_b: \ x^2 + y^2 \leqslant 1 \qquad : 0 \leqslant r \leqslant 1, \qquad 0 \leqslant \varphi \leqslant 2\pi$

a) $$I = \int_{A_a} e^{-(x^2+y^2)}\, dx\, dy = \int_{A_a} e^{-r^2}\, r\, dr\, d\varphi =$$

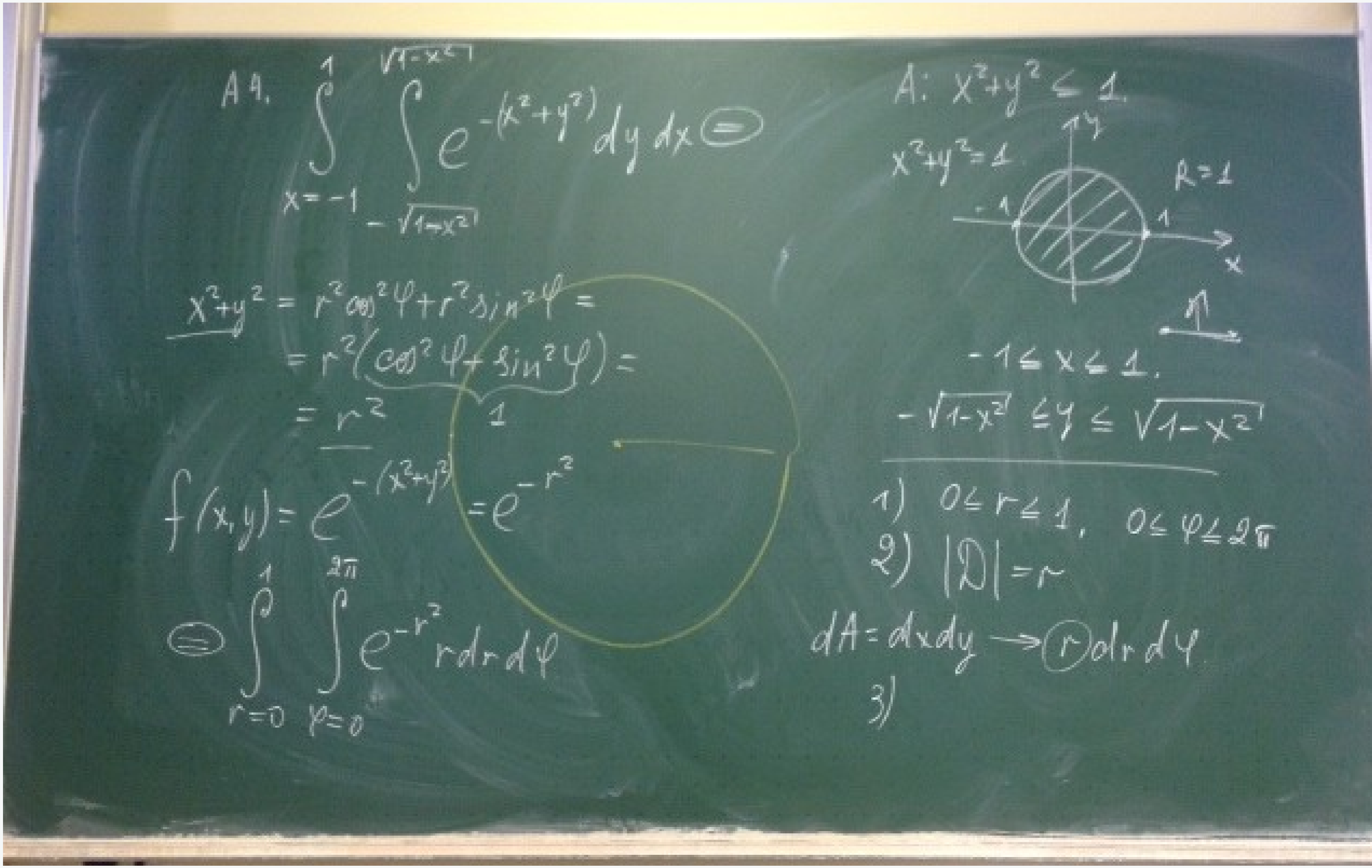
$$= \int_0^{2\pi} d\varphi \int_0^a e^{-r^2}\, r\, dr = 2\pi \int_0^a e^{-r^2}\, r\, dr =$$

$$t = r^2, \qquad \frac{dt}{2} = r\, dr$$

$$= \pi \int_0^{a^2} e^{-t}\, dt = \pi \left[-e^{-t} \right]_0^{a^2} = \pi \left(1 - e^{-a^2} \right) = \pi \left(1 - \frac{1}{e^{a^2}} \right)$$

b) $$I = \int_{A_b} e^{-(x^2+y^2)}\, dx\, dy = \pi \left(1 - \frac{1}{e} \right)$$

Doppelintegral in Polarkoordinaten: Lösung 4



$$I = \int_A e^{-(x^2 + y^2)} (x^2 + y^2)\, dx\, dy\,, \qquad A: \quad x^2 + y^2 \leqslant 4$$

$$x^2 + y^2 \leqslant 4 \quad \Leftrightarrow \quad r \leqslant 2, \quad 0 \leqslant \varphi \leqslant 2\pi$$

$$I = \int_A e^{-(x^2+y^2)} (x^2+y^2)\, dx\, dy = \int_{\varphi=0}^{2\pi} \int_{r=0}^{2} e^{-r^2} r^3\, dr\, d\varphi =$$

$$t = r^2, \qquad \frac{dt}{2} = r\, dr\,, \qquad 0 \leqslant t \leqslant 4$$

$$= \pi \int_0^4 e^{-t}\, t\, dt = \pi \left[-(1 + t)\, e^{-t} \right]_0^4 = \pi \left(1 - 5\, e^{-4} \right)$$

$$\int x\, e^{a x}\, dx = \frac{e^{a x}}{a^2} (a x - 1)$$

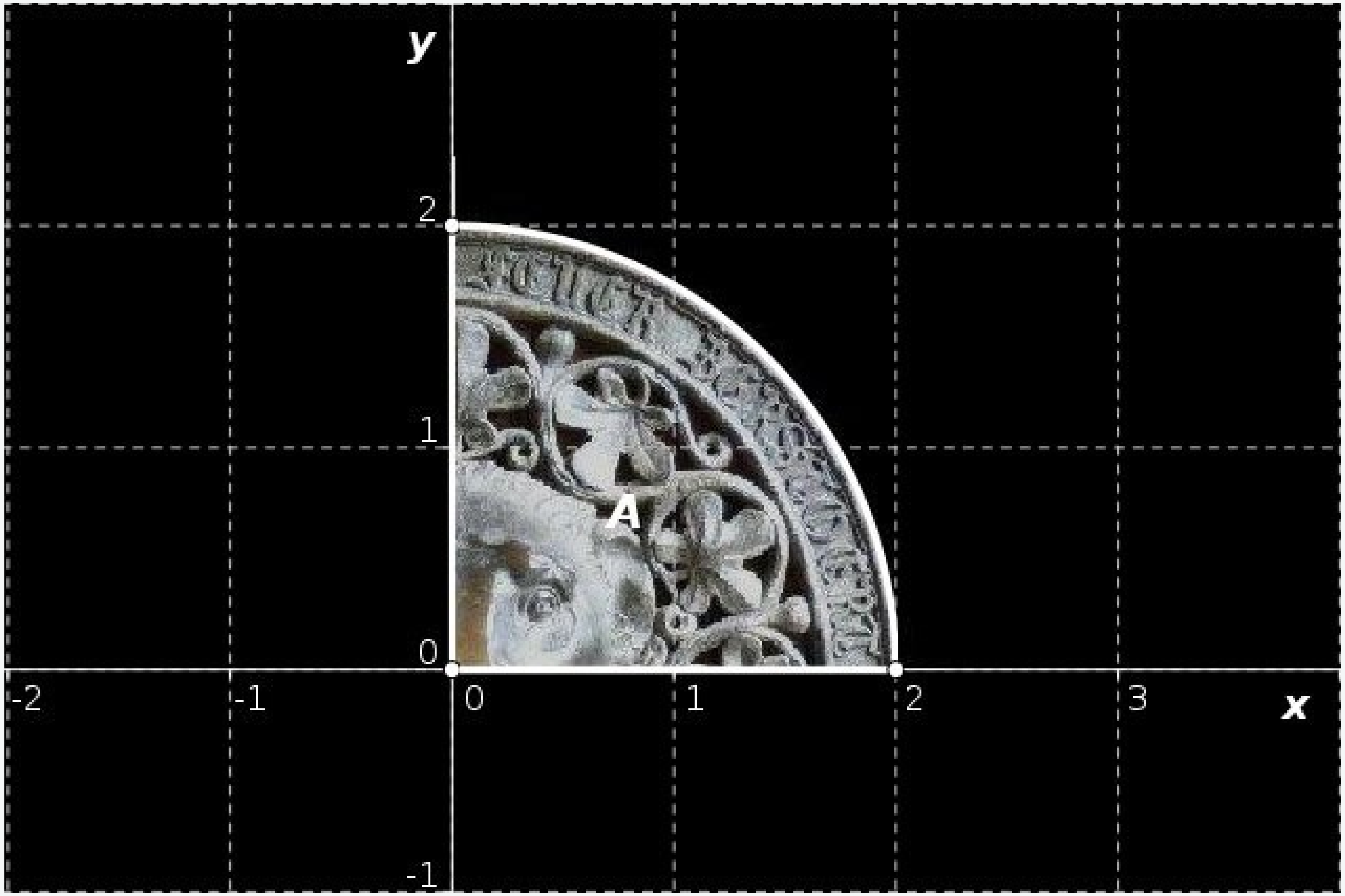


Abb. L6: Darstellung des Integrationsbereiches A

$$A \quad : x^2 + y^2 \leqslant 4, \quad x\,,\, y \geqslant 0 \quad \Leftrightarrow \quad r \leqslant 2, \quad 0 \leqslant \varphi \leqslant \frac{\pi}{2}$$

$$I = \int\limits_A e^{-(x^2+y^2)} \frac{x\,y}{x^2 + y^2}\, dx\, dy =$$

$$= \int\limits_{\varphi=0}^{\pi/2} \int\limits_{r=0}^{2} e^{-r^2} \sin\varphi \cos\varphi \; r\, dr\; d\varphi = \frac{1}{2} \int\limits_{\varphi=0}^{\pi/2} \sin(2\varphi)\, d\varphi \int\limits_{r=0}^{2} e^{-r^2}\, r\, dr =$$

$$= \frac{1}{2} \int\limits_0^2 e^{-r^2}\, r\, dr = \frac{1}{4} \int\limits_0^4 e^{-t}\, dt = \frac{1}{4} (1 - e^{-4})$$

$$\int\limits_{\varphi=0}^{\pi/2} \sin(2\varphi)\, d\varphi = -\left[\frac{1}{2} \cos(2\varphi)\right]_0^{\pi/2} = 1$$

$$t = r^2, \qquad \frac{dt}{2} = r\, dr, \qquad 0 \leqslant t \leqslant 4$$