Complex Numbers, Basics

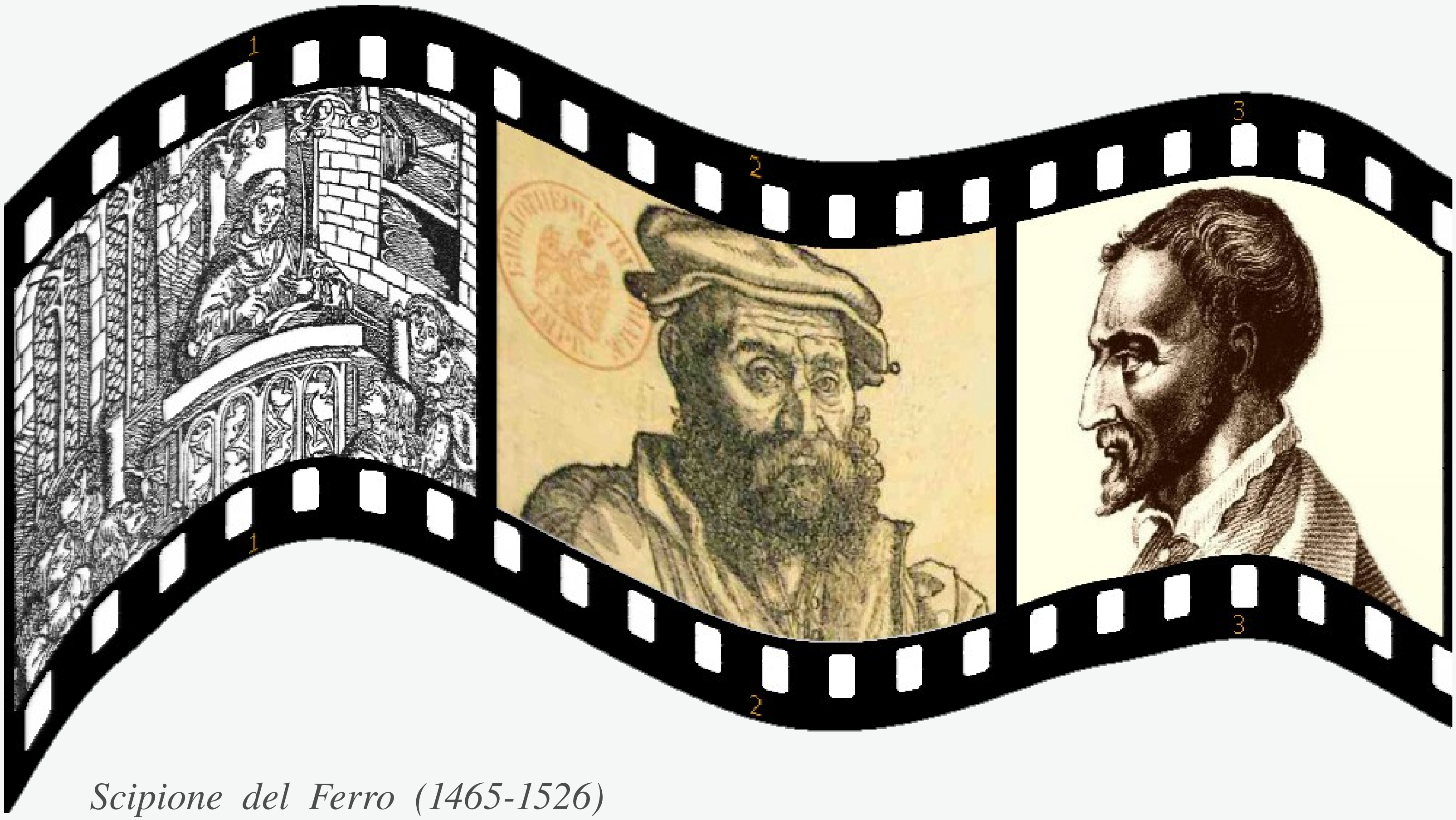
"The shortest path between two truths in the real domain passes through the complex domain." (Jaques Hadamard 1865-1963)

It was one of the early problems of mathematics, that some simple problems could not be solved with the available system of numbers. So at some times it was necessary to introduce new types of numbers.

http://hua.umf.maine.edu/Reading_Revolutions/MagnaCarta/monk-at-workwl.jpg

Mathematicians of the 16th century in particular searched for the solutions of polynomial equations.

Historical roots: the solution of cubic equations



Scipione del Ferro (1465-1526)

Niccolo Tartaglia (1500-1557)

Gerolamo Cardano (1501-1576)

These mathematicians played an important role in the history of the solution of a very old problem: the solution of cubic equations. At their times, the solution of quadratic equations was known already.

Today everybody has to learn at school the formula:

$$x_{1,2} = \frac{1}{2\,a}\left(-b \pm \sqrt{b^2 - 4\,a\,c}\right)$$

which solves the quadratic equation:

$$a\,x^2 + b\,x + c = 0$$

This formula was already known at the beginning of the 16th century, but calculations with negative numbers were not yet really known. There was also some lack of understanding, that a quadratic equation has two solutions.

http://wapedia.mobi/de/Luca_Pacioli

What about equations of higher degrees?

Luca Pacioli (1445-1514), a Franciscan friar, claimed in his book on mathematics in 1494, that equations of the type

$$a\,x^3 + cx = d\,, \qquad a\,x^3 + d = 0$$

can not be solved analytically.

Luca Pacioli (1445-1514)

Portrait of Luca Pacioli, attributed to Jacopo de Barbari, 1495

Luca Pacioli, italien mathematician and Franciscan friar. His book "Summa de Arithmetica . . . ", 1494, is known as most comprehensive mathematical textbook of the renaissance. It contains treatises on geometry, proportions, book keeping, and more. He discussed linear and quadratic equations. His claim that a cubic equation can not be solved numerically stayed unrefuted until 1535.

Niccolo Tartaglia (1500-1557)

Niccolo Tartaglia, Italien arithmetician

Niccolo Tartaglia was an arithmetician in his hometown Brescia as well as in Verona and Venice. In the course of a contest on arithmetics, he worked on the solution of cubic equations. He found a solution which was however published first by Gerolamo Cardano. This solution was an important step which led to improved understanding of algebraic equations, beyond the knowledge of the ancient world.

Niccolo Tartaglia (1500-1557)

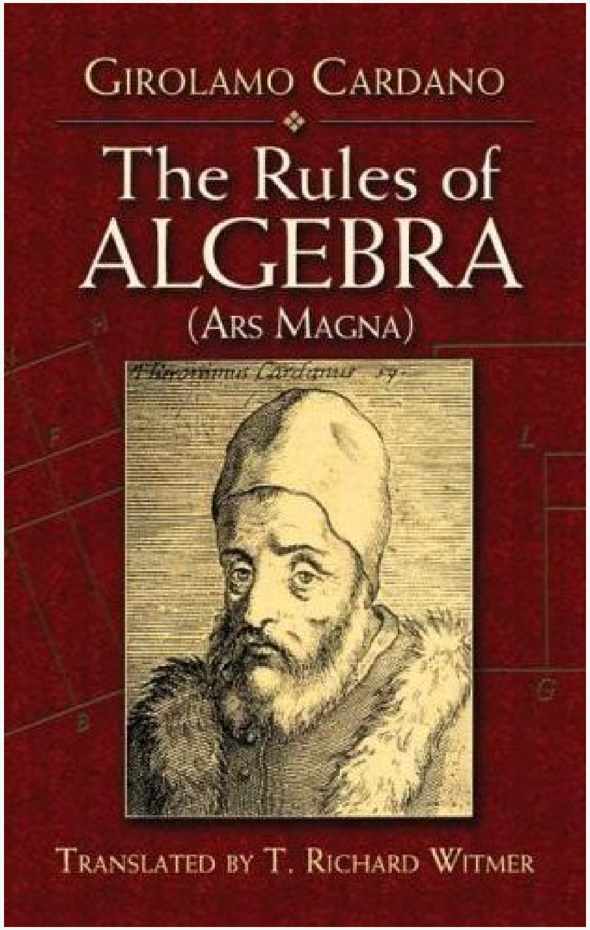
http://www.panoramio.com/photos/original/9163725.jpg

Monument, Niccolo Tartaglia, Brescia

Gerolamo Cardano was an Italien mathematician, physician and philosopher.

Cardano worked on algebra, in particular on the solution of cubic equations. The solution, also quoted as Cardan's formula, originally was found by Niccolo Tartaglia.

Cardano made important contributions to algebra. In his time, mathematicians still avoided the application of negtive numbers, but he did already calculations using roots with negative radicands.



http://www.amazon.com/Rules-Algebra-Magna-Dover-Mathematics/dp/0486458733/ref=pd_bxgy_b_img_b

This book, which appeared in 1545, is a cornerstone in the history of mathematics. It contains the principles of the solutions of cubic and quartic equations.

Rafael Bombelli was the first to show, that calculations with complex numbers can be useful.

He worked many years as engineer and studied algebra in leisure time. In 1572 he published “Algebra”, where he had put together the knowledge of his time on algebra, and he presented the rules of addition and multiplication of complex numbers. He could show, that the formula by Cardano for the solution of cubic equations lead to proper real solutions even then when it contained roots with negative radicands.

Over three centuries, mathematicians were in doubt about the actual meaning of complex numbers. They were considered as pure imagination of human mind. This gets obvious when looking at the adjectives attributed to these numbers. Famous mathematicians called them impossible, fancied, imaginary ...

A reason to enlarge the range of numbers were equations which could not be solved using the numbers known at the time. Thus negative numbers were added to the natural ones to solve an equation like

$$x + 27 = 19$$

Similarly, real numbers were introduced to solve the equation

$$x^2 - 2 = 0$$

If we change the sign in the latter equation, we encounter the problem, that there is no solution to

$$x^2 + 2 = 0 \quad \Leftrightarrow \quad x^2 = -2$$

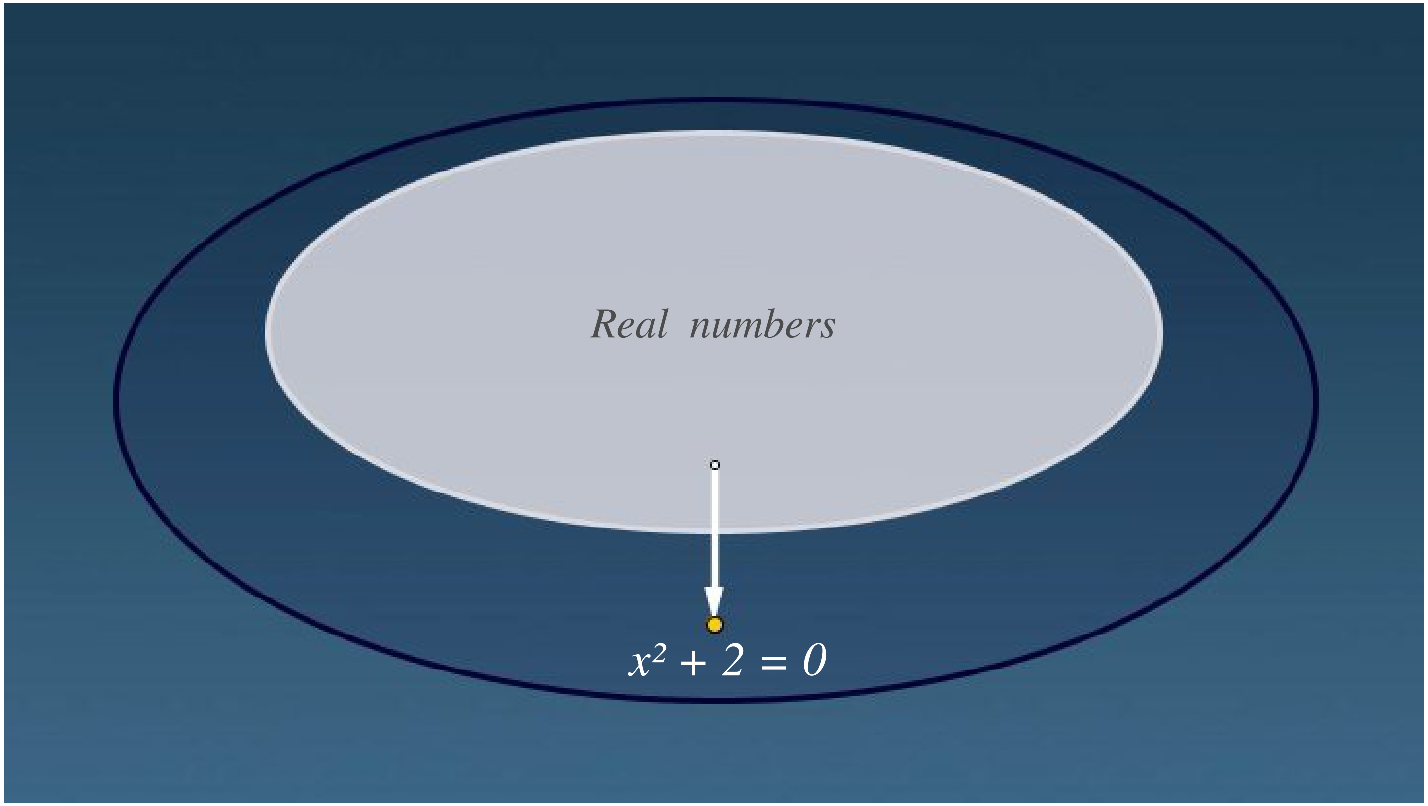
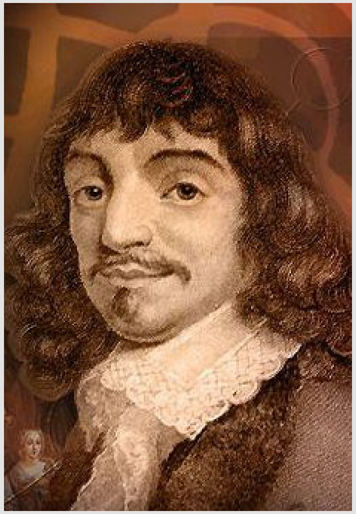


Fig. 1: Extension of the range of numbers to solve the equation $x^2 + 2 = 0$

We add to the set of real numbers another level with the goal to solve all quadratic equations.



René Descartes (1596-1650)

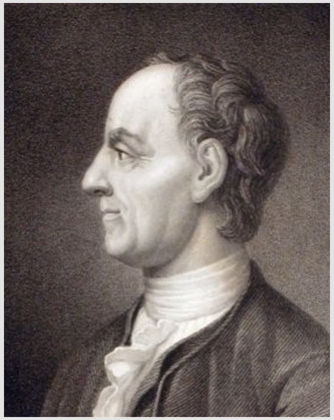
A simple quadratic equation

$$x^2 + 1 = 0 \quad \Leftrightarrow \quad x^2 = -1$$

yields two strange expressions which appear absurd:

$$+\sqrt{-1}\ , \qquad -\sqrt{-1}$$

Descartes thought, that the square root of a negative number is even more meaningless than the negative numbers themselves and called them imaginary numbers.



Leonhard Euler
(1707-1783)

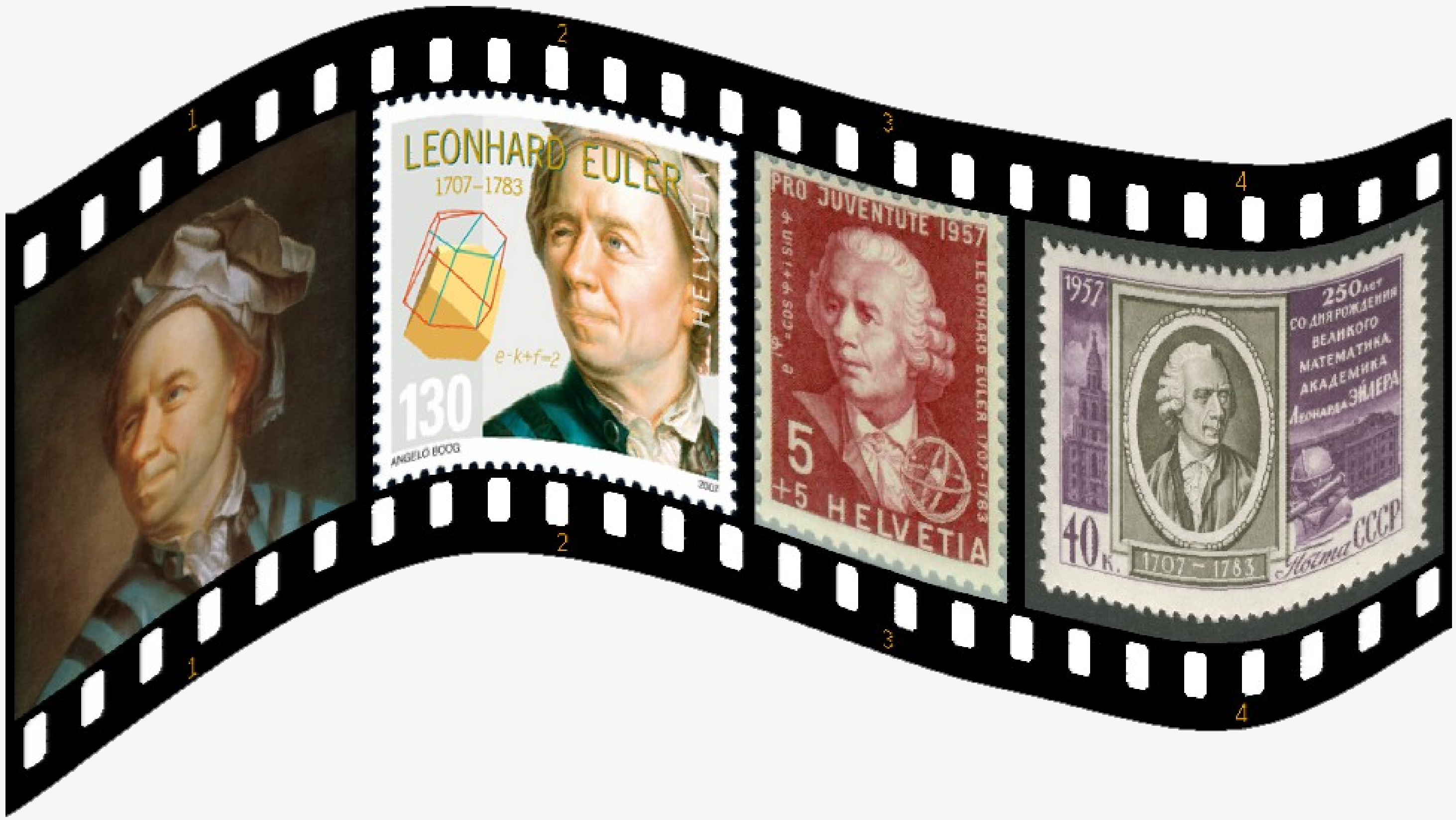
The equation

$$x^2 = -1$$

has no solution on the set of real numbers. This led to the introduction of a new number which solves this equation. As no such number did exist among the known numbers, it was called an imaginary (not existent) unit.

The symbol ***i*** was assigned in 1777 by Leonhard Euler:

$$i^2 = -1$$



Leonhard Euler was a genius in mathematics of the 18th century with enormous productivity. He produced more than 800 scientific articles, half of which were written after he went blind in 1766.

$$i^2 = -1$$

The imaginary unit is often understood as

$$i = \sqrt{-1} \ ,$$

However, this is a merely formal presentation introducing a square root with negative radicand which is not defined on the set of real numbers.

Be aware that the following equation does <u>not</u> hold !

$$i^2 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1} = \sqrt{(-1) \cdot (-1)} = \sqrt{1} = +1$$

Therefor ***i*** can not simply be defined as $\sqrt{-1}$.

<u>Definition of an imaginary number:</u>

$$i\,b\,, \quad (i\,b)^2 = -\,b^2 < 0 \quad b \in \mathbb{R}\,, \quad b \neq 0$$

Gottfried Wilhelm Leibnitz
(1646-1716)

Leibnitz thought i to be a strange mixture of existence and non existence; in his binary system something between 1 (God) and 0 (nothing). he compared the number i with the Holy Spirit: both are not really substantial.

$$x^2 - 4x + 5 = 0$$

The monic form of a quadratic equation:

$$x^2 + px + q = 0 \qquad (p, q \in \mathbb{R})$$

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

$$p = -4, \quad q = 5 \quad \Rightarrow \quad x_{1,2} = 2 \pm \sqrt{-1}$$

This quadratic equation has no real solution. In other words, there is no intersection of the graph of this function and the x-axis.

The solution is then written as

$$x_{1,2} = 2 \pm \sqrt{-1} = 2 \pm i$$

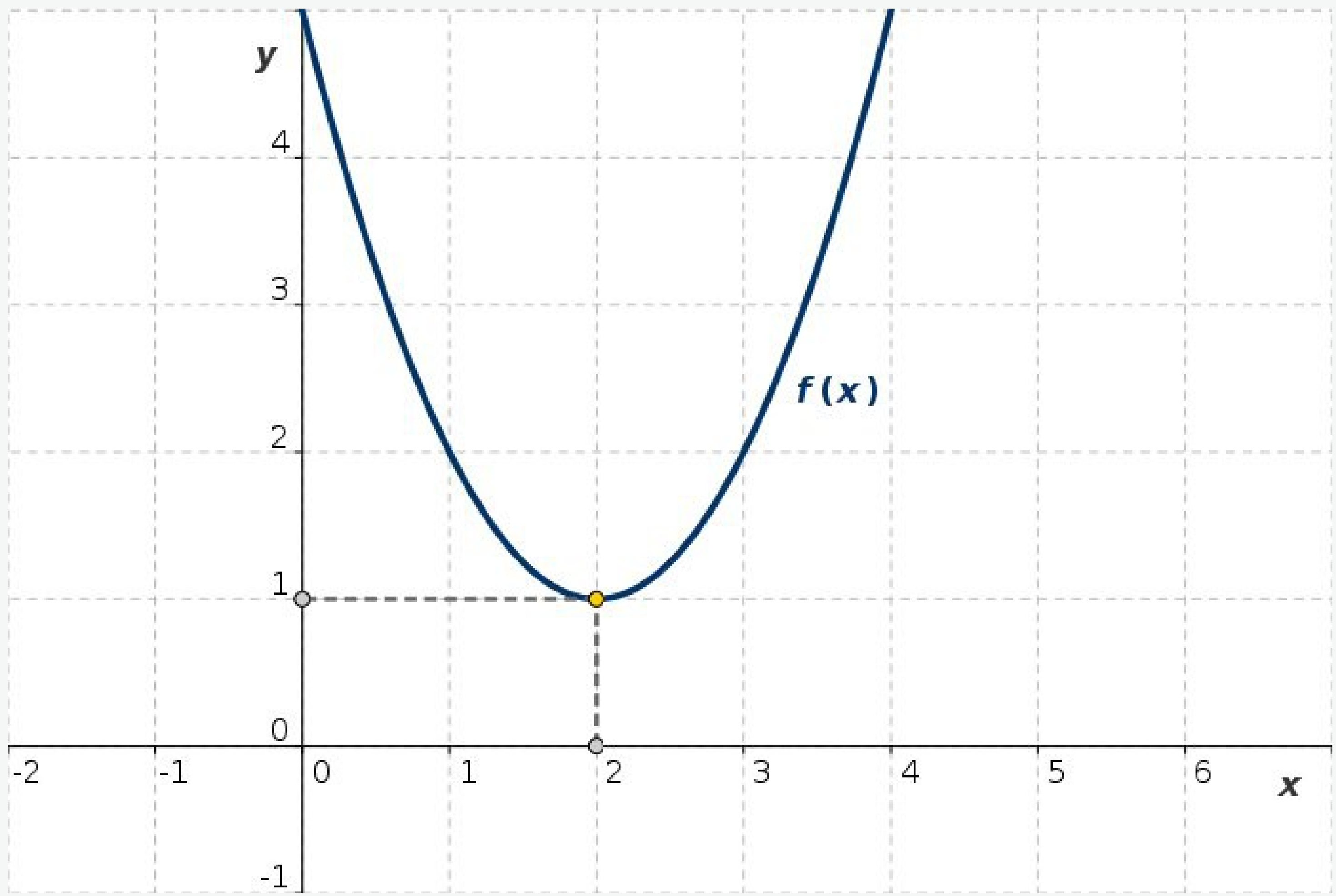


Fig. 2: Graphical representation of the function $f(x) = x^2 - 4x + 5$