http://www.youtube.com/watch?v=9FFDJK8jmms

Derivative of some Functions

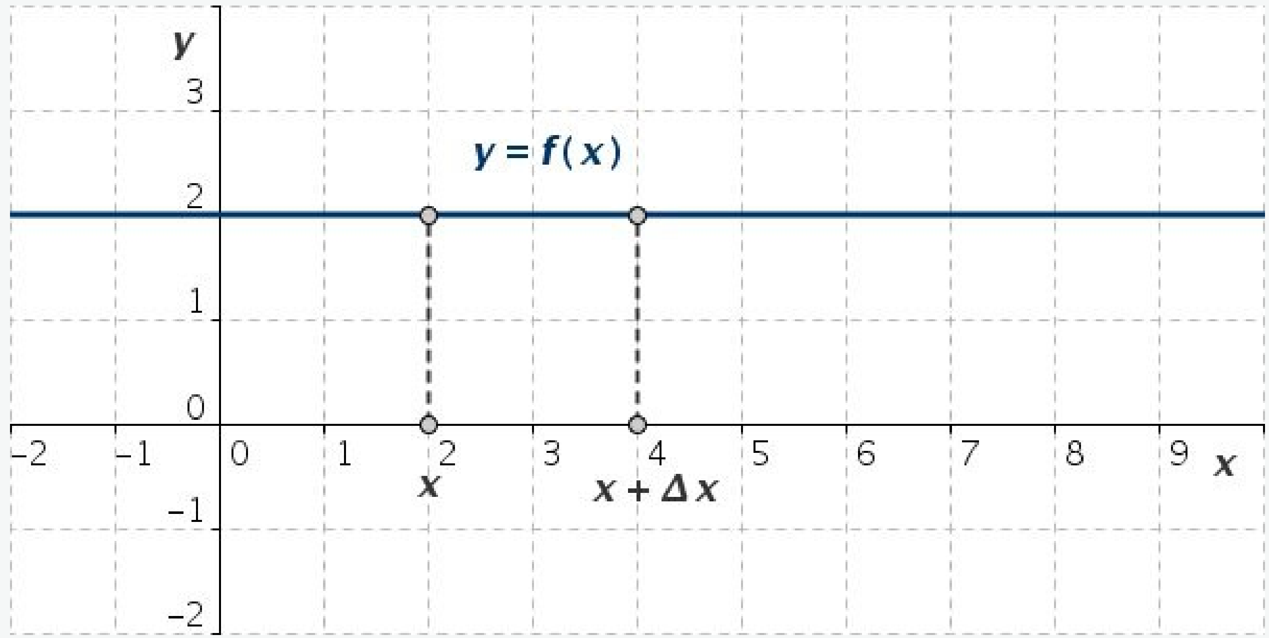

Fig. 1-1: a constant function $f(x) = c$

$$f(x) = c, \qquad c \in \mathbb{R}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = 0$$

The derivative of a constant function is zero.

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Derivative of a linear function

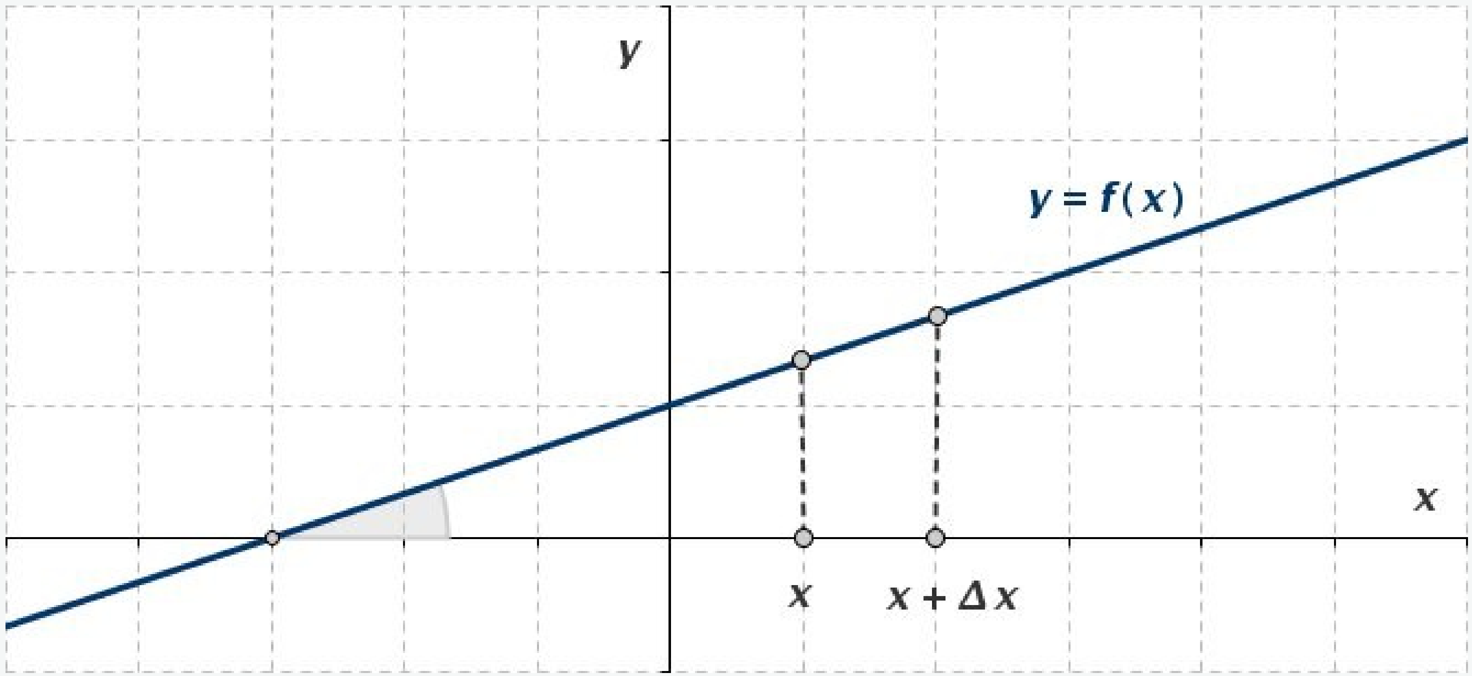

Fig. 1-2: Linear function $f(x) = m\,x + n$ with constant slope m

$$f(x) = m\,x + n\,, \qquad m\,,\ n \in \mathbb{R}\,, \qquad m \neq 0$$

$$f\,'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{m(x + \Delta x) + n - (m\,x + n)}{\Delta x} = m$$

$$f\,'(x) = \frac{d}{dx}(m\,x + n) = m$$

The derivative of a linear function $f(x) = m\,x + n$ is the constant m.

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Derivative of a quadratic function

Derivative of a quadratic function

$$f(x) = a\,x^2 + b\,, \qquad a\,,\ b \in \mathbb{R}\,, \qquad a \neq 0$$

$$f\,'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{a\,(x + \Delta x)^2 + b - (a\,x^2 + b)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{a\,(x^2 + 2\,x\,\Delta x + \Delta x^2) + b - (a\,x^2 + b)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{a\,\Delta x\,(2\,x + \Delta x)}{\Delta x} = \lim_{\Delta x \to 0} a\,(2\,x + \Delta x) = 2\,a\,x$$

$$f\,'(x) = (a\,x^2 + b)\,' = 2\,a\,x$$

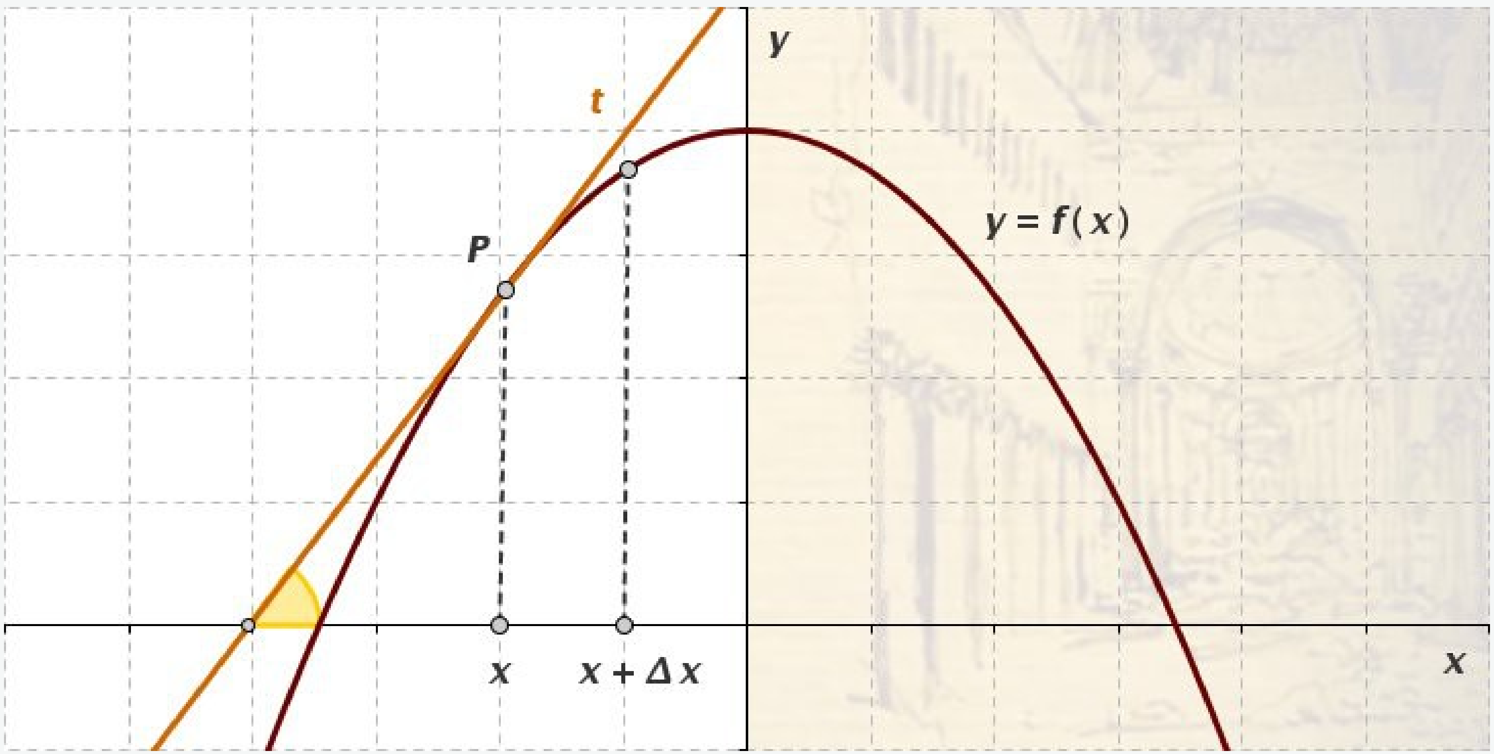

Fig. 1-3: Quadratic function $f(x) = -x^2/3 + 4$ and the slope at point P

At each point $(x, f(x))$ of the curve above, the derivative is $2ax = -2x/3$. Therefore the slope of tangent line is $4/3$ at $x = -2$.

Derivative of a function $y = x^n$

$$f(x) = x^n, \qquad n \in \mathbb{N}, \qquad n > 2$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \times$$

$$\times \left(x^n + C_n^1 \; x^{n-1} \Delta x + C_n^2 \; x^{n-2} (\Delta x)^2 + \ldots + C_n^n (\Delta x)^n - x^n \right) =$$

$$= \lim_{\Delta x \to 0} \left(C_n^1 \; x^{n-1} + C_n^2 \; x^{n-2} \; \Delta x + \ldots \right) = n \; x^{n-1}$$

$$f'(x) = (x^n)' = n \, x^{n-1}$$

Binomial coefficients:

$$C_n^k = \frac{n\,!}{k\,!\,(n-k)\,!} \qquad (k\,,\ n \in \mathbb{N}^*\,, \quad k \leqslant n)$$

$$C_n^0 = 1\,, \qquad C_n^1 = n\,, \qquad C_n^n = 1$$

Factorial:

$$n\,! = 1 \cdot 2 \cdot 3 \ldots n \qquad 0\,! = 1$$

$$f(x) = \cos x\,, \qquad f\,'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{\cos(x + \Delta x) - \cos x}{\Delta x} =$$

$$\cos\alpha - \cos\beta = -2\,\sin\frac{\alpha + \beta}{2}\,\sin\frac{\alpha - \beta}{2}$$

$$= -2 \lim_{\Delta x \to 0} \frac{\sin\dfrac{\Delta x}{2} \cdot \sin\left(x + \dfrac{\Delta x}{2}\right)}{\Delta x} =$$

$$= -\lim_{\Delta x \to 0} \frac{\sin\dfrac{\Delta x}{2}}{\dfrac{\Delta x}{2}} \cdot \lim_{\Delta x \to 0} \sin\left(x + \frac{\Delta x}{2}\right) = -\sin x$$

$$\sin\left(x + \frac{\Delta x}{2}\right) = \sin x\,\cos\frac{\Delta x}{2} + \cos x\,\sin\frac{\Delta x}{2}$$

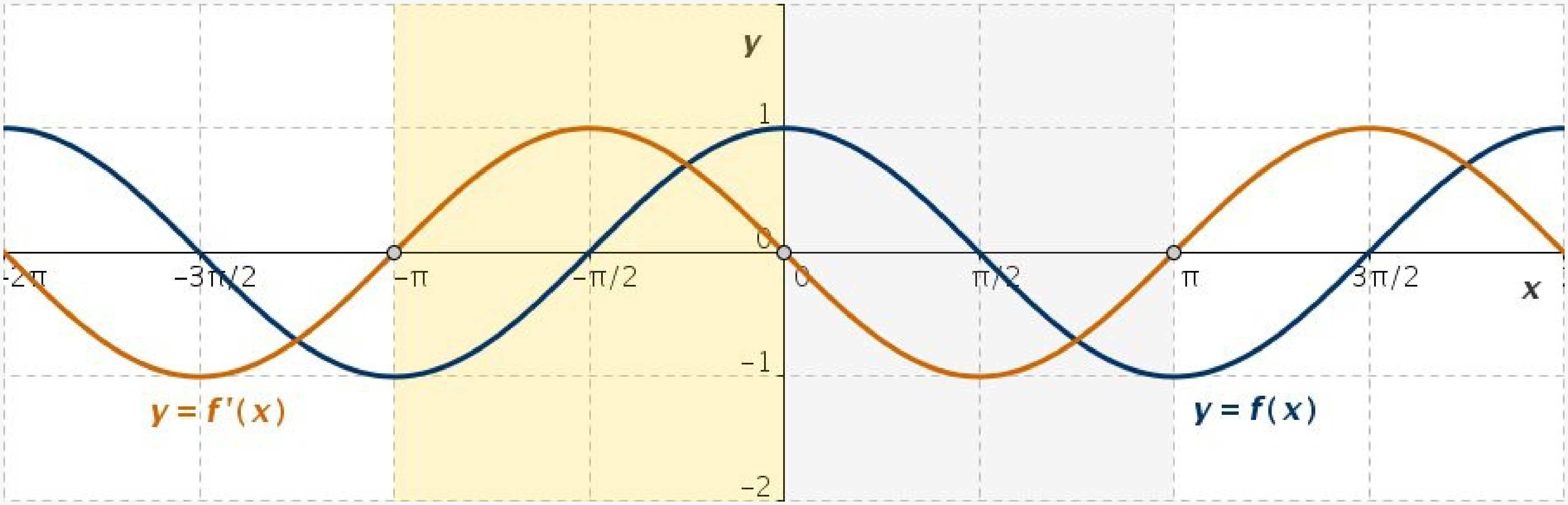

Fig. 1-4: Function f (x) = cos x (blue) and its derivative f '(x) = - sin x (red)

$$f(x) = \tan x, \qquad x = \frac{\pi}{2} + \pi n, \qquad n \in \mathbb{Z}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\tan(x + \Delta x) - \tan x}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left(\frac{\sin(x + \Delta x)}{\cos(x + \Delta x)} - \frac{\sin x}{\cos x} \right) =$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \frac{\sin(x + \Delta x)\cos x - \cos(x + \Delta x)\sin x}{\cos(x + \Delta x)\cos x} =$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \frac{\sin(x + \Delta x)\cos x - \cos(x + \Delta x)\sin x}{\cos(x + \Delta x)\cos x} =$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \frac{\sin(x + \Delta x - x)}{\cos(x + \Delta x)\cos x} = \frac{1}{\cos x} \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} \frac{1}{\cos(x + \Delta x)} =$$

$$= \frac{1}{\cos x} \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} \cdot \lim_{\Delta x \to 0} \frac{1}{\cos(x + \Delta x)} = \frac{1}{\cos^2 x}$$

$$(\tan x)' = \frac{1}{\cos^2 x}$$

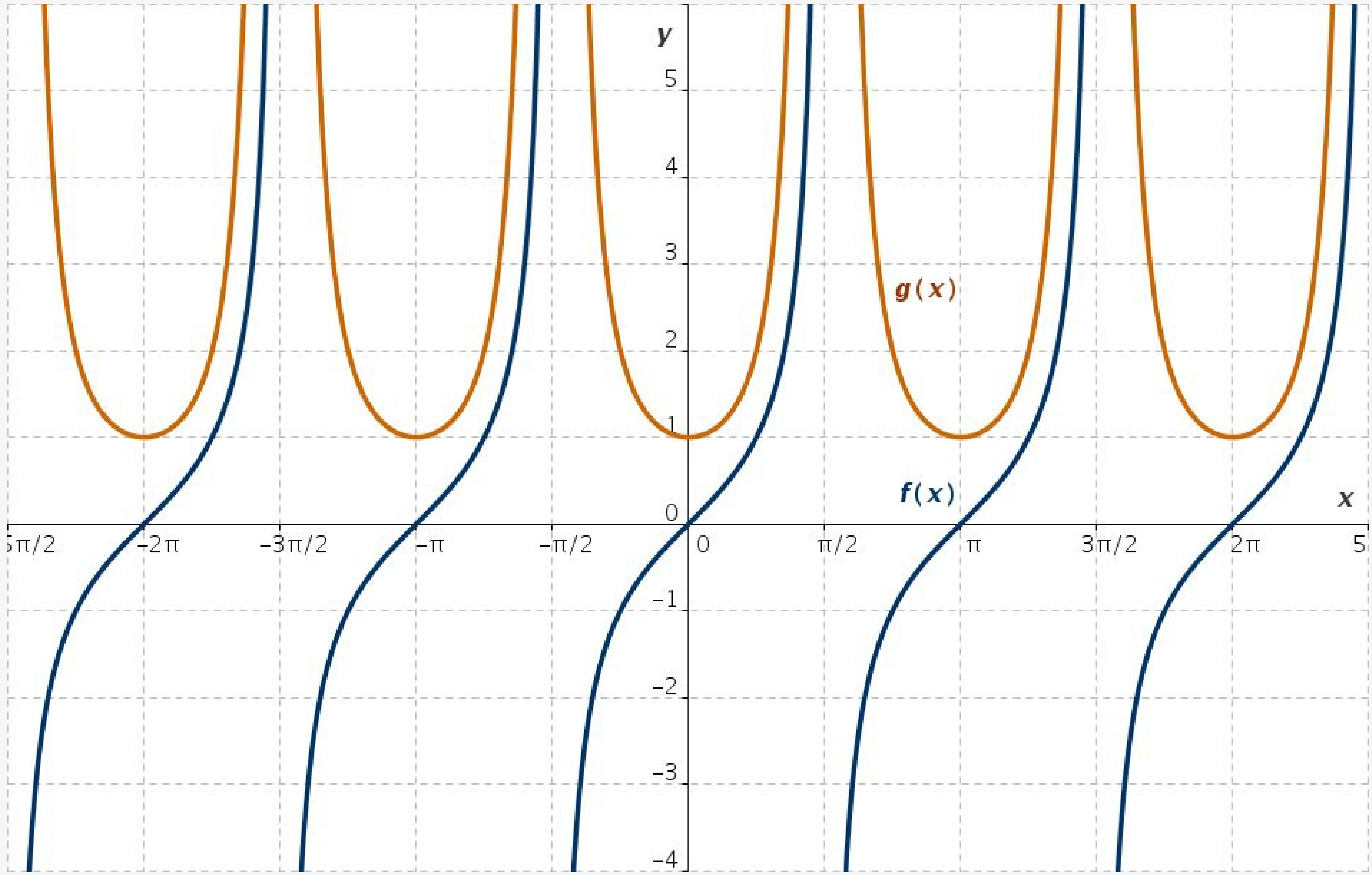

Abb. 1-5: Function $f(x) = \tan x$ (blue) and derivative $f'(x) = 1/\cos^2 x$ (red)

$$f(x) = \sqrt{x}\,, \qquad f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} =$$

$$= \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x \left(\sqrt{x + \Delta x} + \sqrt{x}\right)} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} =$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$$

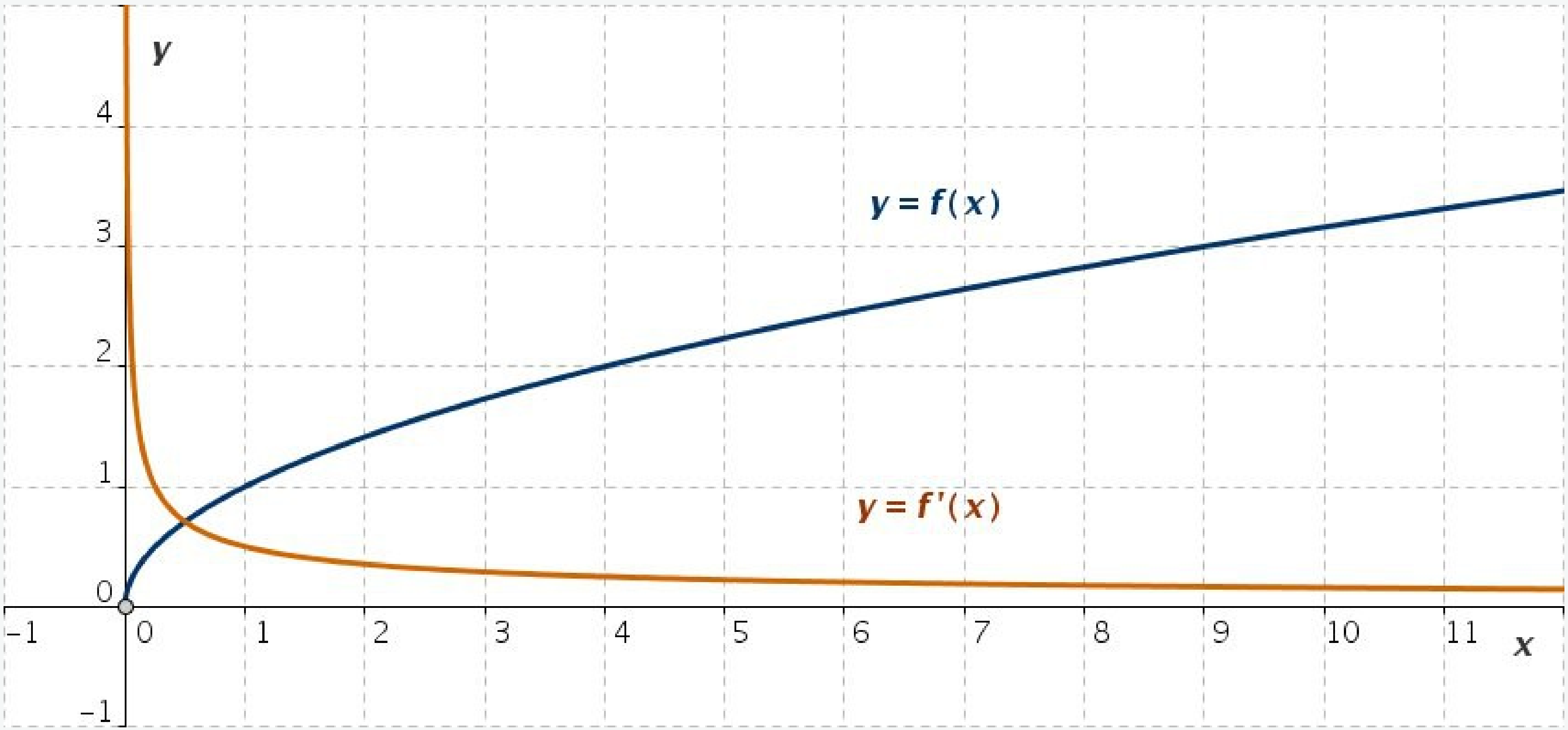

Fig. 1-7: Function $y = f(x)$ (blue) and its derivative $y = f'(x)$ (red)

$$f(x) = \sqrt{x}, \qquad f'(x) = \frac{1}{2\sqrt{x}}$$

Derivative of an Exponential Function

$$f(x) = e^x, \qquad f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{e^{x + \Delta x} - e^x}{\Delta x} = \lim_{\Delta x \to 0} e^x \cdot \frac{e^{\Delta x} - 1}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} e^x \cdot \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = e^x$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{\Delta x} - 1 = \Delta x + \frac{(\Delta x)^2}{2!} + \frac{(\Delta x)^3}{3!} + \ldots$$

$$\frac{e^{\Delta x} - 1}{\Delta x} = 1 + \frac{\Delta x}{2!} + \frac{(\Delta x)^2}{3!} + \ldots$$

$$\lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = 1$$

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$