Complex Numbers: Division

Definition:

The quotient of two complex numbers

$$z_1 = x_1 + i\, y_1, \qquad z_2 = x_2 + i\, y_2$$

is given by

$$\boxed{\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\, \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}}$$

$$\frac{z_1}{z_2} = \frac{z_1 \cdot z_2^*}{z_2 \cdot z_2^*}$$

$$\sqrt{z_2 \cdot z_2^*} = |\, z_2 \,| = \sqrt{x_2^2 + y_2^2}$$

Following fractions should get real denominators:

$$a)\ \frac{1}{1-i}, \qquad b)\ \frac{1}{2+i}, \qquad c)\ \frac{1+2i}{2+3i}$$

$$a)\ \frac{1}{1-i} = \frac{1\cdot(1-i)^*}{(1-i)\cdot(1-i)^*} = \frac{1\cdot(1+i)}{(1-i)\cdot(1+i)} = \frac{1}{2} + \frac{i}{2}$$

$$b)\ \frac{1}{2+i} = \frac{1\cdot(2+i)^*}{(2+i)\cdot(2+i)^*} = \frac{1\cdot(2-i)}{(2+i)\cdot(2-i)} = \frac{2}{5} - \frac{i}{5}$$

$$c)\ \frac{1+2i}{2+3i} = \frac{(1+2i)(2-3i)}{(2+3i)(2-3i)} = \frac{8}{13} + \frac{i}{13}$$

Exercise 1:

Turn the denominators of following fractions into real ones:

$$a)\ \frac{1}{1-2i},\quad \frac{1}{1+2i},\qquad b)\ \frac{1}{2-i},\quad \frac{1}{2+i}$$

$$c)\ \frac{2+i}{2-i},\quad \frac{2-i}{2+i}\qquad d)\ \frac{3+2i}{3-i},\quad \frac{3-i}{3+2i}$$

$$e)\ \frac{1}{(1+i)(2-i)},\quad \frac{1}{(1-i)(2+i)}$$

Solution 1:

$$a)\quad \frac{1}{1-2i} = \frac{1}{5} + \frac{2i}{5}, \qquad \frac{1}{1+2i} = \frac{1}{5} - \frac{2i}{5}$$

$$b)\quad \frac{1}{2-i} = \frac{2}{5} + \frac{i}{5}, \qquad \frac{1}{2+i} = \frac{2}{5} - \frac{i}{5}$$

$$c)\quad \frac{2+i}{2-i} = \frac{3}{5} + \frac{4i}{5}, \qquad \frac{2-i}{2+i} = \frac{3}{5} - \frac{4i}{5}$$

$$d)\quad \frac{3+2i}{3-i} = \frac{7}{10} + \frac{9i}{10}, \qquad \frac{3-i}{3+2i} = \frac{7}{13} - \frac{9i}{13}$$

$$e)\quad \frac{1}{(1+i)(2-i)} = \frac{3}{10} - \frac{i}{10}, \qquad \frac{1}{(1-i)(2+i)} = \frac{3}{10} + \frac{i}{10}$$

Determine with the complex numbers:

$$z_1 = 1 + i\,, \qquad z_2 = 2 + i\,, \qquad z_3 = 1 - 2i$$

$$z_4 = 3 + 2i\,, \qquad z_5 = -2 + 5i\,, \qquad z_6 = -i$$

the expressions:

$$a)\quad \frac{z_1 + z_2}{z_3}\,, \qquad \frac{z_2 + z_3}{z_4 + z_6}\,, \qquad \frac{z_1 + z_2 + z_3}{z_4 - z_2}$$

$$b)\quad \frac{(z_1 - 2\,z_2)\; z_6}{z_2}\,, \qquad \frac{z_1 \cdot z_2^*}{z_3^*}\,, \qquad \frac{z_1 \cdot (z_2^* + 3\,z_6^*)}{z_5 + 7\,z_6}$$

$$c)\quad \frac{1}{z_1} + \frac{1}{z_2}\,, \qquad \frac{z_2^*}{z_1} + \frac{z_1^*}{z_2}\,, \qquad \frac{1}{z_1} + \frac{z_2}{z_1 \cdot z_2^*}$$

$$d)\quad \left(\frac{1}{z_3} - \frac{z_2}{z_3 \cdot z_2^*}\right) z_6\,, \qquad \frac{i\,(z_5 + 3\,z_6)\; z_1^*}{z_1 \cdot z_4^*\,(z_3 - z_6)}\,, \qquad \frac{i}{z_1 \cdot z_2^* \cdot z_6}$$

Division: Solution 2 a-c)

a) $$\frac{z_1 + z_2}{z_3} = -\frac{1}{5} + \frac{8\,i}{5}, \qquad \frac{z_2 + z_3}{z_4 + z_6} = \frac{4}{5} - \frac{3\,i}{5}$$

$$\frac{z_1 + z_2 + z_3}{z_4 - z_2} = 2 - 2\,i$$

b) $$\frac{(z_1 - 2\,z_2)\;z_6}{z_2} = \frac{1}{5} + \frac{7\,i}{5}, \qquad \frac{z_1 \cdot z_2^*}{z_3^*} = 1 - i$$

$$\frac{z_1 \cdot (z_2^* + 3\,z_6^*)}{z_5 + 7\,z_6} = -1 - i$$

c) $$\frac{1}{z_1} + \frac{1}{z_2} = \frac{9}{10} - \frac{7\,i}{10}, \qquad \frac{z_2^*}{z_1} + \frac{z_1^*}{z_2} = \frac{7}{10} - \frac{21\,i}{10}$$

$$\frac{1}{z_1} + \frac{z_2}{z_1 \cdot z_2^*} = \frac{6}{5} - \frac{2\,i}{5}$$

Division: *Solution 2d*

$$\left(\frac{1}{z_3} - \frac{z_2}{z_3 \cdot z_2^*}\right) z_6 = \frac{2}{5}\, z_6 = -\frac{2\,i}{5}$$

$$\frac{i\,(z_5 + 3\,z_6)\; z_1^*}{z_1 \cdot z_4^*\,(z_3 - z_6)} = -\frac{6}{13} - \frac{4\,i}{13}$$

$$\frac{i}{z_1 \cdot z_2^* \cdot z_6} = -\frac{3}{10} + \frac{i}{10}$$