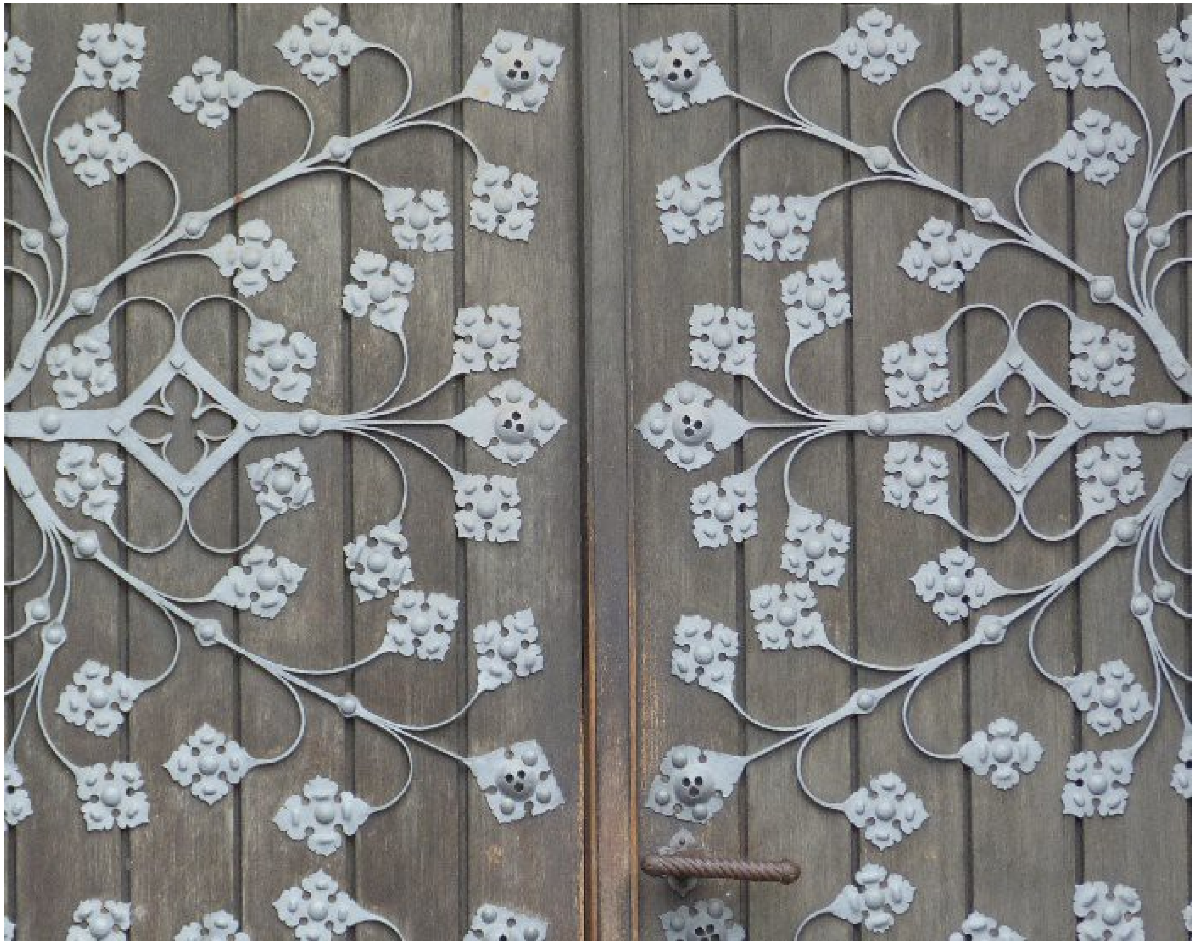

Complex Numbers: Multiplication

Multiplication

$$z_1 \cdot z_2 = (x_1 + i\, y_1) \cdot (x_2 + i\, y_2) =$$

$$= x_1 x_2 + i\, (x_1 y_2 + x_2 y_1) + i^2\, y_1 y_2 =$$

$$= (x_1 x_2 - y_1 y_2) + i\, (x_1 y_2 + x_2 y_1)$$

Definition:

The product of two complex numbers

$z_1 = x_1 + i\, y_1$, $\qquad z_2 = x_2 + i\, y_2$

is given by

$$\boxed{z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + i\, (x_1 y_2 + x_2 y_1)}$$

Multiplication: Exercise 1

Exercise 1:

Determine the following expressions:

$a)$ $(1+5i)\cdot(1-2i)$

$b)$ $(-3+6i)\cdot\left(1-\dfrac{i}{3}\right)$

$c)$ $(4+12i)\cdot\left(-3+\dfrac{i}{2}\right)$

$d)$ $(2+4i)\cdot\left(\dfrac{1}{2}-i\right)\cdot\left(\dfrac{3}{2}+3i\right)$

$e)$ $(3+2i)\cdot(-1+3i)\cdot(4-i)$

$f)$ $(1-2i)\cdot(1+3i)\cdot(-2+i)\cdot(-3+2i)$

$g)$ $(1-2i^2+i)\cdot(1+3i-i^3)$

$a)\quad (1+5i)\cdot(1-2i) = 11+3i$

$b)\quad (-3+6i)\cdot\left(1-\dfrac{i}{3}\right) = -1+7i$

$c)\quad (4+12i)\cdot\left(-3+\dfrac{i}{2}\right) = -18-34i$

$d)\quad (2+4i)\cdot\left(\dfrac{1}{2}-i\right)\cdot\left(\dfrac{3}{2}+3i\right) = \dfrac{15}{2}+15i$

$e)\quad (3+2i)\cdot(-1+3i)\cdot(4-i) = -29+37i$

$f)\quad (1-2i)\cdot(1+3i)\cdot(-2+i)\cdot(-3+2i) = 35-45i$

$g)\quad (1-2i^2+i)\cdot(1+3i-i^3) = -1+13i$

Multiplication: Exercises 2,3

Exercise 2:

Determine using the following numbers

a) $z_1 = 1 + 2i\,, \quad z_2 = 3 - 2i\,, \quad z_3 = 4 - 3i$

b) $z_1 = 5 + i\,, \quad z_2 = -2 + 4i\,, \quad z_3 = 2i$

the terms given below

$$z_1 \cdot z_2\,, \quad 3\,z_1 \cdot z_2^*\,, \quad 2\,z_1^* \cdot z_2\,, \quad z_1^2\,, \quad z_1^* \cdot z_2 \cdot z_3^*$$

$$z_2^2\,, \quad z_1 \cdot z_3^* + z_1^* \cdot z_3\,, \quad (z_2 \cdot z_3)^*\,, \quad z_1 \cdot z_2^* + z_3^*$$

Exercise 3: Determine the product $z \cdot z^*$

Solution 2a:

$$z_1 \cdot z_2 = 7 + 4\,i\,, \qquad 3\;z_1 \cdot z_2^* = -3 + 24\,i\,, \qquad 2\;z_1^* \cdot z_2 = -2 - 16\,i$$

$$z_1^2 = -3 + 4\,i\,, \qquad z_1^* \cdot z_2 \cdot z_3^* = 20 - 35\,i\,, \qquad z_2^2 = 5 - 12\,i$$

$$z_1 \cdot z_3^* + z_1^* \cdot z_3 = -4, \qquad (z_2 \cdot z_3)^* = 6 + 17\,i\,, \qquad z_1 \cdot z_2^* + z_3^* = 3 + 11\,i$$

Solution 2b:

$$z_1 \cdot z_2 = -14 + 18\,i\,, \qquad 3\;z_1 \cdot z_2^* = -18 - 66\,i\,, \qquad 2\;z_1^* \cdot z_2 = -12 + 44\,i$$

$$z_1^2 = 24 + 10\,i\,, \qquad z_1^* \cdot z_2 \cdot z_3^* = 44 + 12\,i\,, \qquad z_2^2 = -12 - 16\,i$$

$$z_1 \cdot z_3^* + z_1^* \cdot z_3 = 4, \qquad (z_2 \cdot z_3)^* = -8 + 4\,i\,, \qquad z_1 \cdot z_2^* + z_3^* = -6 - 24\,i$$

Solution 3:

$$z \cdot z^* = (x + i\,y) \cdot (x - i\,y) = x^2 + y^2 = |\,z\,|^2 \quad \Rightarrow$$

$$\boxed{\sqrt{z \cdot z^*} = |\,z\,| = \sqrt{x^2 + y^2}}$$

Determine the following expressions:

a) $(1+2i)^2$, $(2+4i)^2$

b) $(1-3i)^2$, $(3-9i)^2$

c) $(2+i)^3$, $(4+2i)^3$

d) $(1-2i)^2\cdot(1+2i)^2$, $(1-2i)^2\cdot(1+2i)^3$

e) $(1+3i)^2\cdot(1-3i)^2$, $(1+3i)^2\cdot(1-3i)^3$

f) $(1-i)^2\cdot(1+3i)^2$, $(2-2i)^2\cdot\left(\frac{1}{2}+\frac{3i}{2}\right)^2$

g) $(1+5i)^2\cdot(1+i)^3$, $(1+5i)^3\cdot(1+i)^3$

h) $(8+4i)^2\cdot\left(1-\frac{i}{2}\right)^2$, $(8+4i)^2\cdot\left(1-\frac{i}{2}\right)^3$

a) $(1+2i)^2 = 1^2 + 2\cdot 2i + (2i)^2 = 1 + 4i + 4i^2 = -3 + 4i$

$(2+4i)^2 = 2^2\,(1+2i)^2 = 4\,(-3+4i) = -12 + 16i$

b) $(1-3i)^2 = -8 - 6i = -2\,(4+3i)$

$(3-9i)^2 = -72 - 54i = -18\,(4+3i)$

c) $(2+i)^3 = 2 + 11i$

$(4+2i)^3 = 2^3\,(2+i)^3 = 8\,(2+11i) = 16 + 88i$

d) $(1-2i)^2 \cdot (1+2i)^2 = \big((1-2i)\cdot(1+2i)\big)^2 = 5^2 = 25$

$(1-2i)^2 \cdot (1+2i)^3 = (1-2i)^2 \cdot (1+2i)^2 \cdot (1+2i) =$

$= 25\,(1+2i) = 25 + 50i$

e) $(1+3i)^2 \cdot (1-3i)^2 = \big((1+3i)\cdot(1-3i)\big)^2 = 10^2 = 100$

$(1+3i)^2 \cdot (1-3i)^3 = (1+3i)^2 \cdot (1-3i)^2 \cdot (1-3i) =$

$= 100\,(1-3i) = 100 - 300i$

f) $(1 - i)^2 \cdot (1 + 3i)^2 = 12 + 16i = 4\,(3 + 4i)$

$$(2 - 2i)^2 \cdot \left(\frac{1}{2} + \frac{3i}{2}\right)^2 = 2^2\,(1 - i)^2 \cdot \frac{(1 + 3i)^2}{2^2} = (1 - i)^2$$

$$= (1 - i)^2 \cdot (1 + 3i)^2 = 4\,(3 + 4i)$$

g) $(1 + 5i)^2 \cdot (1 + i)^3 = 28 - 68i = 4\,(7 - 17i)$

$$(1 + 5i)^3 \cdot (1 + i)^3 = 4\,(7 - 17i) \cdot (1 + 5i) =$$

$$= 368 + 72i = 8\,(46 + 9i)$$

h) $(8 + 4i)^2 \cdot \left(1 - \frac{i}{2}\right)^2 = 4^2\,(2 + i)^2 \cdot \frac{(2 - i)^2}{2^2} =$

$$= 4\,(2 + i)^2 \cdot (2 - i)^2 = 100$$

$$(8 + 4i)^2 \cdot \left(1 - \frac{i}{2}\right)^3 = (8 + 4i)^2 \cdot \left(1 - \frac{i}{2}\right)^2 \left(1 - \frac{i}{2}\right) =$$

$$= 100\left(1 - \frac{i}{2}\right) = 100 - 50i$$

Exercise 5: Calculate the first *12* powers of i.

Exercise 6: Prove the following identity of complex numbers:

$$(z_1 \cdot z_2)^* = z_1^* \cdot z_2^*$$

Solution 5:

$$i^1 = i\,, \quad i^2 = -1, \quad i^3 = -i\,, \quad i^4 = 1, \quad i^5 = i\,, \quad i^6 = -1, \; \ldots$$

$$\boxed{i^{1+4n} = i\,, \quad i^{2+4n} = -1, \quad i^{3+4n} = -i\,, \quad i^{4+4n} = 1}$$

Solution 6:

$$z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + i\,(x_1 y_2 + x_2 y_1)$$

$$(z_1 \cdot z_2)^* = (x_1 x_2 - y_1 y_2) - i\,(x_1 y_2 + x_2 y_1)$$

$$z_1^* \cdot z_2^* = (x_1 - i\,y_1) \cdot (x_2 - i\,y_2) = (x_1 x_2 - y_1 y_2) - i\,(x_1 y_2 + x_2 y_1)$$

$$\Rightarrow \quad (z_1 \cdot z_2)^* = z_1^* \cdot z_2^*$$