Polar form of complex numbers

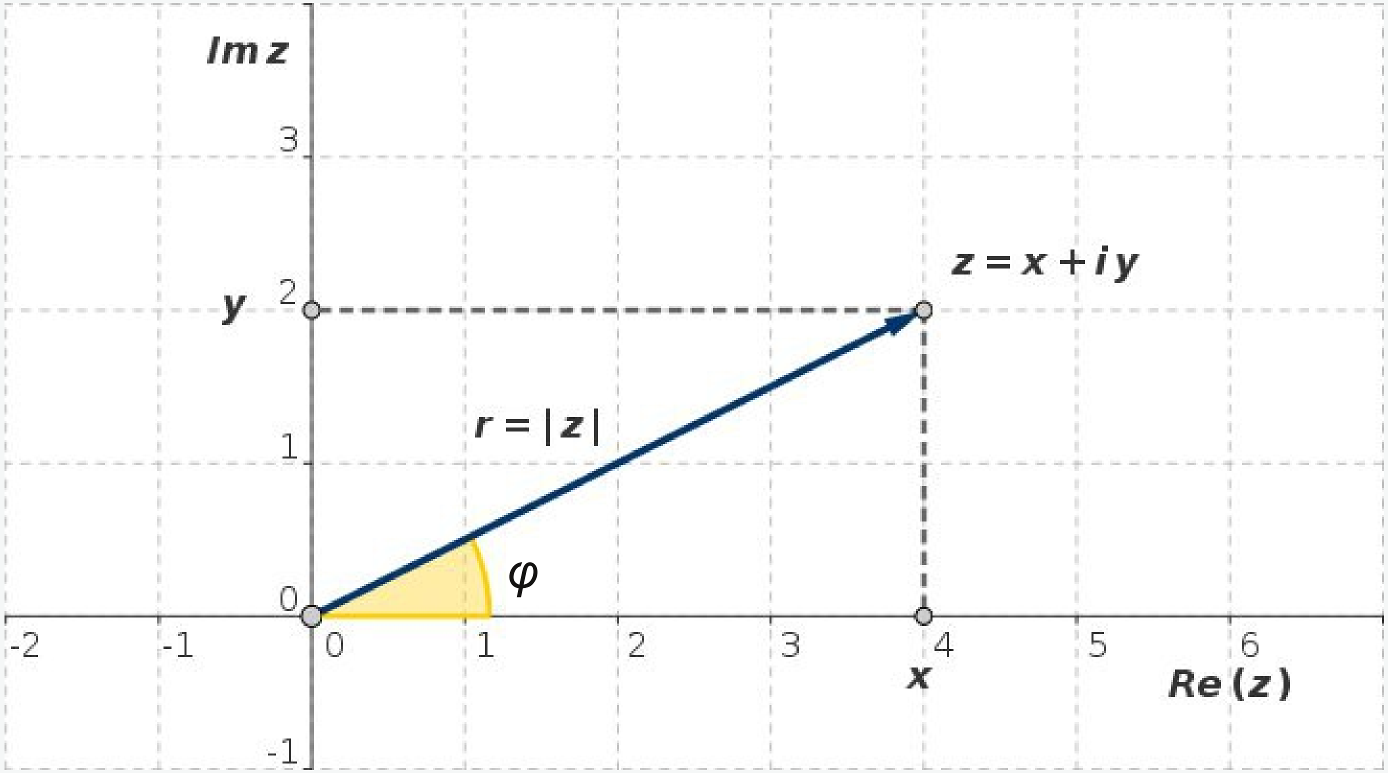

Fig. 1: Trigonometric form of a complex number

Transformation:

$$x = r \cos\varphi, \qquad y = r \sin\varphi$$

Cartesian (algebraic) form: $z = x + i\,y$

Trigonometric (polar) form: $z = r\,(\cos\varphi + i\,\sin\varphi)$

Trigonometric (or polar) form of complex numbers

$z = r\,(\cos\varphi + i\,\sin\varphi)\,,\qquad r\,,\ \varphi$ – polar coordinates

$$r = |\,z\,| = \sqrt{x^2 + y^2}\,,\qquad r \geqslant 0$$

r – magnitude (absolute value) of z

$\varphi = arg\,z\,,\quad 0 \leqslant \varphi \leqslant 2\pi$ – principle value

The argument of a complex number is not uniquely defined:

$$\varphi_k = \varphi + 2\,k\,\pi$$

Represent the complex numbers, given below in trigonometric form, by the position vector in the Gauss plane. Write down these numbers also in Cartesian form.

a) $$z_1 = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$z_2 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

b) $$z_3 = \cos\pi + i\sin\pi,$$

$$z_4 = 2\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$$

c) $$z_5 = 2\sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

$$z_6 = \frac{6}{\sqrt{2}}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

d) $z_7:\ r = 2,\quad \varphi = \dfrac{3\pi}{4},\qquad z_8:\ r = 3,\quad \varphi = \dfrac{2\pi}{3}$

e) $z_9:\ r = \dfrac{3}{2},\quad \varphi = -\dfrac{13}{6}\pi,\qquad z_{10}:\ r = \sqrt{3},\quad \varphi = -\dfrac{\pi}{3}$

f) $z_{11}:\ r = 3,\quad \varphi = 60^\circ,\qquad z_{12}:\ r = 2,\quad \varphi = 210^\circ$

g) $z_{13}:\ r = 2,\quad \varphi = 18^\circ,\qquad z_{14}:\ r = 2,\quad \varphi = -36^\circ$

h) $z_{15}:\ r = 2,\quad \varphi = 162^\circ,\qquad z_{16}:\ r = \sqrt{3},\quad \varphi = 200^\circ$

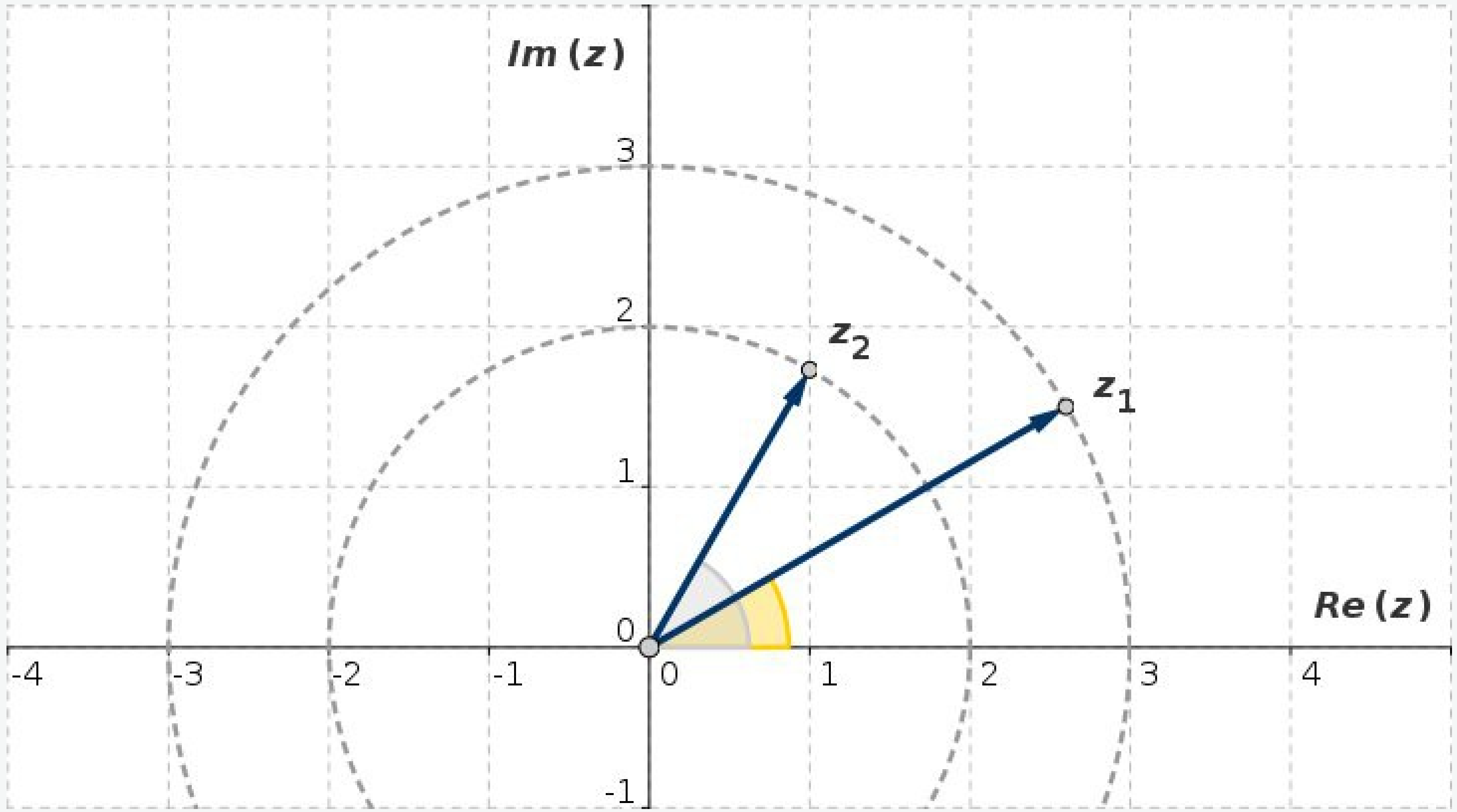

Fig. L1a: Complex numbers in Gauss plane

$$z_1 = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 3\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = \frac{3\sqrt{3}}{2} + \frac{3}{2}\,i$$

$$z_2 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right) = 1 + \sqrt{3}\,i$$

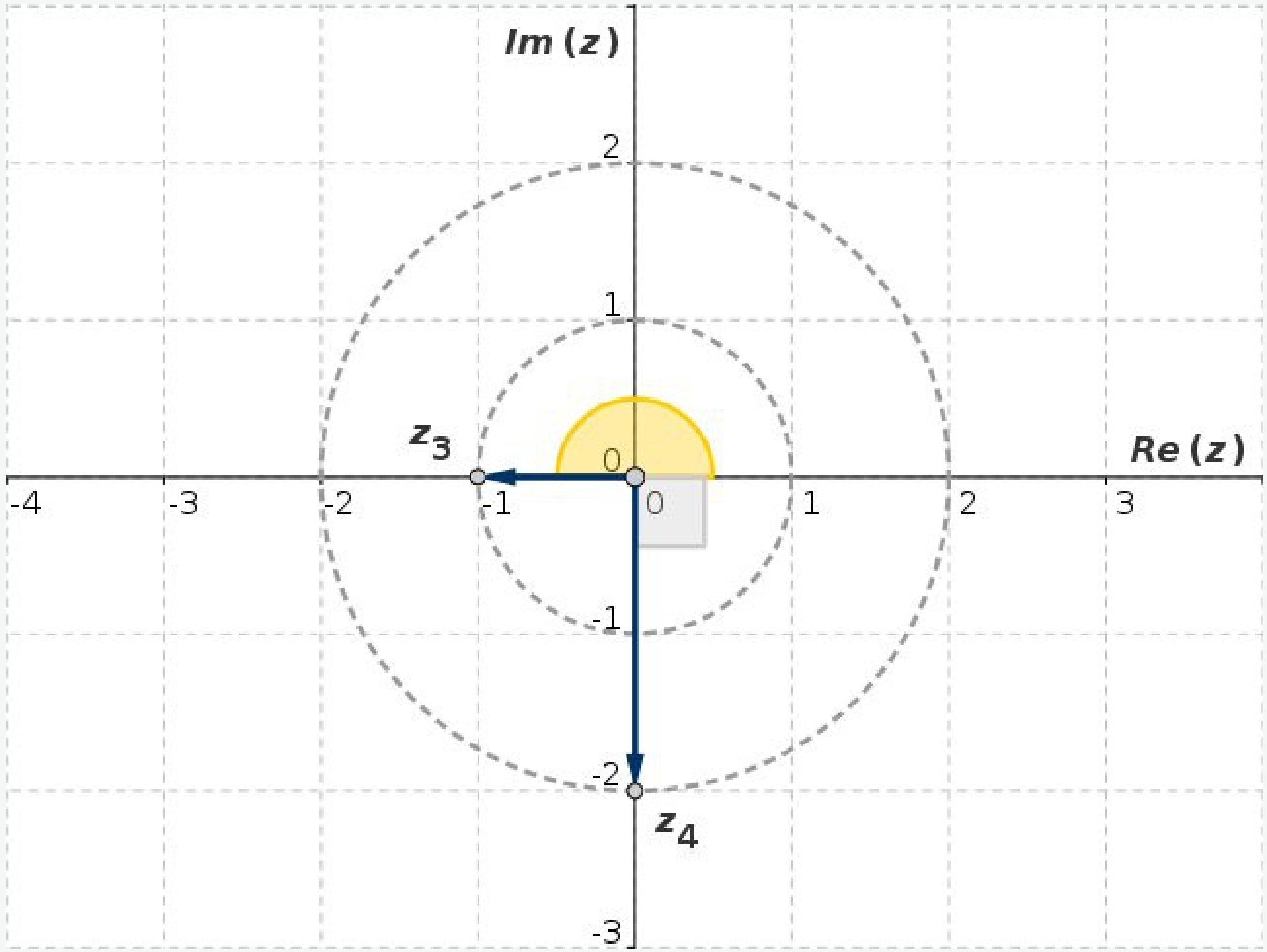

Fig. L1b: Complex numbers in Gauss plane

$$z_3 = \cos\pi + i\sin\pi = -1, \qquad z_4 = 2\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right) = -2\,i$$

Trigonometric form: Solution 1c

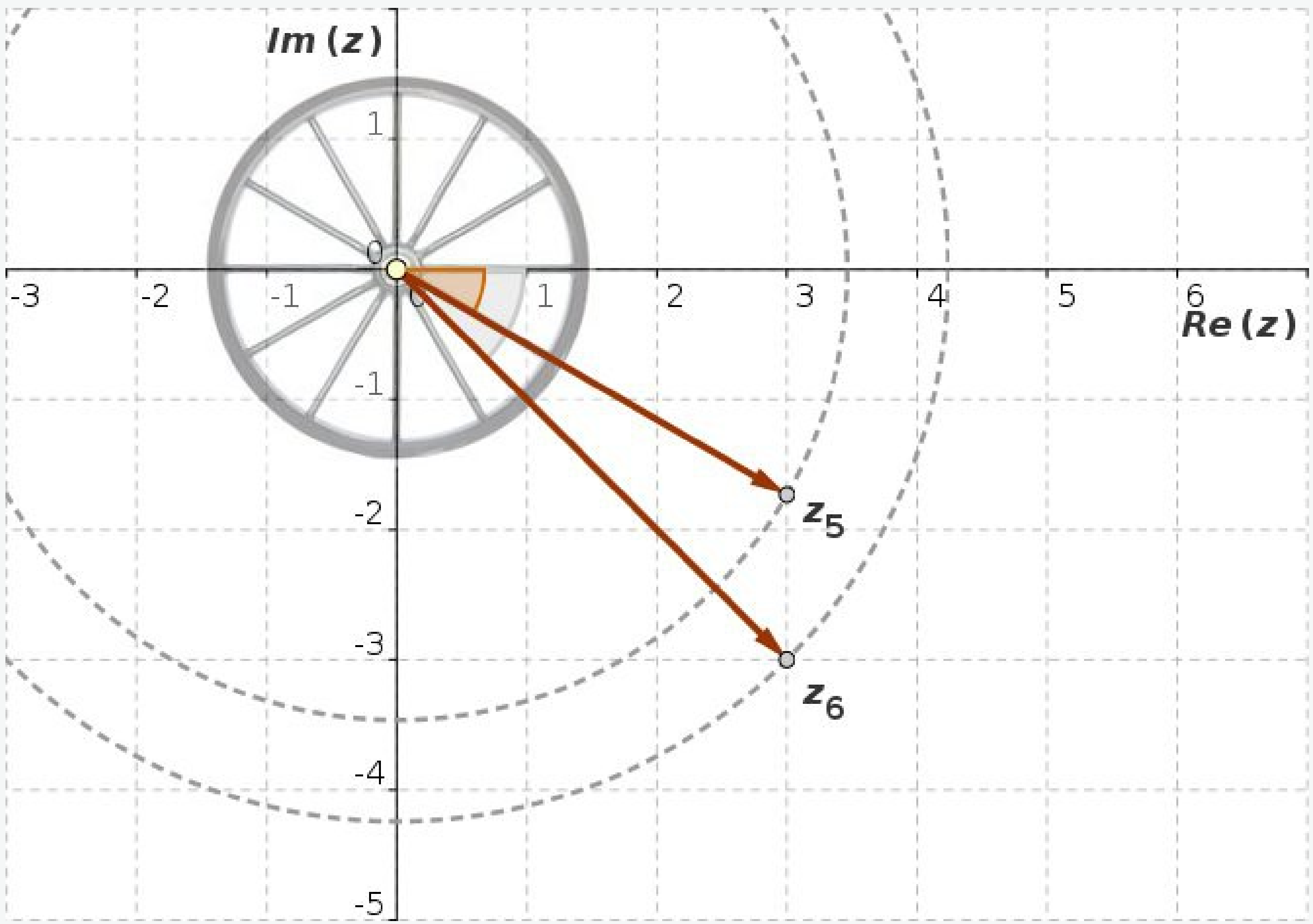

Fig. L1c: Complex numbers in Gauss plane

$$z_5 = 2\sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = 2\sqrt{3}\left(\cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)\right) = 3 - \sqrt{3}\,i$$

$$z_6 = \frac{6}{\sqrt{2}}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = \frac{6}{\sqrt{2}}\left(\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)\right) = 3\,(1 - i)$$

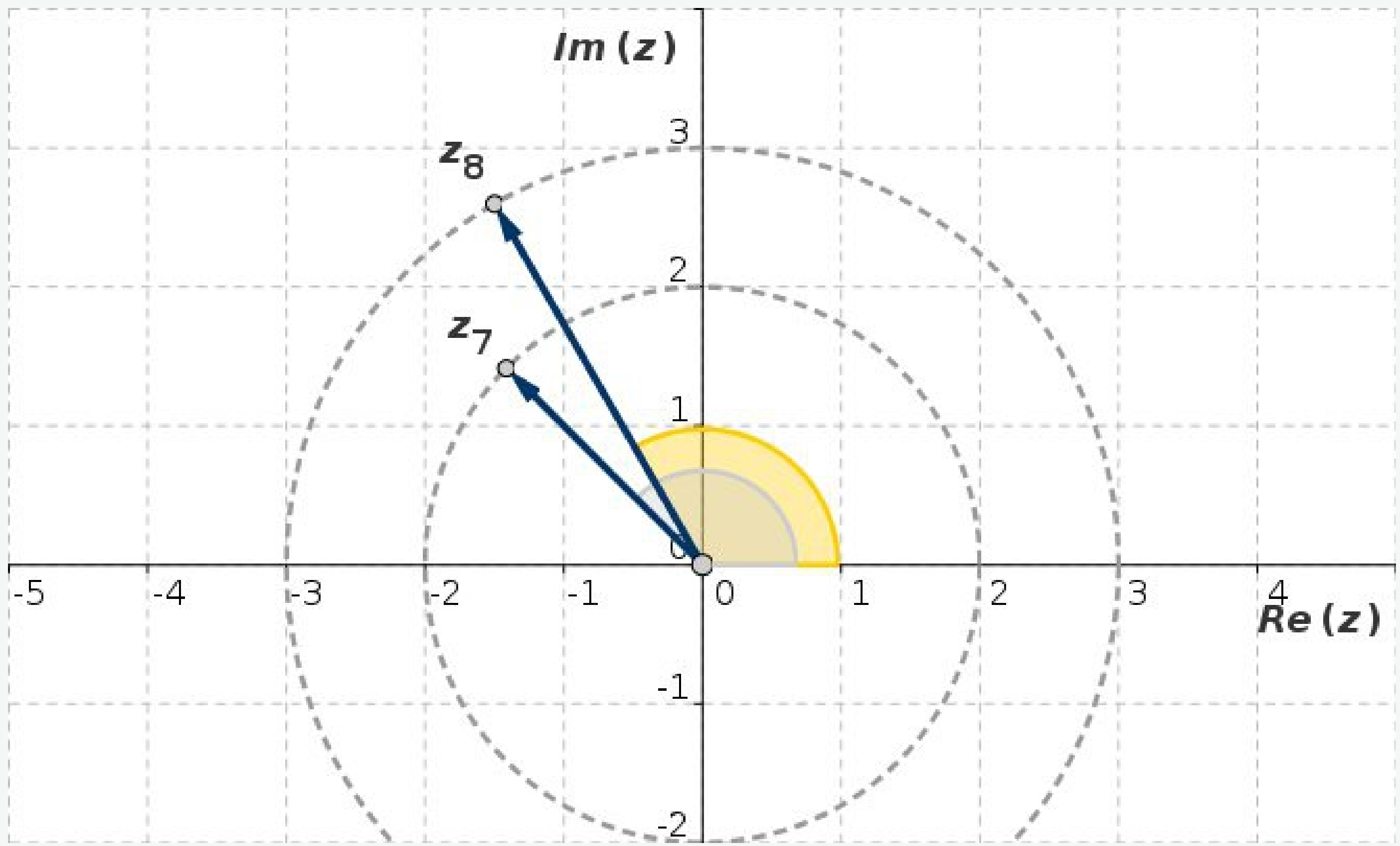

Fig. L1d: Complex numbers in Gauss plane, represented by magnitude and angle

$$z_7 = 2 \left(\cos\left(\frac{3\,\pi}{4}\right) + i \sin\left(\frac{3\,\pi}{4}\right)\right) = -\sqrt{2} + i\,\sqrt{2}$$

$$z_8 = 3 \left(\cos\left(\frac{2\,\pi}{3}\right) + i \sin\left(\frac{2\,\pi}{3}\right)\right) = -\frac{3}{2} + \frac{3\sqrt{3}}{2}\ i$$

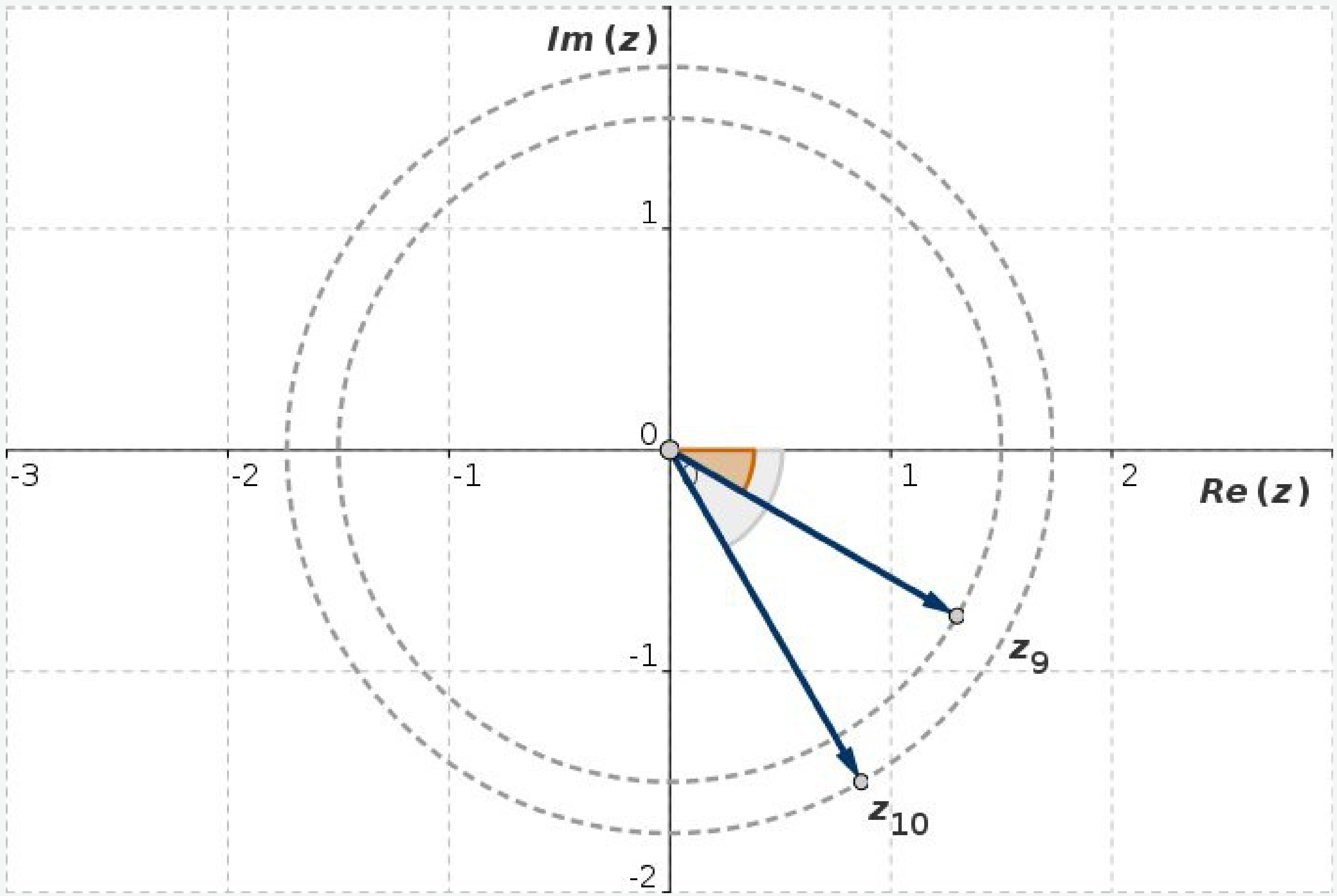

Fig. L1e: Complex numbers in Gauss plane

$$z_9 = \frac{3}{2}\left(\cos\left(-\frac{13}{6}\pi\right) + i\sin\left(-\frac{13}{6}\pi\right)\right) = \frac{3}{2}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = \frac{3}{4}\left(\sqrt{3} - i\right)$$

$$z_{10} = \sqrt{3}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) = \sqrt{3}\left(\cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{3}\right)\right) = \frac{\sqrt{3}}{2} - i\,\frac{3}{2}$$

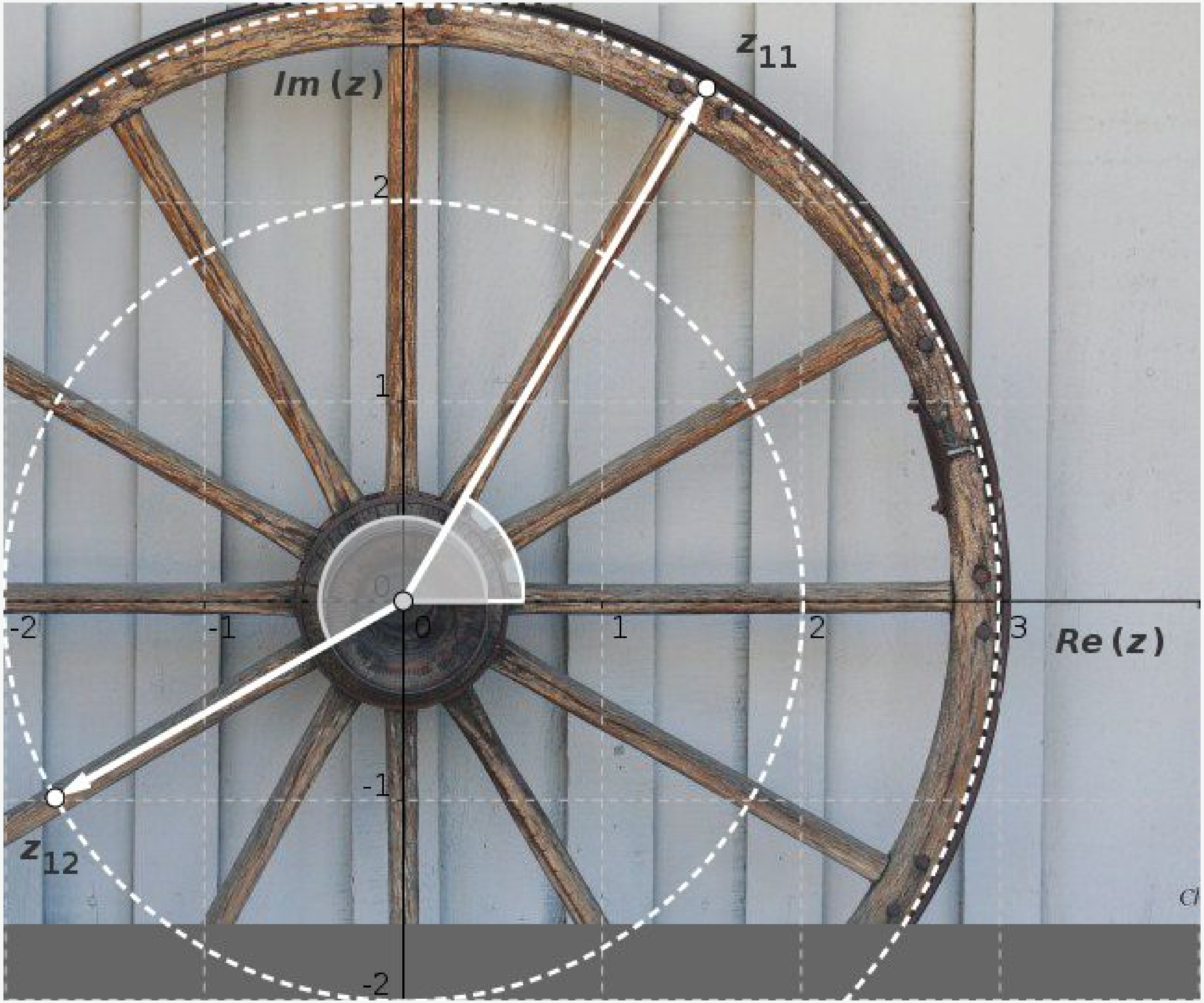

Fig. L1f: Representation of the complex numbers of exercise 1f

$$z_{11} = \frac{3}{2}\,(1 + i\sqrt{3}\,), \qquad z_{12} = -\sqrt{3}\, - i$$

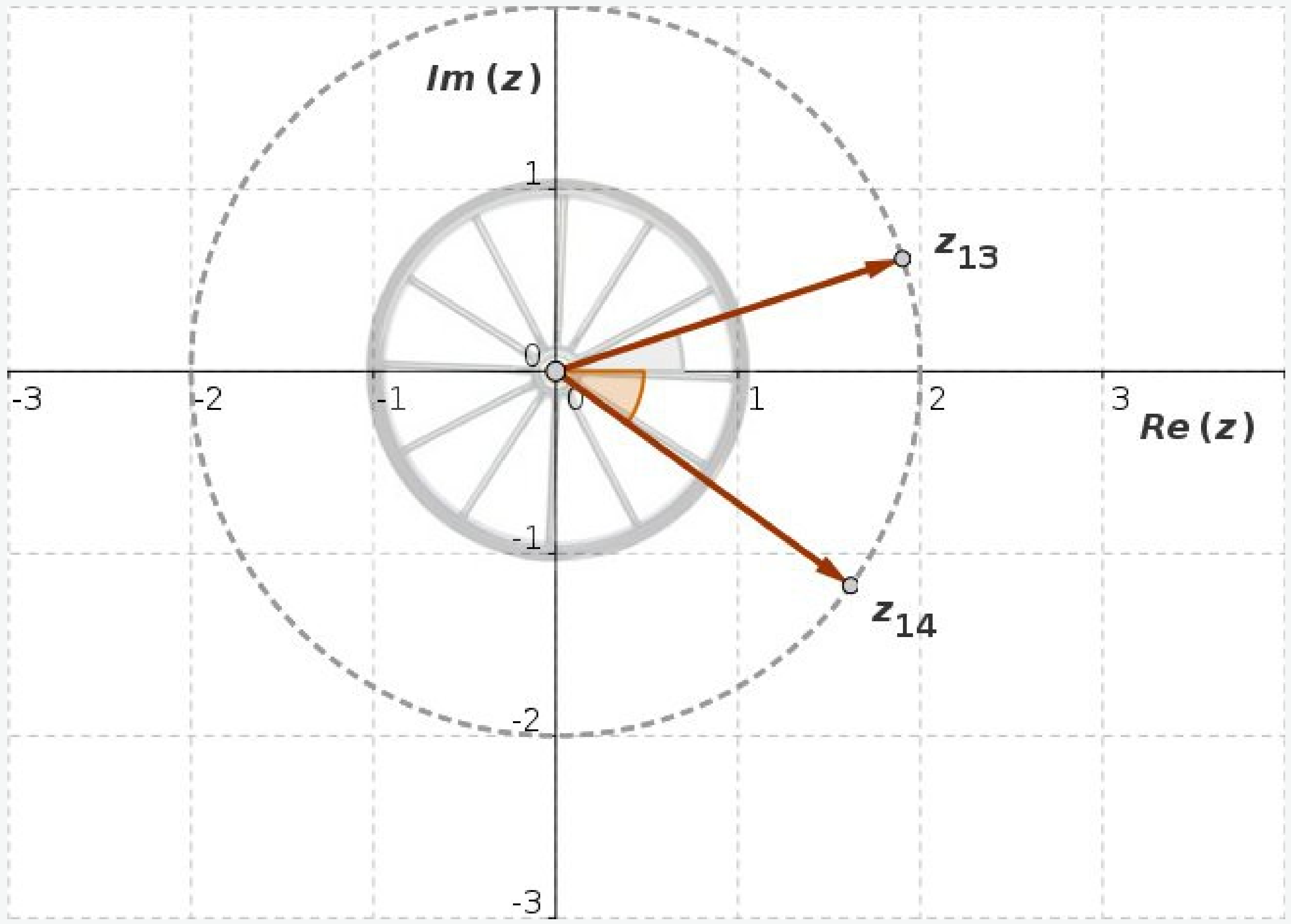

Fig. L1g: Complex numbers in Gauss plane

$$z_{13} = 2\ (\cos(18^\circ) + i\sin(18^\circ)) \simeq 1.902 + i\,0.618$$

$$z_{14} = 2\ (\cos(-36^\circ) + i\sin(-36^\circ)) \simeq 1.618 - i\,1.176$$

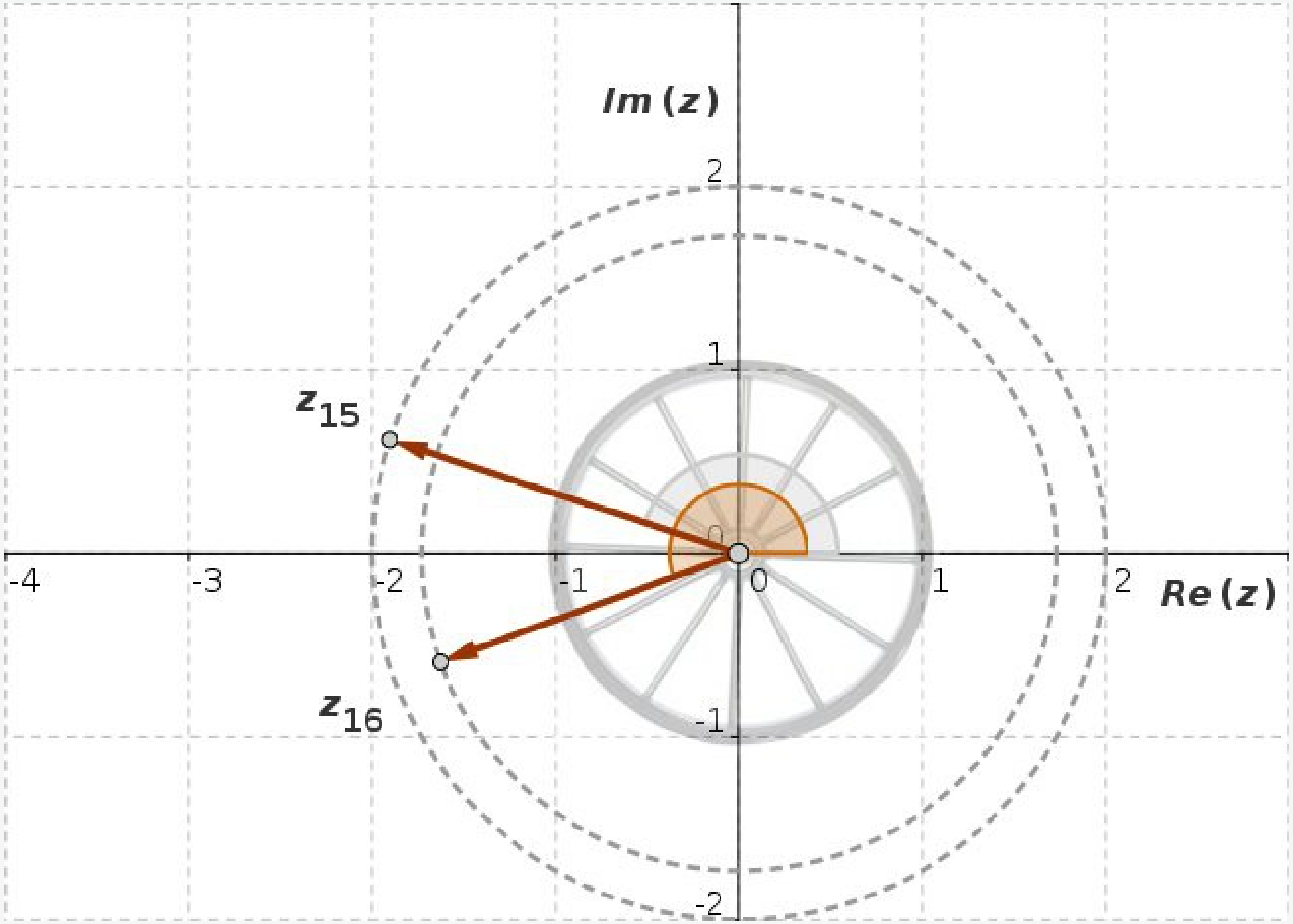

Fig. L1h: Complex numbers in Gauss plane

$$z_{15} = 2\ (\cos(162^\circ) + i\sin(162^\circ)) \simeq -1.902 + i\,0.618$$

$$z_{16} = 2\ (\cos(200^\circ) + i\sin(200^\circ)) \simeq -1.628 - i\,0.592$$

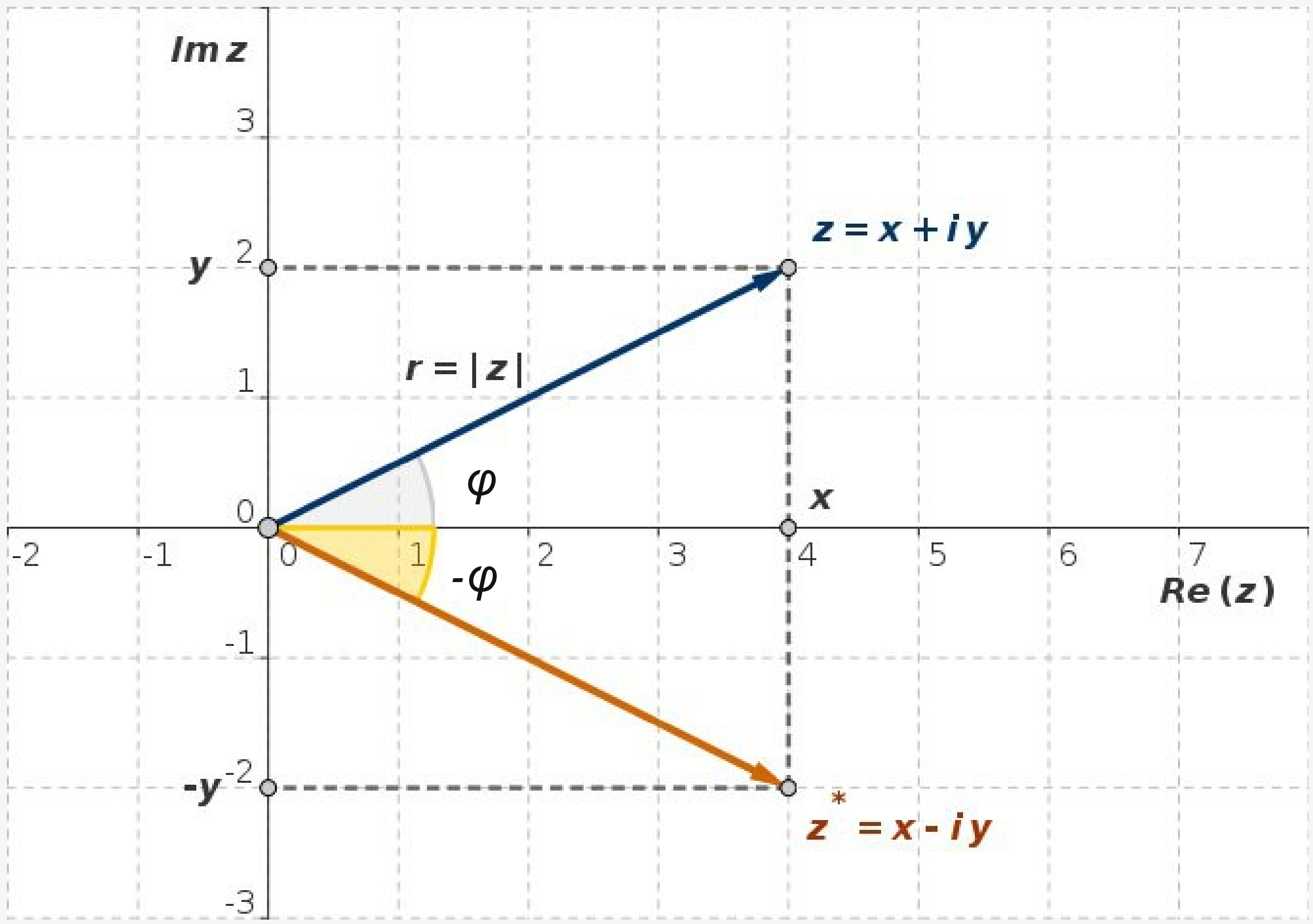

Fig. 2: Trigonometric representation of the complex conjugate number

$$z = r\ (\cos\varphi + i\ \sin\varphi) \quad \rightarrow \quad z^{*} = r\ (\cos\varphi - i\ \sin\varphi)$$

$$\varphi \rightarrow -\varphi \quad \text{or} \quad i \rightarrow -i$$

$$\cos(-\varphi) = \cos\varphi\,, \qquad \sin(-\varphi) = -\sin\varphi$$