Cartesian Form, Polar Form: Conversion

A complex number given in Cartesian form $z = x + i\,y$ can be converted to polar form by the conversion equations, but the quadrant of the complex plane, where the number is located, must be considered with some care.

Quadrant I: $x > 0, \quad y > 0$

Quadrant II: $x < 0, \quad y > 0$

Quadrant III: $x < 0, \quad y < 0$

Quadrant IV: $x > 0, \quad y < 0$

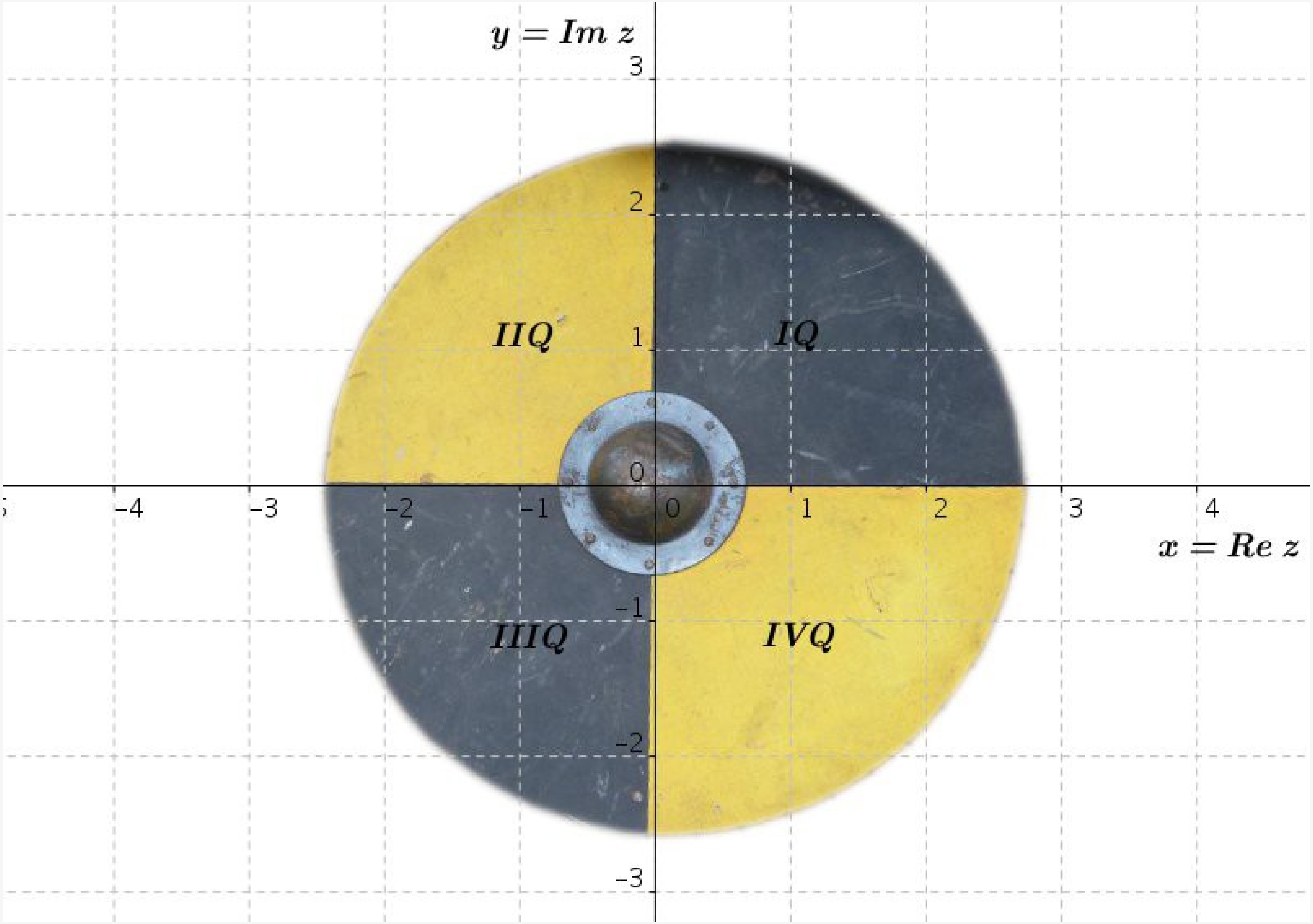

Fig. 3: The 4 quadrants of the Gauss plane

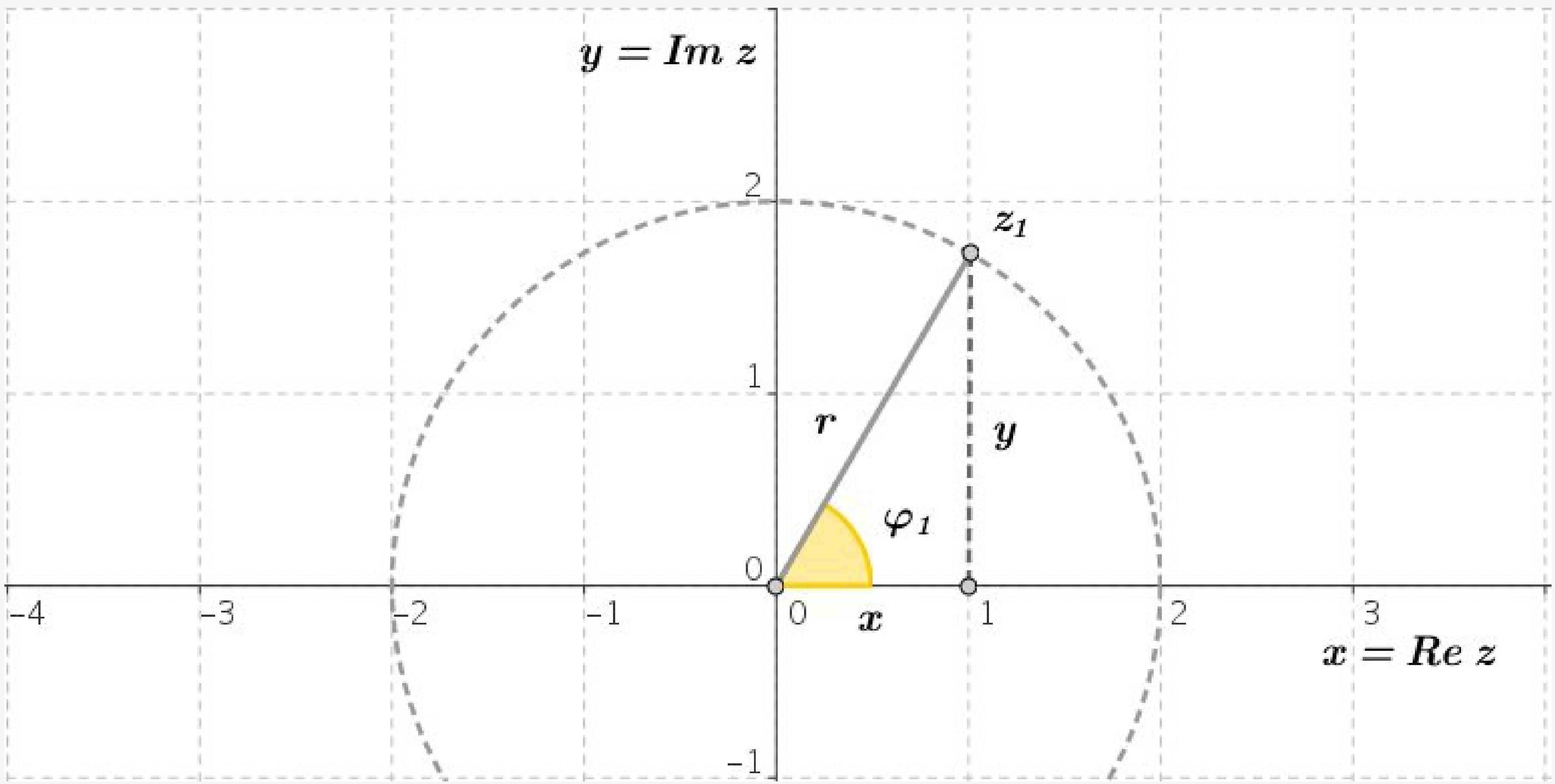

Fig. 4-1: The complex number $1 + \sqrt{3}\, i$ in the Gauss plane

We now transform a complex number given in Cartesian form into polar form. That is, we have to determine the absolute value and the angle of the number.

$$x\,,\quad y \quad \rightarrow \quad r\,,\quad \varphi_1 : \qquad z = x + i\,y \quad \rightarrow \quad z = r\,e^{i\,\varphi_1}$$

$$z_1 = 1 + \sqrt{3}\,i$$

$$z_1 = 1 + \sqrt{3}\, i\,, \qquad x_1 = 1, \qquad y_1 = \sqrt{3}$$

$$r = |\, z_1 \,| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1 + 3} = 2$$

$$\cos\ \varphi_1 = \frac{|\, x_1 \,|}{r} = \frac{1}{2}\,, \qquad \varphi_1 = 60^\circ = \frac{\pi}{3}$$

$$z_1 = 1 + \sqrt{3}\, i = 2\ e^{i\,\frac{\pi}{3}} = 2\left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)\right)$$

The conversion into polar coordinates is straight forward, if the complex number is in the first quadrant, but some more attention is needed, if the numbers are in other quadrants, like e.g.

$$z_2 = -1 + \sqrt{3}\, i\,, \qquad z_3 = -1 - \sqrt{3}\, i\,, \qquad z_4 = 1 - \sqrt{3}\, i$$

These numbers are geometrically presented below. The absolute values agree, but the angles differ as shown in figure 4-2.

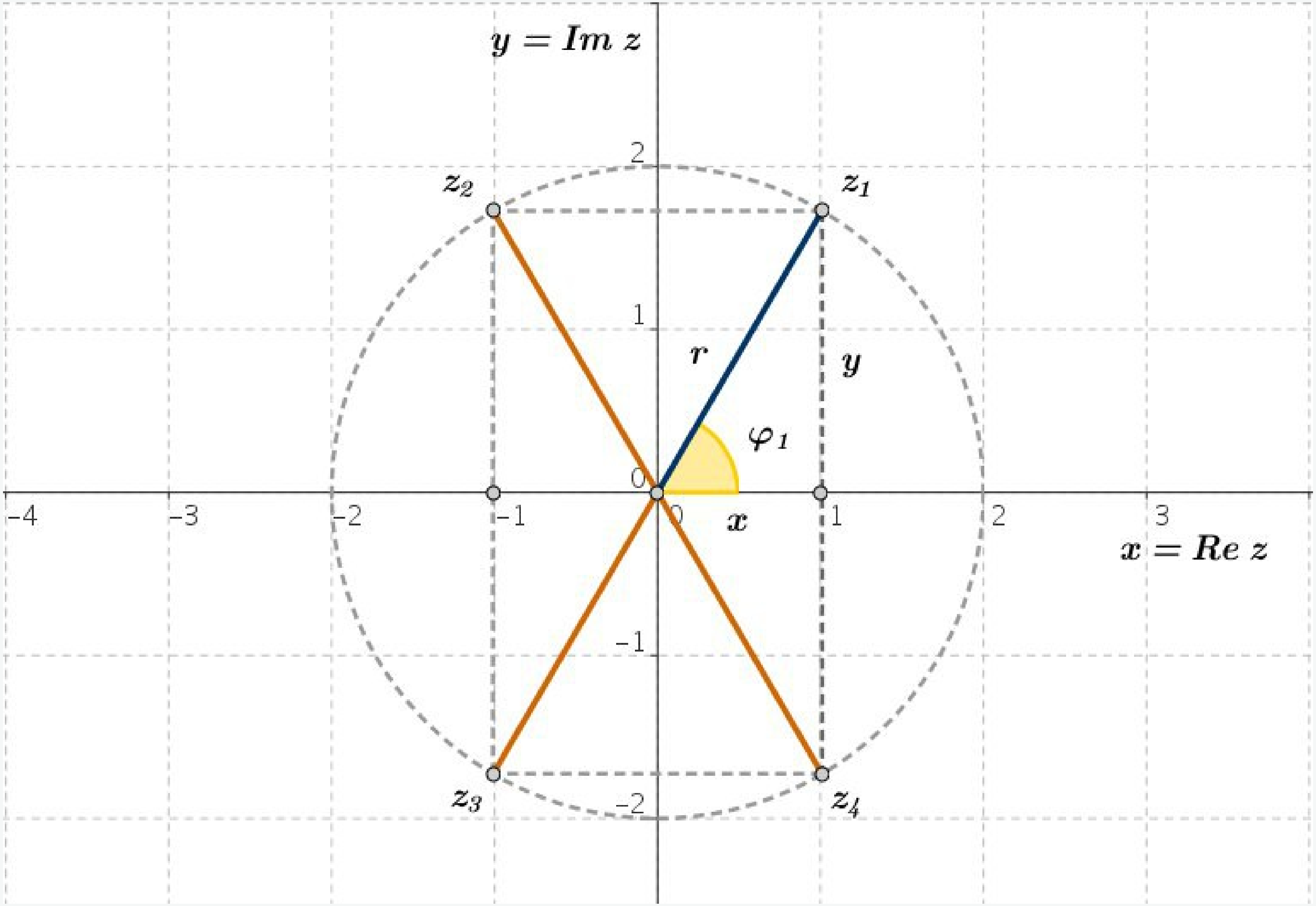

Fig. 4-2: Four complex numbers in different quadrants with same absolute values.

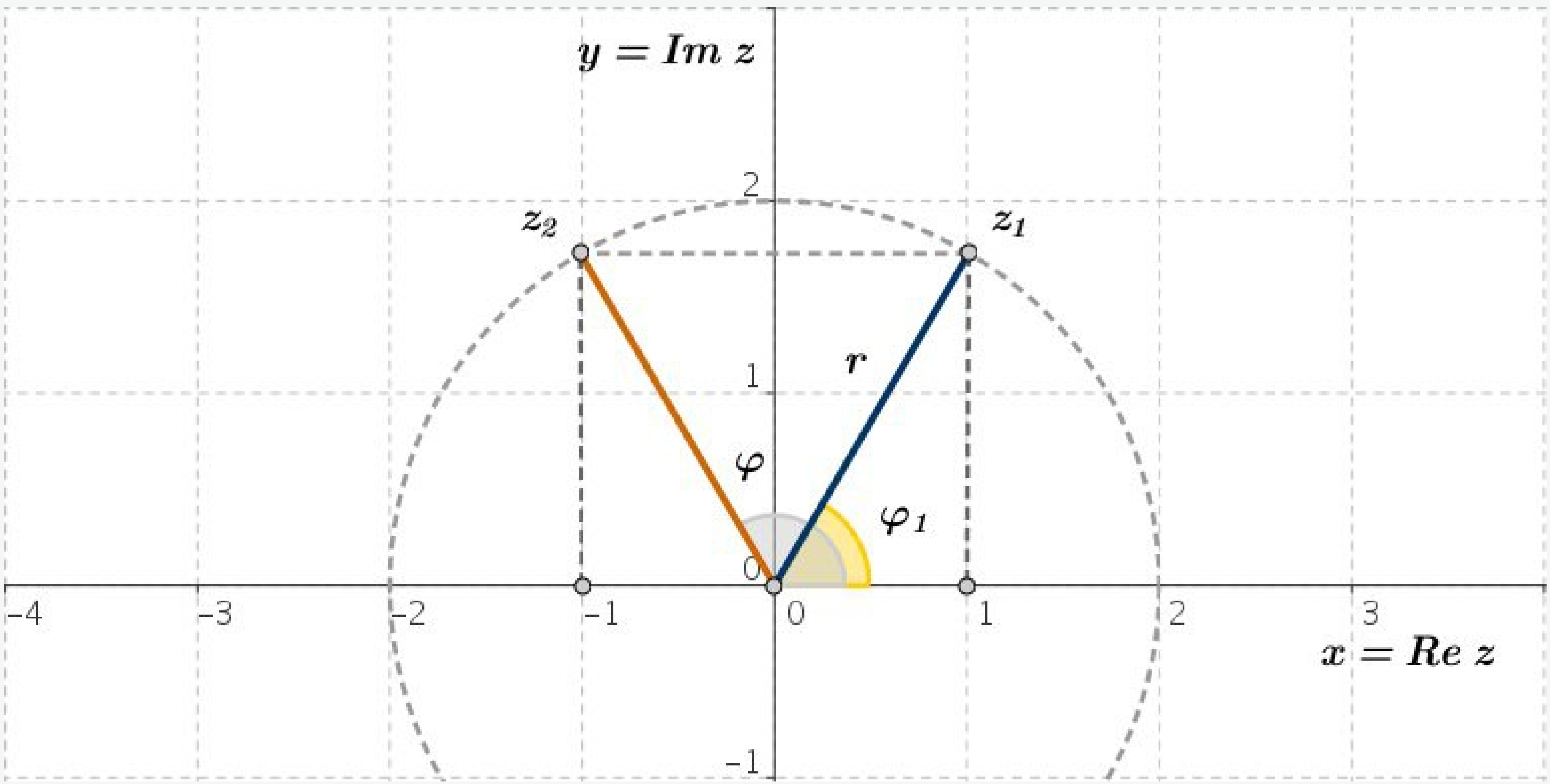

Fig. 4-3: Determination of the polar angle of a complex number

$$z_1 = 1 + \sqrt{3}\, i\,, \qquad z_2 = -1 + \sqrt{3}\, i\,, \qquad |\, z_1 \,| = |\, z_2 \,| = r = 2$$

$$\varphi = \pi - \varphi_1$$

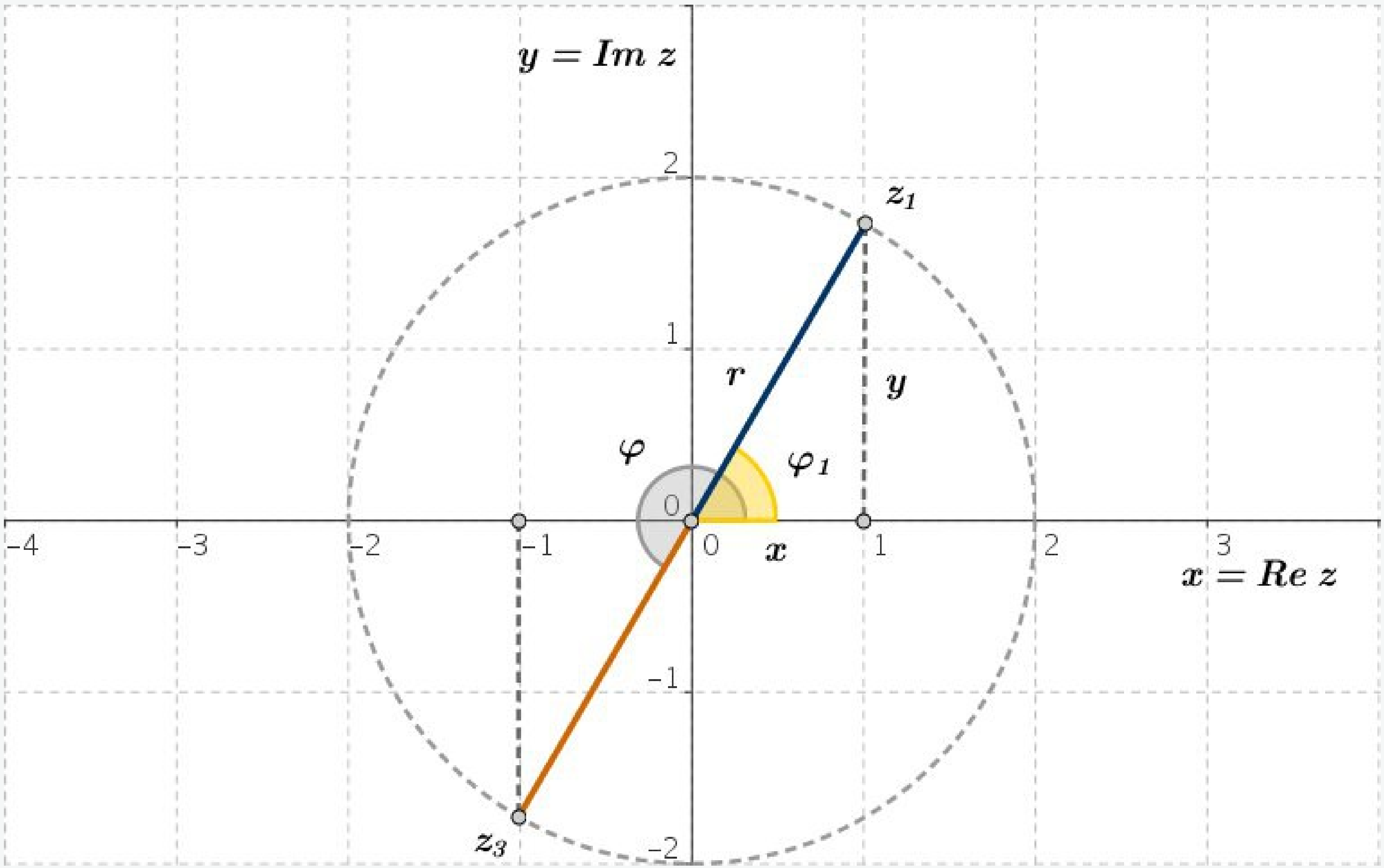

Fig. 4-4: Determination of the polar angle of a complex number

$$z_1 = 1 + \sqrt{3}\, i\,, \qquad z_3 = -1 - \sqrt{3}\, i\,, \qquad |\, z_1 \,| = |\, z_3 \,| = r = 2$$

$$\varphi = \pi + \varphi_1$$

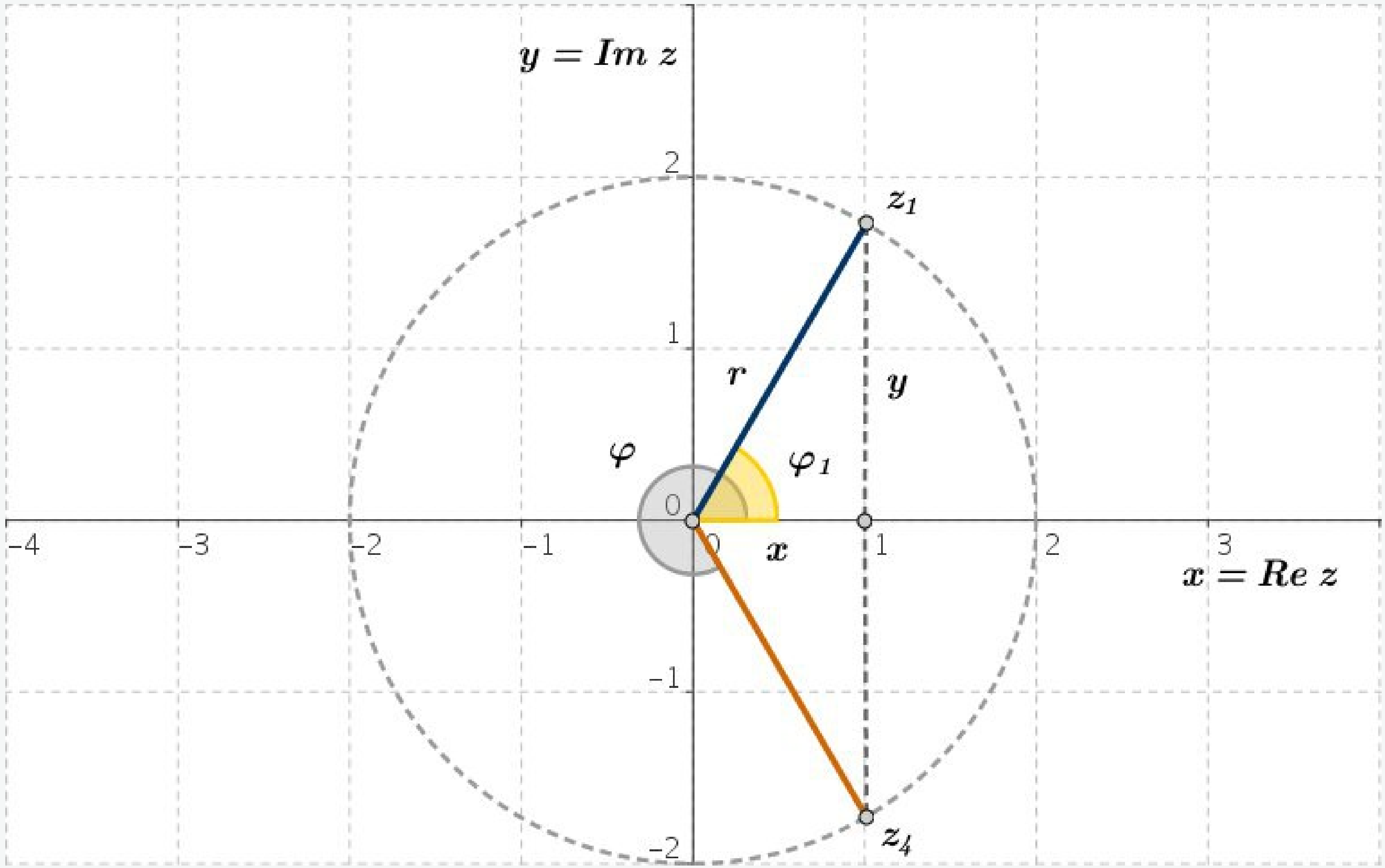

Fig. 4-5: Determination of the polar angle of a complex number

$$z_1 = 1 + \sqrt{3}\, i\,, \qquad z_4 = 1 - \sqrt{3}\, i\,, \qquad |\, z_1 \,| = |\, z_4 \,| = r = 2$$

$$\varphi = 2\pi - \varphi_1$$

$$z = x + i\,y, \qquad z = r\,(\cos\varphi + i\,\sin\varphi) = r\,e^{i\varphi}$$

Conversion in three steps:

1. Determine the absolute value $r = |z| = \sqrt{x^2 + y^2}$

2. Determine an auxiliary angle from one of the relations

$$\cos\varphi_1 = \frac{|x|}{r}, \qquad \sin\varphi_1 = \frac{|y|}{r}, \qquad \varphi_1 \in [0, \frac{\pi}{2})$$

φ_1 – auxiliary angle

3. Get the principle value φ from the auxiliary angle taking into account the signs of x and y:

IQ :	$x > 0,$	$y > 0,$	$\varphi = \varphi_1$
IIQ :	$x < 0,$	$y > 0,$	$\varphi = \pi - \varphi_1$
IIIQ :	$x < 0,$	$y < 0,$	$\varphi = \pi + \varphi_1$
IVQ :	$x > 0,$	$y < 0,$	$\varphi = 2\pi - \varphi_1$

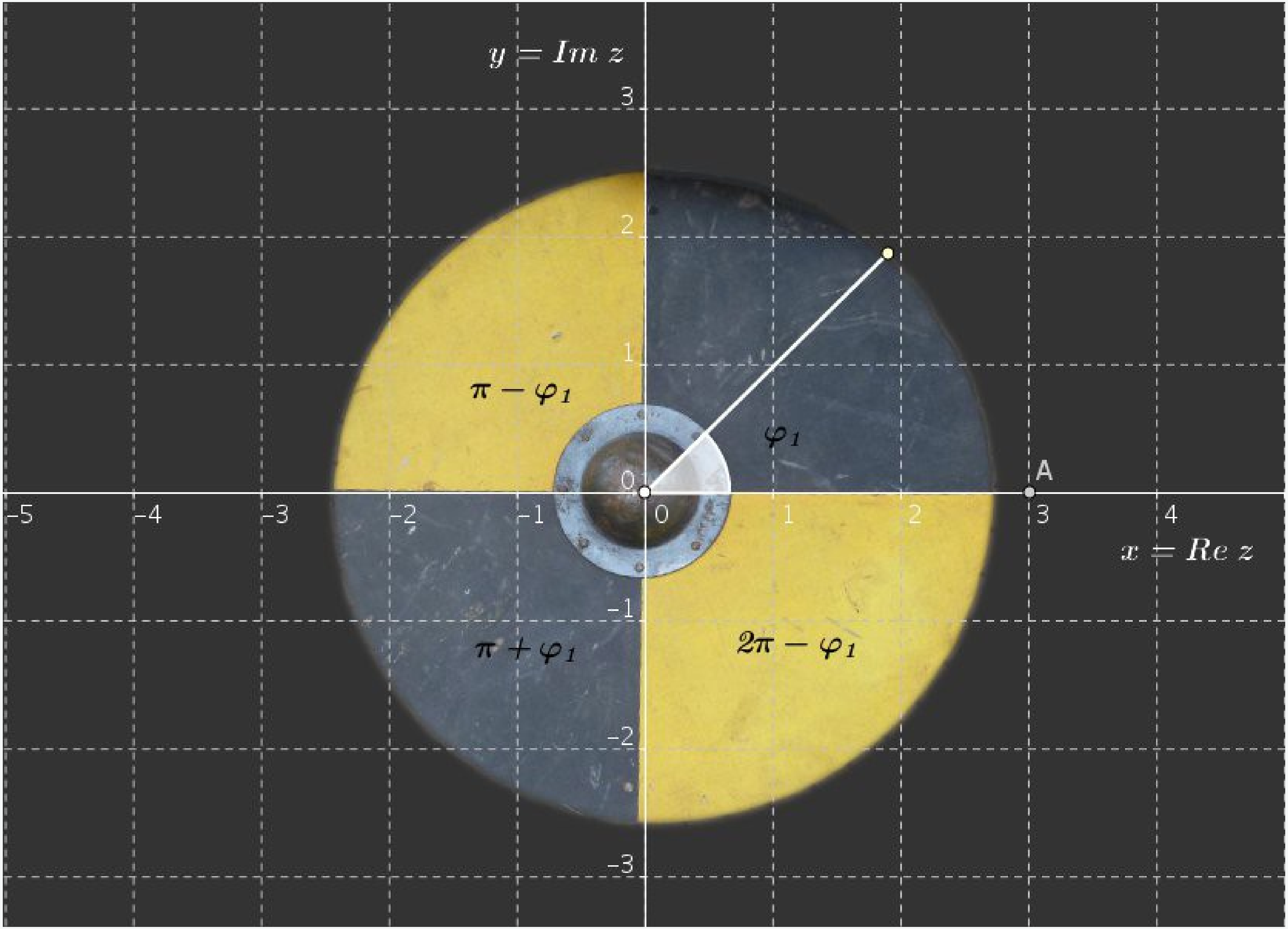

Fig. 4-6: Determination of the polar angle of a complex number