Würzburg

Complex Numbers: Basics

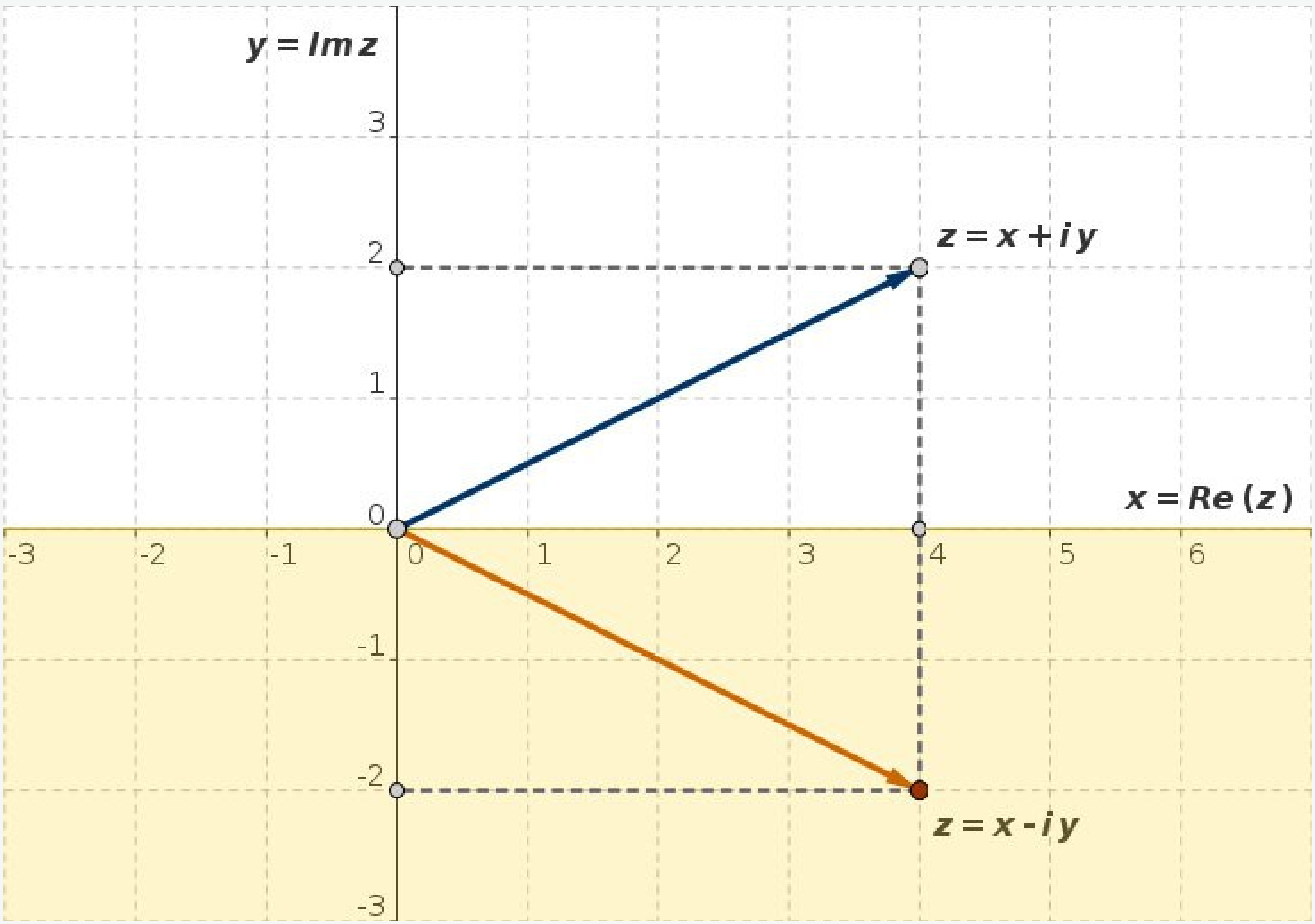

Fig. 7: Complex conjugate of a complex number

$z^* = x - i\,y$ is the complex conjugate of $z = x + i\,y$

$z^* = z$ – real number, $z^* = -z$ – imaginary number , $(z^*)^* = z$

Equality of two complex numbers

Definition:

Two complex numbers $z_1 = x_1 + i\,y_1, \quad z_2 = x_2 + i\,y_2$

are equal, if $x_1 = x_2, \quad y_1 = y_2$

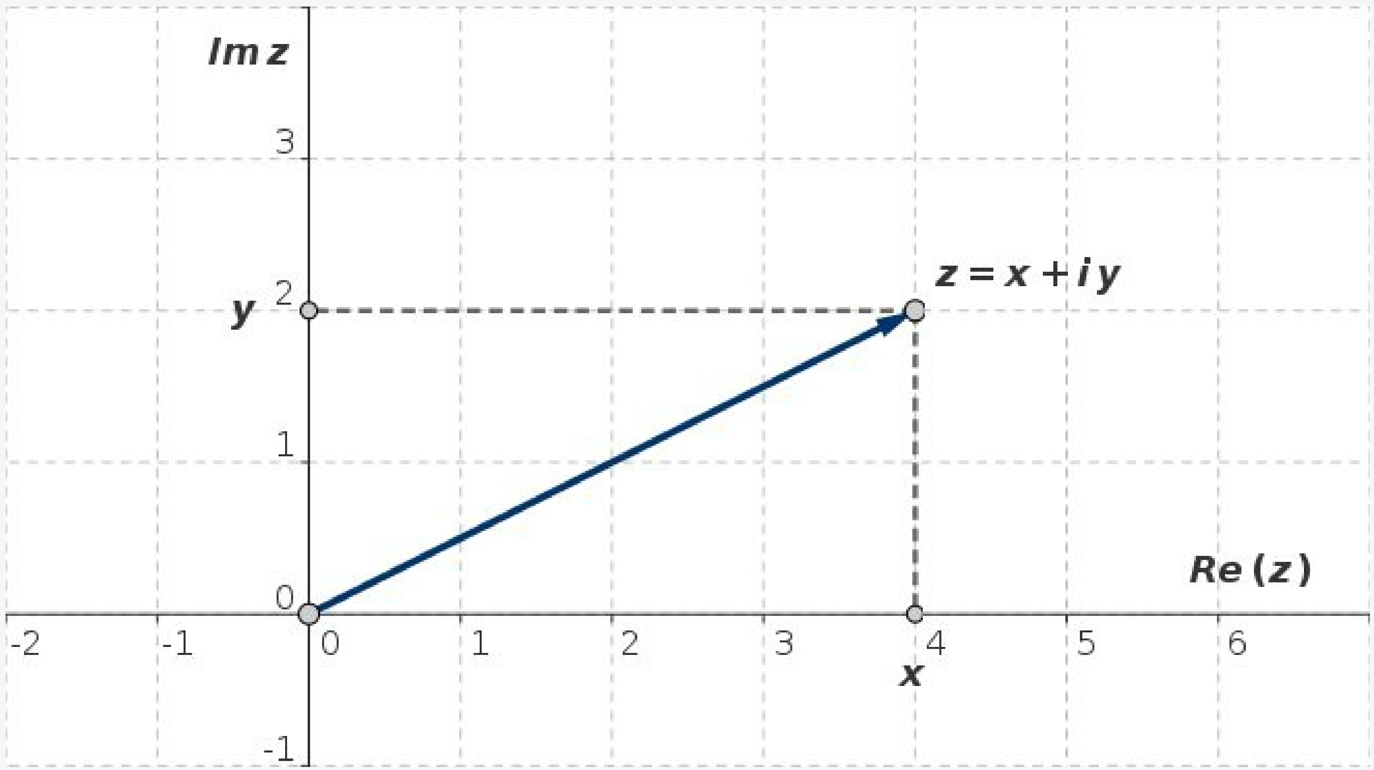

Fig.. 8: Absolute value of a complex number,: length of the position vector

Definition:

The absolute value (or modulus or magnitude) $|z|$ of a complex number $z = x + i\,y$ is given by the length of its position vector

$$|z| = \sqrt{z\;z^*} = \sqrt{x^2 + y^2}\,, \qquad |z| \geqslant 0$$

(application of the Pythagorean theorem).

The following relations hold for a complex number z

$$z \in \mathbb{C}$$

$$|\operatorname{Re} z| \leqslant |z|, \qquad |\operatorname{Im} z| \leqslant |z|$$

Proof:

$$|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} \geqslant \sqrt{(\operatorname{Re} z)^2} = \operatorname{Re} z$$

$$|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} \geqslant \sqrt{(\operatorname{Im} z)^2} = \operatorname{Im} z$$

Exercise 1:

Find the complex conjugates of the given numbers and mark them in the complex plane:

$z_1 = i, \quad z_2 = 2 + i, \quad z_3 = -4 + 2i,$

$z_4 = 3, \quad z_5 = -3 - 1.5i, \quad z_6 = 5 - i$

Exercise 2:

Give the absolute value of the numbers below:

$z_1 = 4 + 3i, \quad z_2 = 3 + 4i, \quad z_3 = 4 - 3i$

$z_4 = 3 - 4i, \quad z_5 = -3 + 4i, \quad z_6 = 5$

$z_7 = -2\sqrt{6} + i, \quad z_8 = -1 - 2\sqrt{6}\, i, \quad z_9 = 5i$

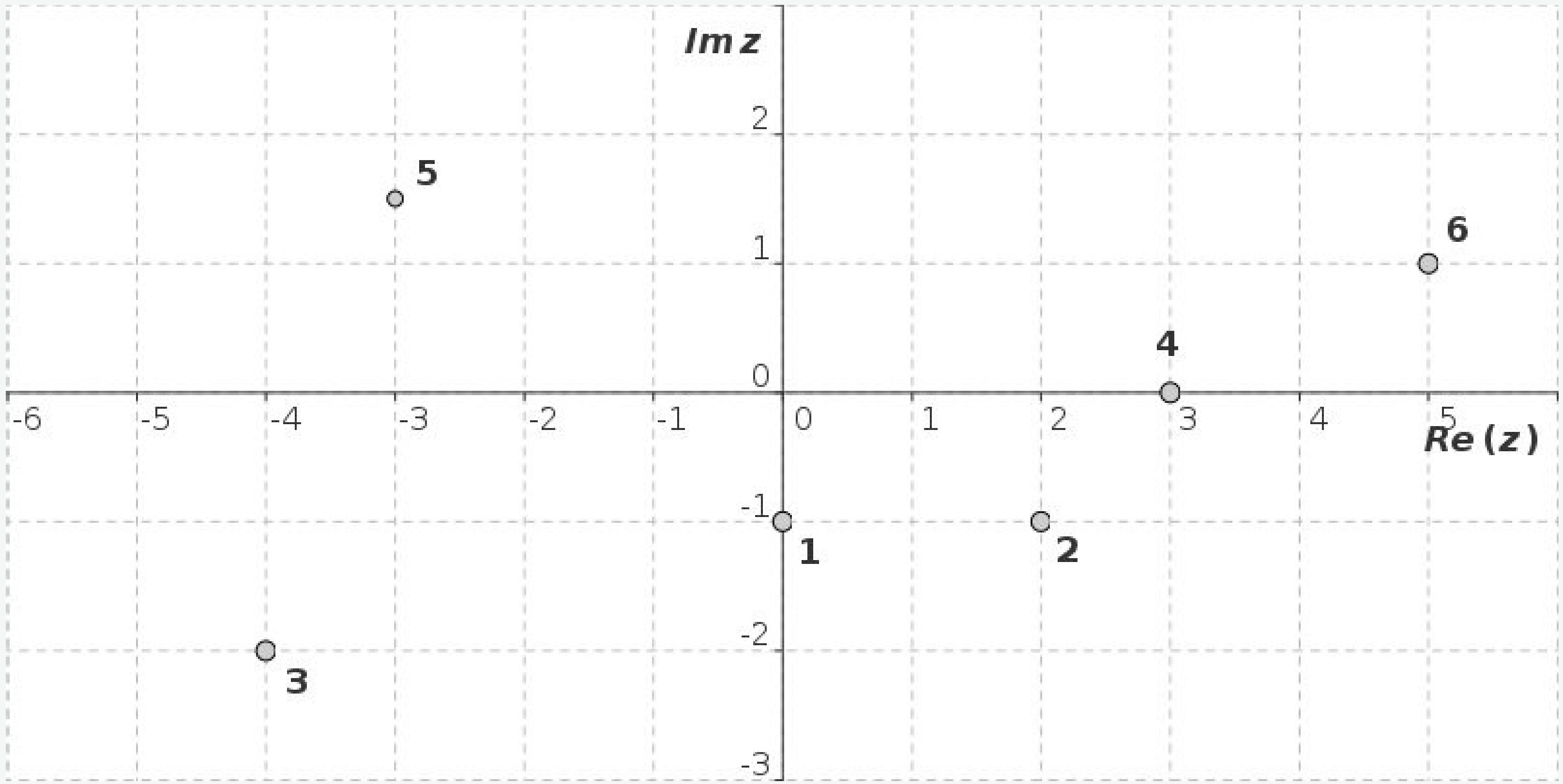

Fig. S1: Complex conjugate of the numbers of exercise 1

$$z_1^* = -i\,, \qquad z_2^* = 2 - i\,, \qquad z_3^* = -4 - 2i\,,$$

$$z_4^* = 3, \qquad z_5^* = -3 + 1.5\,i\,, \qquad z_6^* = 5 + i$$

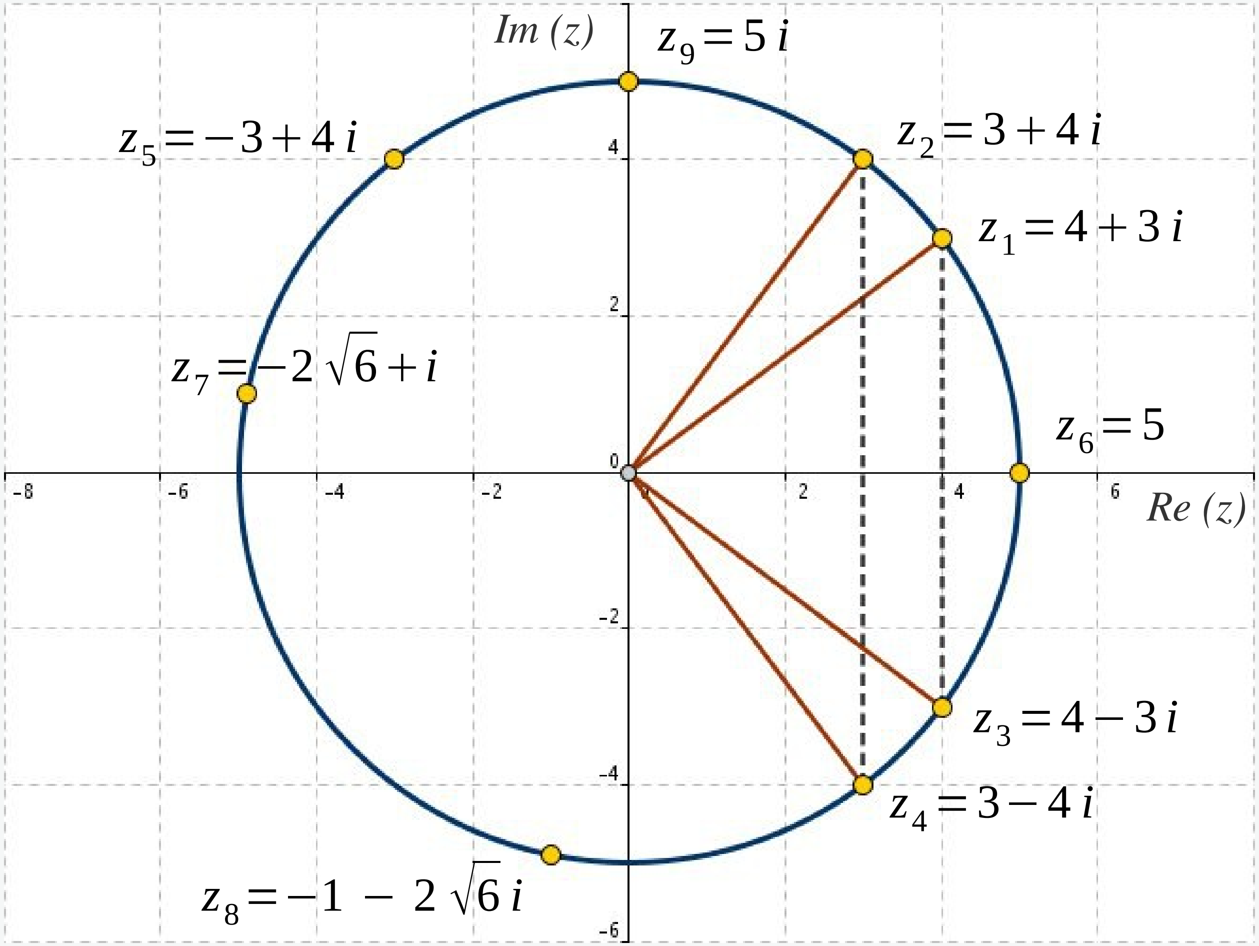

Fig. S2: The complex numbers of exercise 2

All numbers are on a circle of radius $R = 5$, consequently, the absolute value of all of them is 5.

$$|z_1| = |z_2| = \ldots = |z_8| = |z_9| = 5$$

Exercise 3:

Determine the absolute values of the complex numbers given below and draw them on the Gauss plane:

$$z_1 = 2i\,, \quad z_2 = 3 + i^2\,, \quad z_3 = -\sqrt{3} + i\,, \quad z_4 = -2$$

$$z_5 = -\sqrt{3} - i\,, \quad z_6 = \sqrt{2} - \sqrt{2}\,i$$

Exercise 4:

Determine the absolute values of

$$z_1 = -6 + i\,, \quad z_2 = 5 + i^3\,, \quad z_3 = -\sqrt{21} + 2i$$

$$z_4 = -\sqrt{6} - \sqrt{3}\,i\,, \quad z_5 = \sqrt{35} - i^3\,, \quad z_6 = \sqrt{15} + i$$

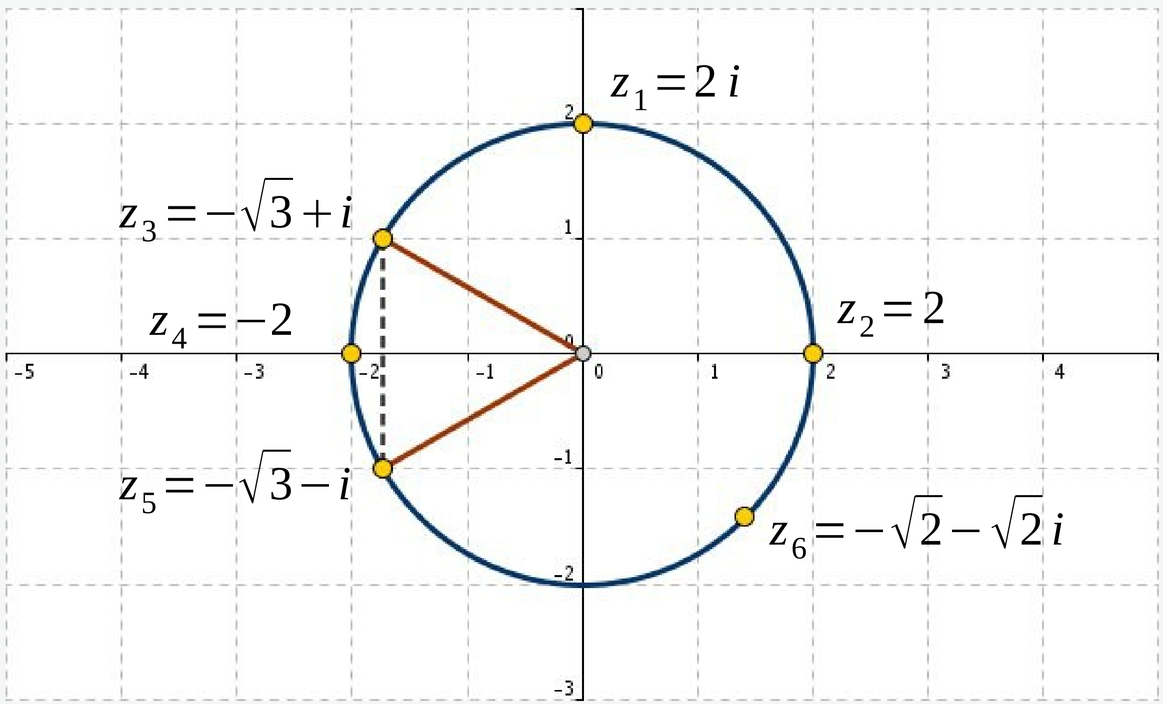

Fig. S3: The complex numbers on the Gauss plane

All numbers are on a circle of radius $R = 2$, therefore their absolute value is 2 :

$$|z_1| = |z_2| = \ldots = |z_6| = 2$$

Basics: Solution 4

$$z_1 = -6 + i\,, \qquad |z_1| = \sqrt{(-6)^2 + 1^2} = \sqrt{37}$$

$$z_2 = 5 + i^3 = 5 - i\,, \qquad |z_2| = \sqrt{5^2 + 1^2} = \sqrt{26}$$

$$z_3 = -\sqrt{21} + 2i\,, \qquad |z_3| = \sqrt{(\sqrt{21})^2 + 2^2} = \sqrt{25} = 5$$

$$z_4 = -\sqrt{6} - \sqrt{3}i\,, \qquad |z_4| = \sqrt{6 + 3} = \sqrt{9} = 3$$

$$z_5 = \sqrt{35} - i^3 = \sqrt{35} + i\,, \qquad |z_5| = \sqrt{35 + 1} = \sqrt{36} = 6$$

$$z_6 = \sqrt{15} + i\,, \qquad |z_6| = \sqrt{15 + 1} = \sqrt{16} = 4$$