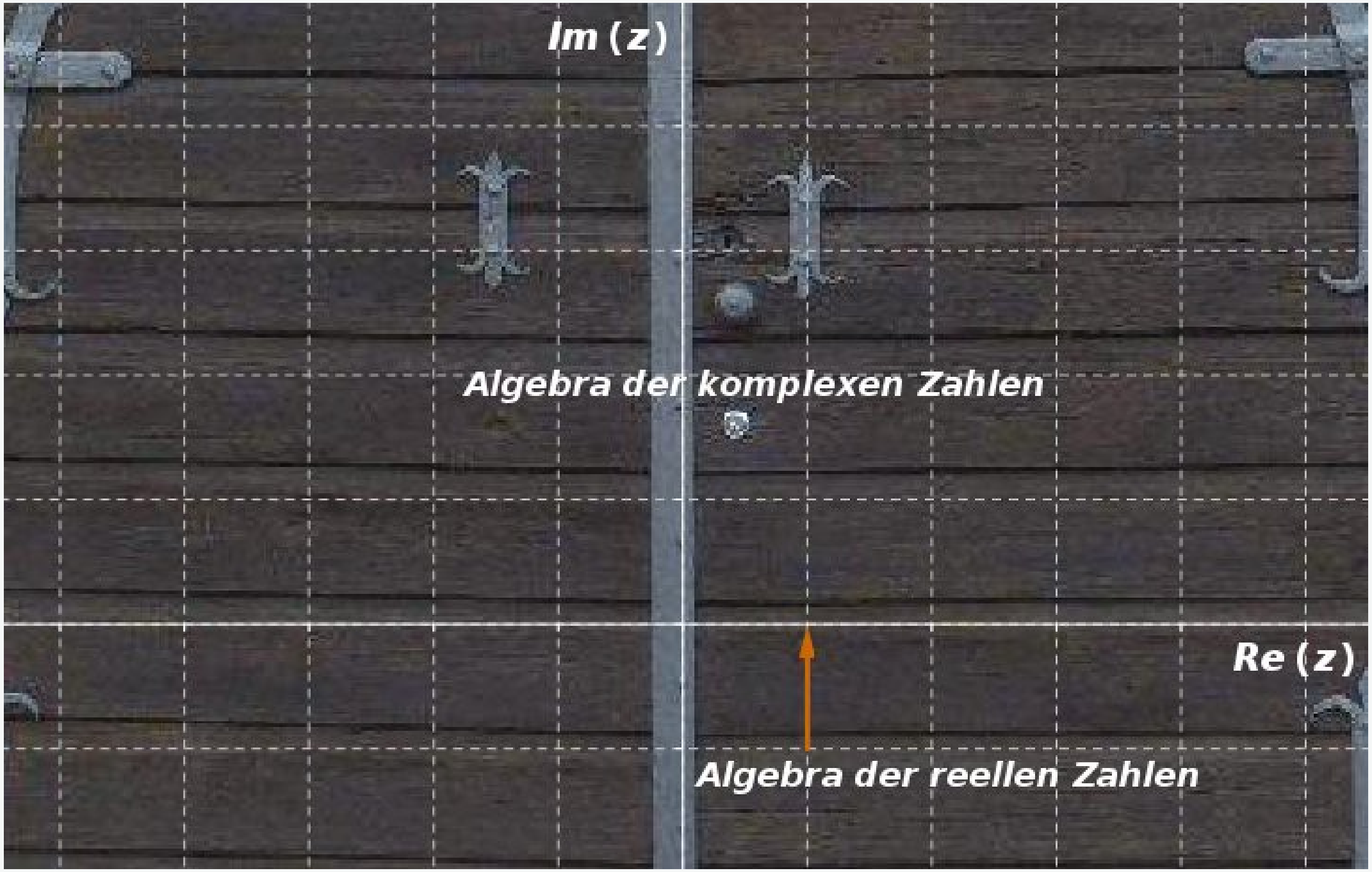
Complex Numbers: Addition, Subtraction



The set of real numbers is a subset of the set of complex numbers

$$\mathbb{R} \subset \mathbb{C}$$

Therefore the arithmetic operations should be defined such, that the rules for complex numbers agree in the real domain with the known rules for real numbers.

Four operations are known on the set of complex numbers:

- addition
- subtraction
- multiplication
- division

The same rules are valid for real and complex numbers.

Exception: order relation

There is no such relation for complex numbers like the following:

$$z_1 < z_2$$

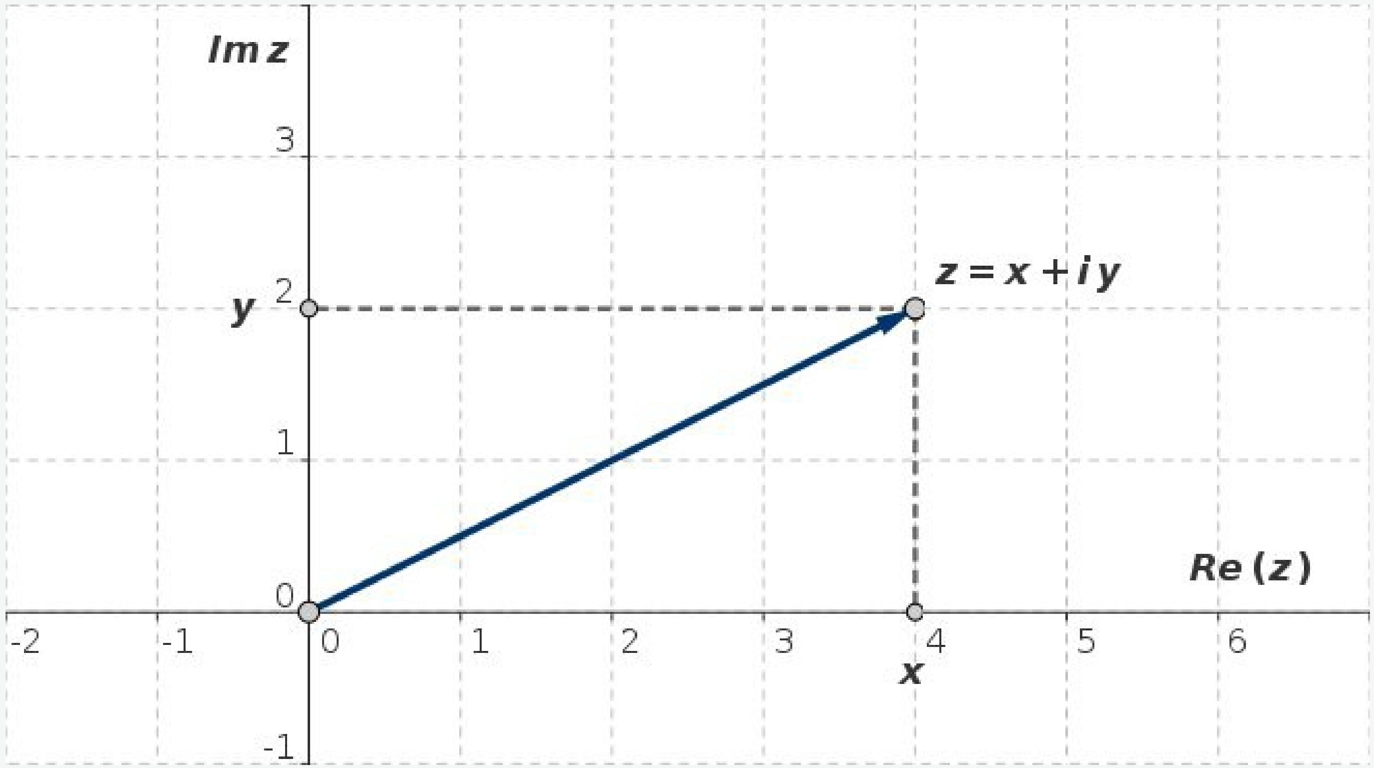


Fig. 1: Vectorial representation of a complex number on the Gauss plane

The vectorial representation of complex numbers is very important to describe addition and subtraction.

$$z = x \cdot 1 + y \cdot i \,, \qquad 1 = (1,\ 0) \qquad i = (0,\ 1)$$

$$|\, z \,| = \sqrt{x^2 + y^2}$$

The sum (difference) of complex numbers is obtained by addition (subtraction) of their real components, like the rule for *2*-dimensional vectors:

$$z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$$

$$z_1 - z_2 = (x_1 - x_2) + i\,(y_1 - y_2)$$

Example:

$z_1 = 2 + 3i\,, \qquad z_2 = 3 - i$

$z_1 + z_2 = (2 + 3) + i\,(3 - 1) = 5 + 2i$

$z_1 - z_2 = (2 - 3) + i\,(3 + 1) = -1 + 4i$

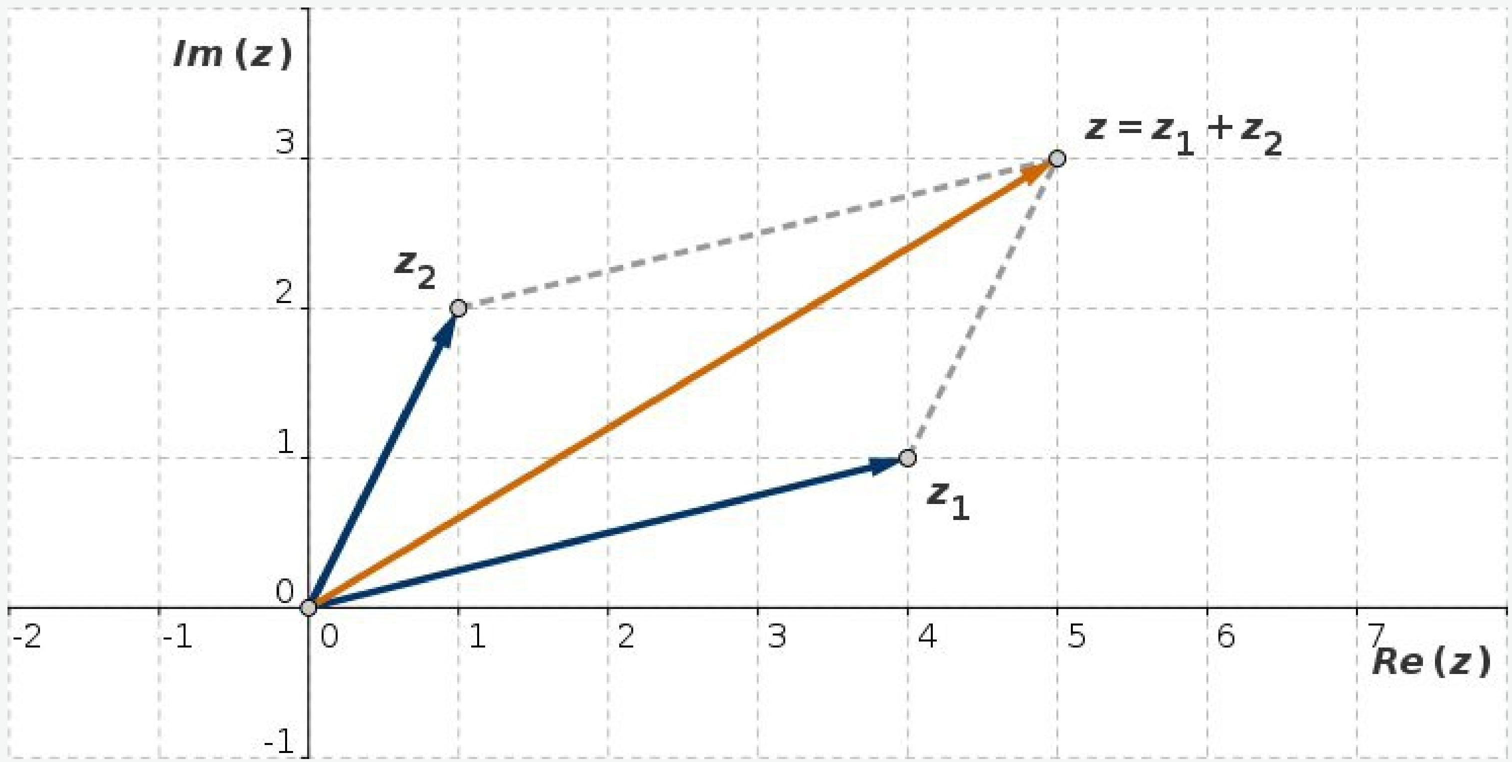


Fig. 2: Geometrical representation of the addition of two complex numbers (parallelogram rule)

$$z_1 = 4 + i\,, \qquad z_2 = 1 + 2\,i$$

$$z_1 + z_2 = (4 + 1) + i\,(1 + 2) = 5 + 3\,i$$

Addition and subtraction are easy operations <u>only</u> in Cartesian form.

Exercise 1:

a) $(1 + 5i) + (2 - 3i)$

b) $(12 - 2i) + (7 - i)$

c) $(21 + 3i) + (2 - i) + (-11 + 3i)$

d) $(31 - 1.5i) - (21 - 3.5i)$

e) $(12.4 + 1.7i) - (9.53 + 4.89i)$

f) $(19 + 2.7i) + 3(1 - i) - (30 + 8i)$

g) $(a + bi) - (b + 2ci) + (-3a + 2bi)$

h) $\left(5\alpha - \frac{2\beta}{3} i\right) - 6\left(\frac{\beta}{2} + \frac{\gamma}{3} i\right) - \left(3\alpha + \frac{\beta}{3} i\right)$

a) $(1+5i)+(2-3i)=3+2i$

b) $(12-2i)+(7-i)=19-3i$

c) $(21+3i)+(2-i)+(-11+3i)=12+5i$

d) $(31-1.5i)-(21-3.5i)=10+2i$

e) $(12.4+1.7i)-(9.53+4.89i)=2.87-3.19i$

f) $(19+2.7i)+3(1-i)-(30+8i)=-8-8.3i$

g) $(a+bi)-(b+2ci)+(-3a+2bi)=-2a-b+(3b-2c)i$

h) $$\left(5\alpha-\frac{2\beta}{3}i\right)-6\left(\frac{\beta}{2}+\frac{\gamma}{3}i\right)-\left(3\alpha+\frac{\beta}{3}i\right)=$$

$$=2\alpha-3\beta-(\beta+2\gamma)i$$

Exercise 2:

Determine the following expressions for the complex number $z = x + i\,y$:

$$\frac{1}{2}\left(z + z^{*}\right), \qquad \frac{1}{2\,i}\left(z - z^{*}\right)$$

Exercise 3:

Determine the following expressions

$$z_1 + z_2, \qquad z_1 - z_2, \qquad z_1 + 3\,z_2 + 2\,z_3^{*}$$

for the complex numbers given below:

a) $z_1 = 2 + 3\,i, \quad z_2 = 4 - 2\,i, \quad z_3 = 1 + i$

b) $z_1 = 5 + 7\,i, \quad z_2 = -3 - i, \quad z_3 = 2.5 - 0.5\,i$

c) $z_1 = \frac{1}{3} + \frac{2}{3}\,i, \quad z_2 = \frac{1}{2} - \frac{3}{2}\,i, \quad z_3 = \frac{3}{4} - \frac{1}{4}\,i$

Solution 2: $\frac{1}{2}\left(z + z^*\right) = x = \mathrm{Re}(z)$, $\frac{1}{2i}\left(z - z^*\right) = y = \mathrm{Im}(z)$

Solution 3:

a) $z_1 = 2 + 3i$, $z_2 = 4 - 2i$, $z_3 = 1 + i$

$z_1 + z_2 = 6 + i$, $z_1 - z_2 = -2 + 5i$, $z_1 + 3z_2 + 2z_3^* = 16 - 5i$

b) $z_1 = 5 + 7i$, $z_2 = -3 - i$, $z_3 = 2.5 - 0.5i$

$z_1 + z_2 = 2 + 6i$, $z_1 - z_2 = 8(1 + i)$, $z_1 + 3z_2 + 2z_3^* = 1 + 5i$

c) $z_1 = \frac{1}{3} + \frac{2}{3}i$, $z_2 = \frac{1}{2} - \frac{3}{2}i$, $z_3 = \frac{3}{4} - \frac{1}{4}i$

$z_1 + z_2 = \frac{5}{6}(1 - i)$, $z_1 - z_2 = \frac{1}{6}(-1 + 13i)$

$z_1 + 3z_2 + 2z_3^* = \frac{10}{3}(1 - i)$

Addition, Subtraction: Exercise 4

Determine the following expressions

$$\frac{z_1}{2} - \frac{z_2}{3}, \quad z_1 - z_2 - (z_2^* - z_1^*), \quad \frac{z_1}{2} - z_3^* + 2(z_2 - z_2^*)$$

$$\frac{i}{2}(z_1 - z_1^*) - i(z_3 - z_3^*) + \frac{i^3}{2}(z_2 - z_2^*)$$

$$\frac{i}{2}(z_1 + z_1^*) + \frac{i^3}{2}(z_3 + z_3^*) + \frac{i^5}{3}(z_2 + z_2^*)$$

$$i(z_1 + z_1^*) + i^2(z_2 - z_2^*) + i^6(z_3 - z_3^*)$$

for the complex numbers given below:

a) $z_1 = 1 + 2i, \quad z_2 = 3 - i, \quad z_3 = 2 + \frac{2}{3}i$

b) $z_1 = 1 + i, \quad z_2 = 3 + i, \quad z_3 = 4 - i$

c) $z_1 = \sqrt{2} + \sqrt{3}\,i, \quad z_2 = \sqrt{3} + i\sqrt{2}, \quad z_3 = \sqrt{6} - i\sqrt{3}$

$$\frac{z_1}{2} - \frac{z_2}{3} = -\frac{1}{2} + \frac{4}{3}\,i$$

$$z_1 - z_2 - (z_2^* - z_1^*) = -4$$

$$\frac{z_1}{2} - z_3^* + 2\,(z_2 - z_2^*) = -\frac{3}{2} - \frac{7}{3}\,i$$

$$\frac{i}{2}\,(z_1 - z_1^*) - i\,(z_3 - z_3^*) + \frac{i^3}{2}\,(z_2 - z_2^*) = -\frac{5}{3}$$

$$\frac{i}{2}\,(z_1 + z_1^*) + \frac{i^3}{2}\,(z_3 + z_3^*) + \frac{i^5}{3}\,(z_2 + z_2^*) = i$$

$$i\,(z_1 + z_1^*) + i^2\,(z_2 - z_2^*) + i^6\,(z_3 - z_3^*) = \frac{8}{3}\,i$$

$$\frac{z_1}{2} - \frac{z_2}{3} = -\frac{1}{2} + \frac{i}{6}$$

$$z_1 - z_2 - (z_2^* - z_1^*) = -4$$

$$\frac{z_1}{2} - z_3^* + 2(z_2 - z_2^*) = -\frac{7}{2} + \frac{7}{2}\, i$$

$$\frac{i}{2}(z_1 - z_1^*) - i\,(z_3 - z_3^*) + \frac{i^3}{2}(z_2 - z_2^*) = -2$$

$$\frac{i}{2}(z_1 + z_1^*) + \frac{i^3}{2}(z_3 + z_3^*) + \frac{i^5}{3}(z_2 + z_2^*) = -i$$

$$i\,(z_1 + z_1^*) + i^2\,(z_2 - z_2^*) + i^6\,(z_3 - z_3^*) = 2\,i$$

$$\frac{z_1}{2} - \frac{z_2}{3} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + i\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{3}\right)$$

$$z_1 - z_2 - (z_2^* - z_1^*) = 2\,(\sqrt{2} - \sqrt{3})$$

$$\frac{z_1}{2} - z_3^* + 2\,(z_2 - z_2^*) = \frac{1}{\sqrt{2}} - \sqrt{6} + i\left(4\sqrt{2} - \frac{\sqrt{3}}{2}\right)$$

$$\frac{i}{2}\,(z_1 - z_1^*) - i\,(z_3 - z_3^*) + \frac{i^3}{2}\,(z_2 - z_2^*) = -3\sqrt{3} + \sqrt{2}$$

$$\frac{i}{2}\,(z_1 + z_1^*) + \frac{i^3}{2}\,(z_3 + z_3^*) + \frac{i^5}{3}\,(z_2 + z_2^*) = i\left(\sqrt{2} - \sqrt{6} + \frac{2}{\sqrt{3}}\right)$$

$$i\,(z_1 + z_1^*) + i^2\,(z_2 - z_2^*) + i^6\,(z_3 - z_3^*) = 2\sqrt{3}\; i$$

Determine the following expressions

$$2\,z_1 + 3\,z_2\,, \qquad 4\,z_1 - 2\,z_2^*$$

for the complex numbers given below:

a) $z_1 = 2 \cdot e^{i\,30^\circ}, \qquad z_2 = 4 \cdot e^{-i\,60^\circ}$

b) $z_1 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$z_2 = 8\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

c) $z_1 = 5 \cdot e^{i\,90^\circ}, \qquad z_2 = 4 \cdot e^{-i\,90^\circ}$

d) $z_1 = 6 \cdot e^{i\,60^\circ}, \qquad z_2 = -4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

e) $z_1 = -2 \cdot e^{i\,\pi}, \qquad z_2 = 3 \cdot e^{0 \cdot i}$

First, the numbers are transformed to algebraic form:

$$a)\quad z_1 = 2 \cdot e^{i\,30^\circ} = \sqrt{3} + i\,, \qquad z_2 = 4 \cdot e^{-i\,60^\circ} = 2 - 2\sqrt{3}\; i$$

$$2\,z_1 + 3\,z_2 = 2\sqrt{3} + 6 + (2 - 6\sqrt{3})\,i = 9.464 - 8.392\; i$$

$$4\,z_1 - 2\,z_2^* = 4\,(\sqrt{3} - 1) + 4\,(1 - \sqrt{3})\,i = 2.928\,(1 - i)$$

$$b)\quad z_1 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\;(1 + i)$$

$$z_2 = 8\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = 4\sqrt{2}\;(-1 + i)$$

$$2\,z_1 + 3\,z_2 = -10\sqrt{2} + 14\sqrt{2}\,i = 2\sqrt{2}\;(-5 + 7\,i) =$$

$$= -14.142 + 19.799\; i$$

$$4\,z_1 - 2\,z_2^* = 12\sqrt{2}\;(1 + i) = 16.971\;(1 + i)$$

Addition, Subtraction: Solutions 5c,d

c) $z_1 = 5 \cdot e^{i\,90^\circ} = 5\,i\,, \qquad z_2 = 4 \cdot e^{-i\,90^\circ} = -4\,i$

$2\,z_1 + 3\,z_2 = -2\,i\,, \qquad 4\,z_1 - 2\,z_2^* = 12\,i$

d) $z_1 = 6 \cdot e^{i\,60^\circ} = 3\,(1 + \sqrt{3}\;i)$

$$z_2 = -4\left(\cos\frac{2\,\pi}{3} + i\sin\frac{2\,\pi}{3}\right) = 2\,(1 - \sqrt{3}\,i)$$

$2\,z_1 + 3\,z_2 = 12$

$4\,z_1 - 2\,z_2^* = 8\,(1 + \sqrt{3}\,i) = 8 + 13.856\,i$

e) $z_1 = -2 \cdot e^{i\,\pi} = 2\,, \qquad z_2 = 3 \cdot e^{0 \cdot i} = 3$

$2\,z_1 + 3\,z_2 = 13, \qquad 4\,z_1 - 2\,z_2^* = 2$

Exercise 6:

Prove the following identity

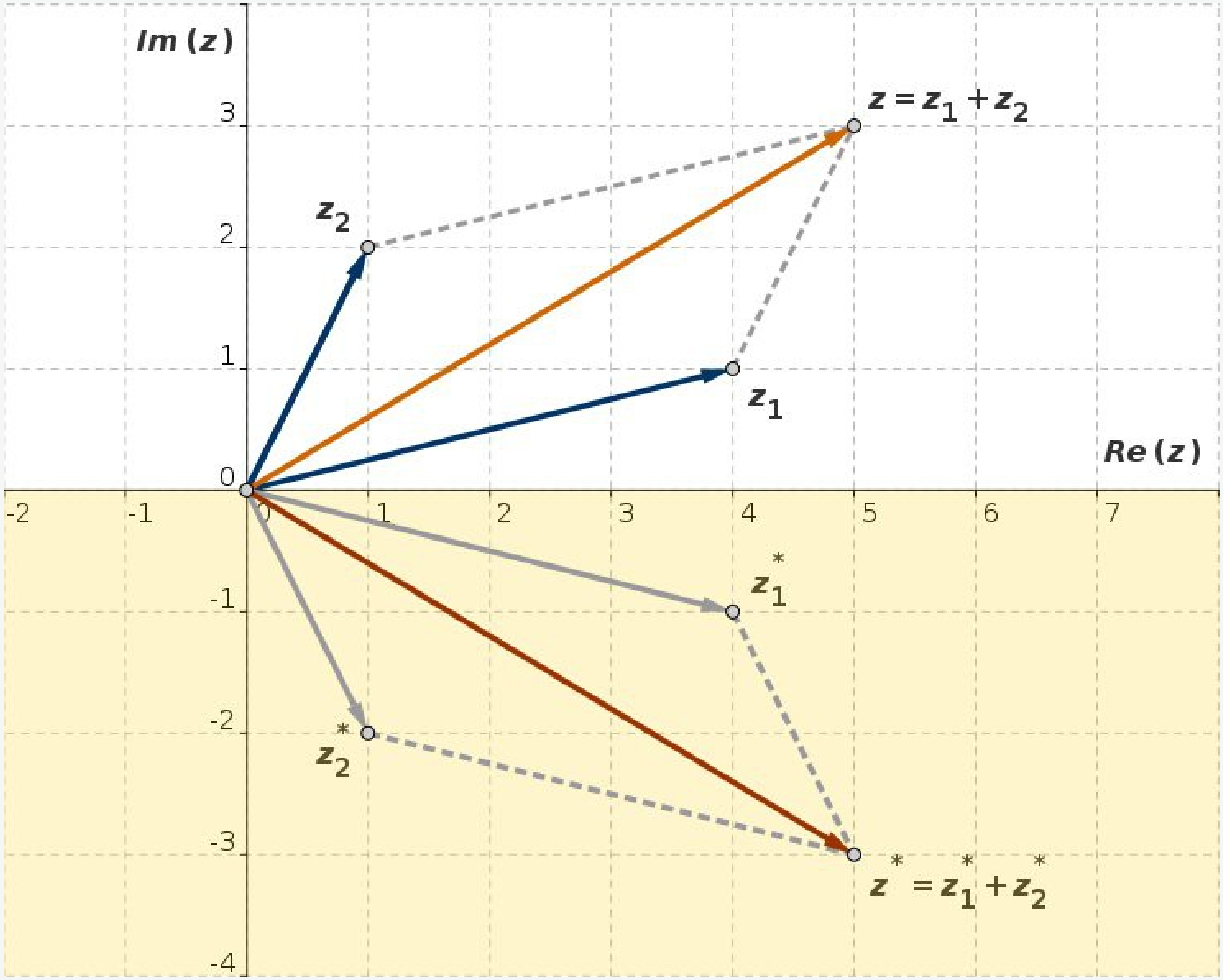
$$(z_1 + z_2)^* = z_1^* + z_2^*$$

Give thje corresponding geometrical representation on the complex plane:

Exercise 7:

Solve the following complex equation:

$$(4 + 2i)\ x + (5 - 3i)\ y = 13 + i$$



$$(z_1 + z_2)^* = (x_1 + x_2 + i\,(y_1 + y_2))^* = x_1 + x_2 - i\,(y_1 + y_2) =$$
$$= (x_1 - i\,y_1) + (x_2 - i\,y_2) \;=\; z_1^* + z_2^*$$

To solve such a complex equation means to solve for the real x and y.

$$(4+2i)\,x+(5-3i)\,y=13+i \quad \Leftrightarrow$$

$$4x+5y+i\,(2x-3y)=13+i$$

$$z_1=z_2: \quad x_1=x_2 \;\wedge\; y_1=y_2 \quad \Rightarrow$$

$$\begin{cases} 4x+5y=13 \\ 2x-3y=1 \end{cases} \qquad x=2, \quad y=1$$