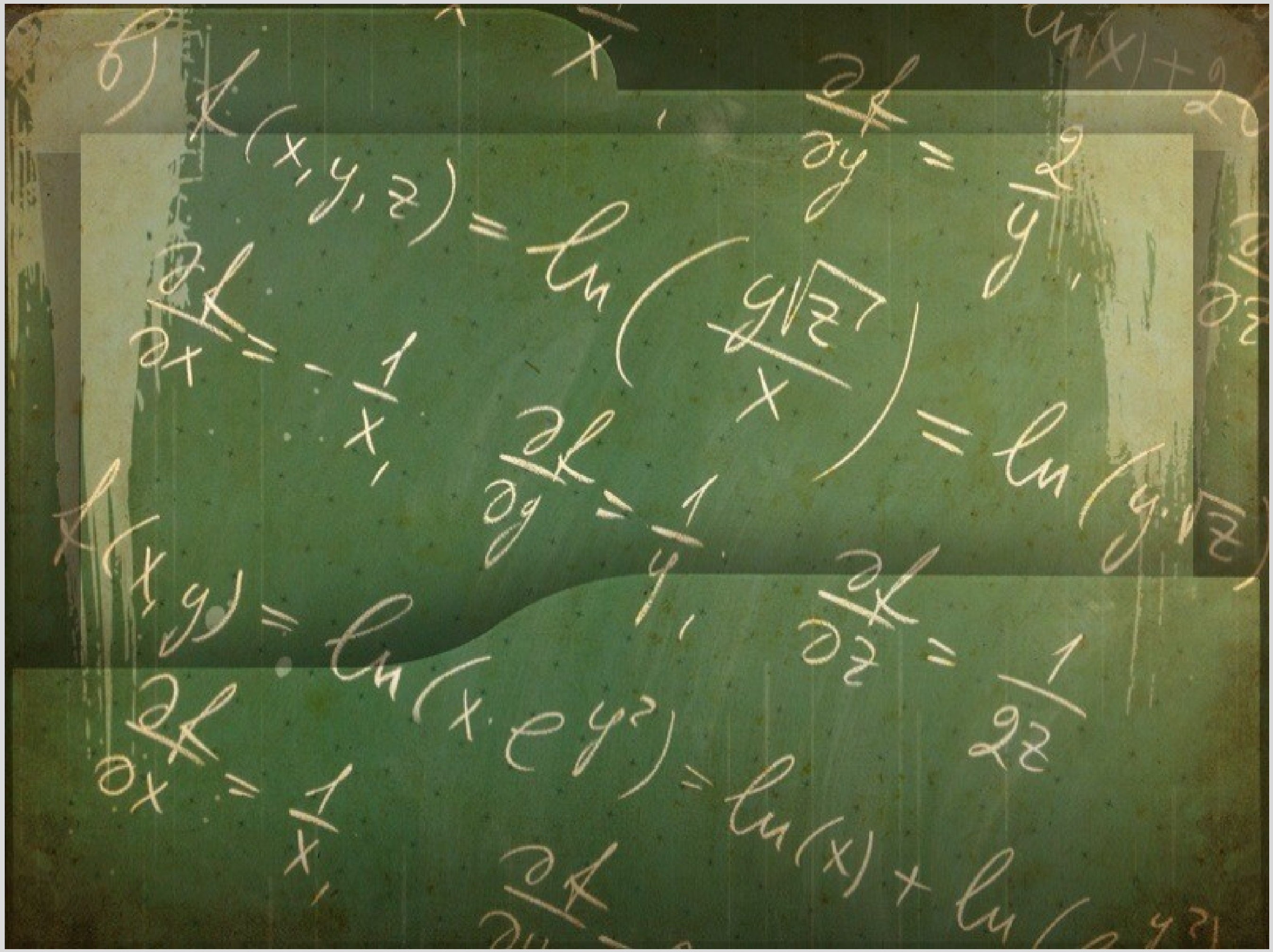

Partielle Ableitungen: Aufgaben

Partielle Ableitungen: Aufgabe 10

Bestimmen Sie die partiellen Ableitungen 1. Ordnung

a) $f(x, y) = x^2 y, \qquad f(x, y) = x y^2$

b) $f(x, y) = e^{x y^3}, \qquad$ c) $f(x, y) = 4 \dfrac{x}{y^5}$

d) $f(x, y) = (2x - y)^2 + \ln(x y)$

e) $f(x, y) = x y^2 \cdot (\sin x + \sin y)$

f) $f(x, y) = \sin(x^2 - y)$

g) $f(x, y) = \ln\left(2x + \dfrac{4}{y}\right)$

h) $f(x, y) = \ln(x + y^2) - e^{2xy} + 3x$

i) $f(x, y, z) = e^{x-y} \cos(5z)$

$$a)\quad f(x, y) = x^2 y, \qquad \frac{\partial f}{\partial x} = 2xy, \qquad \frac{\partial f}{\partial y} = x^2$$

$$f(x, y) = x y^2, \qquad \frac{\partial f}{\partial x} = y^2, \qquad \frac{\partial f}{\partial y} = 2xy$$

$$b)\quad f(x, y) = e^{x y^3}, \qquad \frac{\partial f}{\partial x} = y^3 e^{x y^3}, \qquad \frac{\partial f}{\partial y} = 3 x y^2 e^{x y^3}$$

$$c)\quad f(x, y) = 4\,\frac{x}{y^5}, \qquad \frac{\partial f}{\partial x} = \frac{4}{y^5}, \qquad \frac{\partial f}{\partial y} = -20\,\frac{x}{y^6}$$

$$d)\quad f(x, y) = (2x - y)^2 + \ln(x y) = (2x - y)^2 + \ln x + \ln y$$

$$\frac{\partial f}{\partial x} = 4\,(2x - y) + \frac{1}{x}, \qquad \frac{\partial f}{\partial y} = -2\,(2x - y) + \frac{1}{y}$$

$$f(x, y) = x\,y^2 \cdot (\sin x + \sin y) = u \cdot v\,, \quad u = x\,y^2\,, \quad v = \sin x + \sin y$$

$$\frac{\partial f}{\partial x} = \frac{\partial (u \cdot v)}{\partial x} = \frac{\partial u}{\partial x} \cdot v + \frac{\partial v}{\partial x} \cdot u \qquad f_x = u_x\,v + v_x\,u$$

$$\frac{\partial f}{\partial y} = \frac{\partial (u \cdot v)}{\partial y} = \frac{\partial u}{\partial y} \cdot v + \frac{\partial v}{\partial y} \cdot u \qquad f_y = u_y\,v + v_y\,u$$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\,x\,y^2 = y^2\,, \qquad v_x = \frac{\partial v}{\partial x} = \frac{\partial}{\partial x} \cdot (\sin x + \sin y) = \cos x$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\,x\,y^2 = 2\,x\,y\,, \qquad v_y = \frac{\partial v}{\partial y} = \frac{\partial}{\partial y} \cdot (\sin x + \sin y) = \cos y$$

$$f_x = y^2 \cdot (\sin x + \sin y) + x\,y^2 \cdot \cos x$$

$$f_y = 2\,x\,y \cdot (\sin x + \sin y) + x\,y^2 \cdot \cos y$$

$$f(x, y) = \sin(x^2 - y)$$

Wir führen zunächst die *'Hilfsvariable'* ein, erhalten die 'äußere' Funktion und wenden die Kettenregel an:

$$f = f(x, y) = F(u(x, y))$$

$$\frac{\partial f}{\partial x} = \frac{dF}{du}\frac{\partial u}{\partial x}, \qquad \frac{\partial f}{\partial y} = \frac{dF}{du}\frac{\partial u}{\partial y}$$

$u = x^2 - y$ – Hilfsvariable

$f(x, y) = \sin(x^2 - y) = F(u(x, y)) = \sin u$ – äußere Funktion

$$\frac{dF}{du} = \frac{d}{du}\sin u = \cos u = \cos(x^2 - y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2 - y) = 2x, \qquad \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2 - y) = -1$$

$$\frac{\partial f}{\partial x} = 2x \cdot \cos(x^2 - y), \qquad \frac{\partial f}{\partial y} = -\cos(x^2 - y)$$

$$f(x, y) = \ln\left(2x + \frac{4}{y}\right) = F(u) = \ln u, \qquad u = 2x + \frac{4}{y}$$

$$\frac{dF}{du} = \frac{d}{du}(\ln u) = \frac{1}{u} = \frac{1}{2x + \frac{4}{y}} = \frac{y}{2(2 + xy)}$$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(2x + \frac{4}{y}\right) = 2, \qquad u_y = \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(2x + \frac{4}{y}\right) = -\frac{4}{y^2}$$

$$\frac{\partial f}{\partial x} = \frac{dF}{du} \cdot \frac{\partial u}{\partial x} = \frac{y}{2 + xy}$$

$$\frac{\partial f}{\partial y} = \frac{dF}{du} \cdot \frac{\partial u}{\partial y} = -\frac{2}{y(2 + xy)}$$

h) $f(x, y) = \ln(x + y^2) - e^{2xy} + 3x$

$$\frac{\partial f}{\partial x} = \frac{1}{x + y^2} - 2y\, e^{2xy} + 3, \qquad \frac{\partial f}{\partial y} = \frac{2y}{x + y^2} - 2x\, e^{2xy}$$

i) $f(x, y, z) = e^{x-y} \cos(5z)$

$$\frac{\partial f}{\partial x} = e^{x-y} \cos(5z), \qquad \frac{\partial f}{\partial y} = -e^{x-y} \cos(5z),$$

$$\frac{\partial f}{\partial z} = -5\, e^{x-y} \sin(5z)$$

Berechnen Sie die folgenden partiellen Ableitungen

$$a)\quad \frac{\partial}{\partial m}\left(\frac{1}{2}\, m\, v^2\right); \qquad \frac{\partial}{\partial v_0}\,(v_0 + a\,t); \qquad \frac{\partial}{\partial T}\left(\frac{2\,\pi\, r}{T}\right)$$

$$b)\quad \frac{\partial}{\partial t}\,\sin\,(c\,t - 5\,x); \qquad \frac{\partial}{\partial M}\left(\frac{2\,\pi\, r^{3/2}}{\sqrt{G\,M}}\right)$$

$$c)\quad \frac{\partial}{\partial \beta}\;\frac{e^{x\,\beta - 3}}{2\,y\,\beta + 5}; \qquad \frac{\partial}{\partial v}\;\frac{m_0}{\sqrt{1 - v^2/c^2}}$$

$$a)\quad \frac{\partial}{\partial m}\left(\frac{1}{2}\, m\, v^2\right) = \frac{v^2}{2}; \qquad \frac{\partial}{\partial v_0}\,(v_0 + a\, t) = 1; \qquad \frac{\partial}{\partial T}\left(\frac{2\pi r}{T}\right) = -\frac{2\pi r}{T^2}$$

$$b)\quad \frac{\partial}{\partial t}\,\sin\,(c\, t - 5\, x) = c\cos\,(c\, t - 5\, x); \qquad \frac{\partial}{\partial M}\left(\frac{2\pi r^{3/2}}{\sqrt{G\, M}}\right) = -\frac{\pi\, r^{3/2}}{\sqrt{G\, M^3}}$$

$$c)\quad \frac{\partial}{\partial \beta}\;\frac{e^{x\beta - 3}}{2\, y\beta + 5} = e^{x\beta - 3}\,\frac{(2\, x\, y\beta + 5\, x - 2\, y)}{(2\, y\beta + 5)^2}$$

$$\frac{\partial}{\partial v}\;\frac{m_0}{\sqrt{1 - v^2/c^2}} = \frac{m_0\, v}{c^2\,\sqrt{(1 - v^2/c^2)^3}}$$

Aufgabe 12:

Die Gleichung eines idealen Gases $p = R\,\dfrac{T}{V}$

erlaubt es, jede Variable als Funktion der anderen Variablen darzustellen

$$V = V(T,\ p), \qquad T = T(p,\ V), \qquad p = p(V,\ T)$$

Zeigen Sie, dass

$$\frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial p} \cdot \frac{\partial p}{\partial V} = -1$$

Aufgabe 13:

Berechnen Sie die partielle Ableitung nach x der Funktion $z = z\,(x, y)$, die durch folgende Gleichung bestimmt wird

$$y\,z - \ln z = x + y$$

Partielle Ableitungen: *Lösung 13*

$$y\,z - \ln z = x + y$$

$$\frac{\partial}{\partial x}(y\,z) - \frac{\partial}{\partial x}\ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y\,\frac{\partial z}{\partial x} - \frac{1}{z}\,\frac{\partial z}{\partial x} = 1 + 0$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1 \quad \Rightarrow \quad \frac{\partial z}{\partial x} = \frac{z}{y\,z - 1}$$

Bestimmen Sie die partiellen Ableitungen der folgenden Funktionen:

$a)\quad f(x, y) = x^2\, y + 2\, x^5\, y$

$b)\quad f(x, y) = \sin(5\, x^3\, y - 3\, x\, y^2)$

$c)\quad f(x, y) = \ln(y\, e^{x\, y})$

$d)\quad f(x, y) = \sqrt{x^2 - y} + \ln(x\, y)$

a) $f(x, y) = x^2 y + 2x^5 y$

$$f_x = 2xy + 10x^4 y, \qquad f_y = x^2 + 2x^5 = x^2 (1 + 2x^3)$$

b) $f(x, y) = \sin(5x^3 y - 3xy^2)$

$$f_x = 3y\,(5x^2 - y)\cos(5x^3 y - 3xy^2)$$

$$f_y = x\,(5x^2 - 6y)\cos(5x^3 y - 3xy^2)$$

c) $f(x, y) = \ln(y\, e^{xy}), \qquad f_x = y, \qquad f_y = x + \dfrac{1}{y}$

d) $f(x, y) = \sqrt{x^2 - y} + \ln(xy)$

$$f_x = \frac{x}{\sqrt{x^2 - y}} + \frac{1}{x}, \qquad f_y = -\frac{1}{2\sqrt{x^2 - y}} + \frac{1}{y}$$

Bestimmen Sie die partiellen Ableitungen der folgenden Funktionen:

a) $f(x, y, z) = \ln\left(x\,y^2 z^3\right)$

b) $f(x, y, z) = \ln\left(\dfrac{y\sqrt{z}}{x}\right)$

c) $f(x, y) = x\,y \cdot e^{x\,y}$

d) $f(x, y) = \ln\left(\sqrt{x\,y^3}\right)$

e) $f(x, y) = \ln\left(x\,e^{y^2}\right)$

f) $f(x, y, z) = \ln\left(\sqrt{x\,z} \cdot e^{-y}\right)$

g) $f(x, y) = \ln\left(\dfrac{e^{y^2}}{x\,e^{-x}}\right)$

Eine partielle Differentialgleichung ist eine Differentialgleichung, die partielle Ableitungen enthält. Prüfen Sie, ob die Funktion $z = f(x, y)$

$$z = e^{(x + \sqrt{3}\, y)/4} - 4x - 2y - 4 - 2\sqrt{3}$$

diese partielle Differentialgleichung erfüllt:

$$\frac{\partial z}{\partial x} + \sqrt{3}\,\frac{\partial z}{\partial y} - z = 4x + 2y$$

$$z = e^{(x + \sqrt{3}\, y)/4} - 4\, x - 2\, y - 4 - 2\sqrt{3}$$

$$\frac{\partial z}{\partial x} = \frac{1}{4}\, e^{(x + \sqrt{3}\, y)/4} - 4$$

$$\frac{\partial z}{\partial y} = \frac{\sqrt{3}}{4}\, e^{(x + \sqrt{3}\, y)/4} - 2$$

$$\frac{\partial z}{\partial x} + \sqrt{3}\, \frac{\partial z}{\partial y} = \frac{1}{4}\, e^{(x + \sqrt{3}\, y)/4} - 4 + \frac{3}{4}\, e^{(x + \sqrt{3}\, y)/4} - 2\sqrt{3} =$$

$$= e^{(x + \sqrt{3}\, y)/4} - 4 - 2\sqrt{3}$$

Einsetzen des Ausdrucks für z ergibt die Differentialgleichung

$$\frac{\partial z}{\partial x} + \sqrt{3}\, \frac{\partial z}{\partial y} = z + 4\, x + 2\, y$$