http://www.youtube.com/watch?v=WSt3y8NZT_4&feature=channel

Rules for Derivatives

Scalar multiple: $y = C \cdot f(x)$ $\qquad y' = C \cdot f'$

Sum:

$$y = f_1(x) + f_2(x) + \ldots + f_n(x), \qquad y' = f'_1 + f'_2 + \ldots + f'_n$$

Exercise 1: Determine the derivatives of the functions given below:

a) $y = x^3 - 5x^2 + 3x - 2$

b) $y = x^2 - \dfrac{2}{x^3} + 5\sqrt[3]{x^2}$

c) $y = 3\sin x - 4\cos x$

d) $y = x^5 - 2\cos x + 3e^x + \ln x$

Exercise 2: Determine first to fifth derivative:

$$y = 3x^5 - 4x^3 + 2x^2 - 1, \qquad y', \; y'', \; y''', \; y^{(4)}, \; y^{(5)}$$

Solution 1:

a) $y' = 3x^2 - 10x + 3$

b) $y = x^2 - \dfrac{2}{x^3} + 5\sqrt[3]{x^2} = x^2 - 2x^{-3} + 5x^{2/3}$

$y' = 2x + 6x^{-4} + \dfrac{10}{3}x^{-1/3} = 2x + \dfrac{6}{x^4} + \dfrac{10}{3\sqrt[3]{x}}$

c) $y' = 3\cos x + 4\sin x$

d) $y' = 5x^4 + 2\sin x + 3e^x + \dfrac{1}{x}$

Solution 2: $y = 3x^5 - 4x^3 + 2x^2 - 1, \quad y' = 3 \cdot 5x^4 - 4 \cdot 3x^2 + 2 \cdot 2x$

$y'' = 3 \cdot 5 \cdot 4x^3 - 4 \cdot 3 \cdot 2x + 2 \cdot 2 \cdot 1, \quad y''' = 3 \cdot 5 \cdot 4 \cdot 3x^2 - 4 \cdot 3!$

$y^{(4)} = 3 \cdot 5 \cdot 4 \cdot 3 \cdot 2x, \quad y^{(5)} = 3 \cdot 5! = 360$

Product rule: $y = u \cdot v \qquad y' = (u \cdot v)' = u' \cdot v + v' \cdot u$

$$u = u(x), \qquad v = v(x)$$

Exercise 3: Determine the derivatives of the functions below using the product rule:

a) $y = (x^2 + 1)(x^3 - 1)$

b) $y = 2 \sin x \cos x$

c) $y = x \ln x$

d) $y = \sin^2 x$

e) $y = \sin^2 x + \cos^2 x$

f) $y = \sqrt{x} \cos x$

$a)\quad y' = 5\,x^4 + 3\,x^2 - 2\,x$

$b)\quad y' = 2\,(\cos^2 x - \sin^2 x) = 2\,\cos 2\,x$

$c)\quad y' = 1 + \ln x$

$d)\quad y' = 2\,\sin x\,\cos x = \sin(2\,x)$

$e)\quad y' = 0, \qquad \sin^2 x + \cos^2 x = 1,$

$f)\quad y' = \dfrac{1}{2\sqrt{x}}\,(\cos x - 2\,x\,\sin x)$

Quotient rule: $y = \dfrac{u}{v}$ $\qquad y' = \dfrac{u' \cdot v - v' \cdot u}{v^2}$

$$u = u(x), \qquad v = v(x)$$

Exercise 4: Determine the derivatives of the functions below using the quotient rule:

$a)\quad y = \dfrac{x}{1 - x^2}, \qquad b)\quad y = \dfrac{x}{1 - x^n}$

$c)\quad y = \dfrac{a^2 - x^2}{a^2 + x^2}, \qquad d)\quad y = \dfrac{\sqrt{x}}{a - \sqrt{x}}$

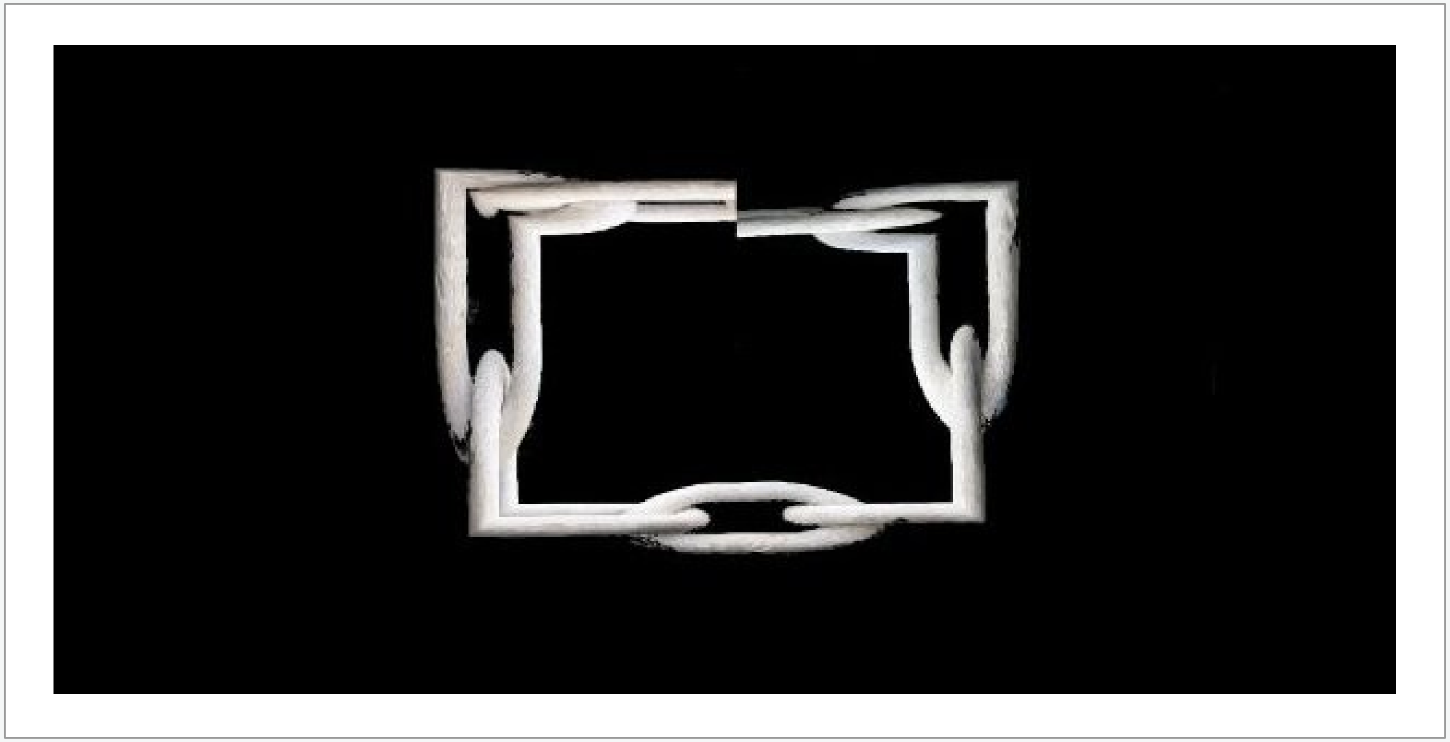
$e)\quad y = \dfrac{1}{\ln x}, \qquad f)\quad y = \tan x$

Quotient rule: *Solution 4*

$$a)\quad y' = \frac{1 + x^2}{(1 - x^2)^2}, \qquad b)\quad y' = \frac{1 + (n-1)\,x^n}{(1 - x^n)^2}$$

$$c)\quad y' = -\frac{4\,a^2\,x}{(a^2 + x^2)^2}, \qquad d)\quad y' = \frac{a}{2\sqrt{x}\,(a - \sqrt{x})^2}$$

$$e)\quad y' = -\frac{1}{x\,(\ln x)^2}, \qquad f)\quad y' = \frac{1}{\cos^2 x}$$



How can we determine the derivatives of the functions below?

$$f(x) = \cos^2 x, \qquad g(x) = \cos^4 x, \qquad h(x) = \sqrt[4]{\cos x}$$

It is possible to use the product rule for function *f (x)*. For *g (x)* this approach is quite cumbersome, and for *h (x)* the product rule does not work.

So we need a further rule for derivatives.

http://www.flickr.com/photos/9184647@N02/2062425056/

Fig. 1-1: Transformation of function $y = f(x)$ by a suitable substitution

We can try to transform the function $f(x)$ by a substitution $u = u(x)$ into a more simple and more elementary function.

Chain rule

$$y = f(x) \longrightarrow u = u(x) \longrightarrow y = F(u)$$

Substitution

$y = F(u)$ – outer function

$u = u(x)$ – inner function

Chain rule:

The derivative of a composite function $y = F(u(x)) = f(x)$ is given by the derivative of the outer function times the derivative of the inner function:

$$y' = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$\frac{dy}{du}$ – outer derivative, $\frac{du}{dx}$ – inner derivative

$y = f(x)$	$y = F(u)$
$y = \cos^2 x$	$y = u^2$
$y = \cos^4 x$	$y = u^4$
$y = \sqrt[4]{\cos x}$	$y = u^{\frac{1}{4}}$

Fig. 1-2: Transformations of functions by the substitution $u = \cos x$

$$y = f(x) \quad \to \quad u = u(x) \quad \to \quad y = F(u)$$

$$y = \sin(3x) \quad \to \quad u = 3x \quad \to \quad y = \sin u$$

$$y = (x^2 + 2)^5 \quad \to \quad u = x^2 + 2 \quad \to \quad y = u^5$$

$$y = \sqrt{\frac{2x - 1}{x + 1}} \quad \to \quad u = \frac{2x - 1}{x + 1} \quad \to \quad y = u^{1/2}$$

$$y = \ln(x^3 - 7) \quad \to \quad u = x^3 - 7 \quad \to \quad y = \ln u$$

Chain rule: Example 1

$$y = \sin(x^3 - 2x)$$

This is not an elementary sine function due to the polynomial in the argument. In such cases one can proceed as follows:

- The given function is decomposed into more basic functions. One tries to find such a substitution, that the derivative of the new function can be obtained using the known rules.

$$y = \sin u, \qquad u = x^3 - 2x$$

- One determines the inner and outer derivatives:

$$\frac{dy}{du} = \cos u, \qquad \frac{du}{dx} = 3x^2 - 2$$

- Finally the product gives the result:

$$y' = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u \cdot (3x^2 - 2) = (3x^2 - 2)\cos(x^3 - 2x)$$

Chain rule: Example 2

$$y = \ln[\sin(x^2 + 3)]$$

<u>1st substitution:</u> $u = u(x) = x^2 + 3, \quad y = \ln(\sin u)$

It is still not easy to get the derivative. We need a further substitution:

<u>2nd substitution:</u> $v = v(u) = \sin u\,, \quad y = \ln v$

$$\Rightarrow \quad y = \ln v\,, \quad v = \sin u\,, \quad u = x^2 + 3$$

$$y' = \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} = \frac{1}{v} \cdot \cos u \cdot (2x) = \frac{\cos u}{\sin u} \cdot (2x) =$$

$$= 2x \cot u = 2x \cot(x^2 + 3)$$

$$y' = \frac{d}{dx} \ln[\sin(x^2 + 3)] = 2x \cot(x^2 + 3)$$

Determine the first derivative of the following functions:

a) $f(x) = (x+3)^3, \qquad g(x) = (x-6)^5$

b) $f(x) = (2x-5)^4, \qquad g(x) = (5x-11)^{12}$

c) $f(x) = \dfrac{1}{(x-2)^3}, \qquad g(x) = \dfrac{2}{(x+12)^7}$

d) $f(x) = \dfrac{3}{(2x-3)^5}, \qquad g(x) = \dfrac{2}{3(4x+1)^3}$

e) $f(x) = \sqrt{x+3}, \qquad g(x) = \sqrt{x-9}$

f) $f(x) = \sqrt{2x-6}, \qquad g(x) = \sqrt{x^2+4}$

g) $f(x) = \sqrt[3]{x+4}, \qquad g(x) = \sqrt[4]{2x+1}$

Chain rule: Solution 5 a-d

$$a)\ f(x) = (x+3)^3, \qquad f'(x) = 3(x+3)^2$$

$$g(x) = (x-6)^5, \qquad g'(x) = 5(x-6)^4$$

$$b)\ f(x) = (2x-5)^4, \qquad f'(x) = 4(2x-5)^3 \cdot 2 = 8\,(2x-5)^3$$

$$g(x) = (5x-11)^{12}, \qquad g'(x) = 12\,(5x-11)^{11} \cdot 5 = 60\,(5x-11)^{11}$$

$$c)\ f(x) = \frac{1}{(x-2)^3} = (x-2)^{-3}, \qquad f'(x) = -3(x-2)^{-4} = -\frac{3}{(x-2)^4}$$

$$g(x) = \frac{2}{(x+12)^7} = 2\,(x+12)^{-7}, \qquad g'(x) = -14(x+12)^{-8} = -\frac{14}{(x+12)^8}$$

$$d)\ f(x) = \frac{3}{(2x-3)^5} = 3\,(2x-3)^{-5},$$

$$f'(x) = -15\,(2x-3)^{-6} \cdot 2 = -30\,(2x-3)^{-6} = -\frac{30}{(2x-3)^6}$$

$$g(x) = \frac{2}{3(4x+1)^3} = \frac{2}{3}\,(4x+1)^{-3},$$

$$g'(x) = \frac{2}{3}\,(4x+1)^{-4} \cdot (-3) \cdot 4 = -8\,(4x+1)^{-4} = -\frac{8}{(4x+1)^4}$$

e) $f(x) = \sqrt{x+3} = (x+3)^{1/2}$, $\quad f'(x) = \frac{1}{2}(x+3)^{-1/2} = \frac{1}{2\sqrt{x+3}}$

$g(x) = \sqrt{x-9} = (x-9)^{1/2}$, $\quad g'(x) = \frac{1}{2}(x-9)^{-1/2} = \frac{1}{2\sqrt{x-9}}$

f) $f(x) = \sqrt{2x-6} = (2x-6)^{1/2}$, $\quad f(x) = \frac{2}{2}(2x-6)^{-1/2} = \frac{1}{\sqrt{2x-6}}$

$g(x) = \sqrt{x^2+4} = (x^2+4)^{1/2}$, $\quad g'(x) = \frac{1}{2}(x^2+4)^{-1/2}\,2x = \frac{x}{\sqrt{x^2+4}}$

g) $f(x) = \sqrt[3]{x+4} = (x+4)^{1/3}$, $\quad f'(x) = \frac{1}{3(x+4)^{2/3}} = \frac{1}{3\sqrt[3]{(x+4)^2}}$

$g(x) = \sqrt[4]{2x+1}$, $\quad g'(x) = \frac{1}{2\sqrt[4]{(2x+1)^3}}$

Determine the first derivative of the following functions:

$a)\quad f(x) = 5\cos(3x), \qquad g(x) = \cos(\sqrt{x})$

$b)\quad f(x) = \sin(x^2 + 3), \qquad g(x) = \sin(x^3 - \sqrt{2x})$

$c)\quad f(x) = \ln(x^4 + 5), \qquad g(x) = \ln(x^3 - 2x + 3)$

$d)\quad f(x) = \dfrac{2}{(x^2 + 3)^2}, \qquad g(x) = \left(\dfrac{x^3 + 2x}{x}\right)^4$

$e)\quad f(x) = \sqrt{2x - \sin(3x)}, \qquad g(x) = \sqrt{x^3 + \cos(2x)}$

$f)\quad f(x) = (x + 2)\sqrt{2x - 3}, \qquad g(x) = \sqrt[5]{x^2 + 7x}$

a) $f(x) = 5\cos(3x)$, $f'(x) = -15\sin(3x)$

$g(x) = \cos(\sqrt{x})$, $g'(x) = -\dfrac{\sin\sqrt{x}}{2\sqrt{x}}$

b) $f(x) = \sin(x^2 + 3)$, $f'(x) = 2x\cos(x^2 + 3)$

$g(x) = \sin(x^3 - \sqrt{2x})$, $g'(x) = \left(3x^2 - \dfrac{1}{\sqrt{2x}}\right) \cdot \cos(x^3 - \sqrt{2x})$

c) $f(x) = \ln(x^4 + 5)$, $f'(x) = \dfrac{4x^3}{x^4 + 5}$

$g(x) = \ln(x^3 - 2x + 3)$, $g'(x) = \dfrac{3x^2 - 2}{x^3 - 2x + 3}$

d) $f(x) = \dfrac{2}{(x^2 + 3)^2}$, $f'(x) = -\dfrac{8x}{(x^2 + 3)^3}$

$g(x) = \left(\dfrac{x^3 + 2x}{x}\right)^4$, $g'(x) = 8x\,(x^2 + 2)^3$

e) $f(x) = \sqrt{2x - \sin(3x)}$, $\quad f'(x) = \dfrac{2 - 3\cos(3x)}{2\sqrt{2x - \sin(3x)}}$

$g(x) = \sqrt{x^3 + \cos(2x)}$, $\quad g'(x) = \dfrac{3x^2 - 2\sin(2x)}{2\sqrt{x^3 + \cos(2x)}}$

f) $f(x) = (x + 2)\sqrt{2x - 3}$, $\quad f'(x) = \dfrac{3x - 1}{\sqrt{2x - 3}}$

$g(x) = \sqrt[5]{x^2 + 7x} = \left(x^2 + 7x\right)^{1/5}$, $\quad g'(x) = \dfrac{2x + 7}{5\left(x^2 + 7x\right)^{4/5}}$