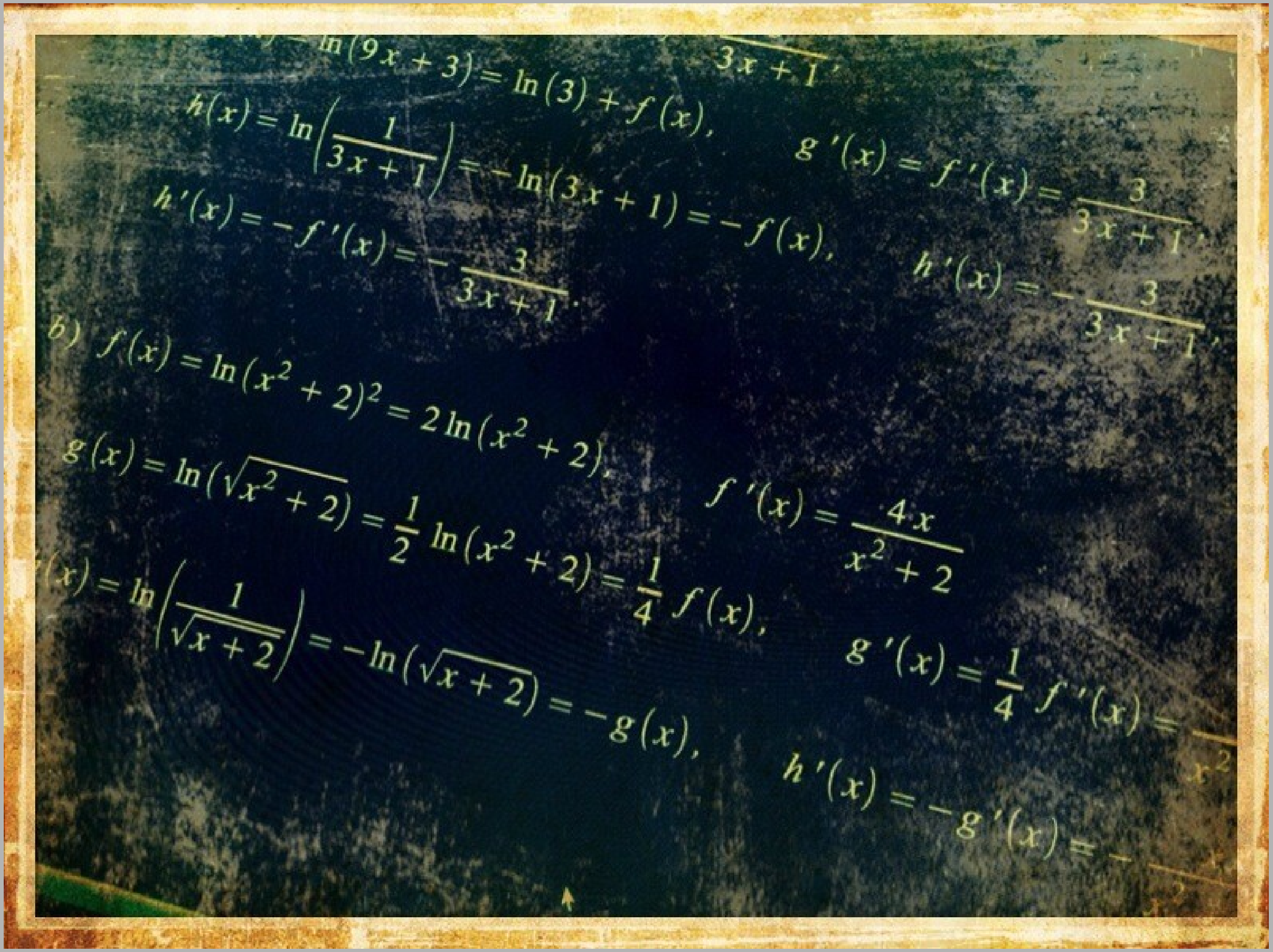


Derivatives: *Exercises*
Part 1

a) $f(x) = \frac{1}{\sqrt[3]{x}} + \sqrt{2x+1}$, $\quad g(x) = \frac{1}{x\sqrt[4]{x}} + \sqrt{x^3 + 2x}$

b) $f(x) = (1 + 2\sqrt{x})(1 - 2\sqrt{x})$, $\quad g(x) = (1 + x^2)(1 - x^2)$

c) $f(x) = \sqrt{x}\ln(x)$, $\quad g(x) = (1 - x^2)\ln(x - 1)$

d) $f(x) = (4x + 7)^3$, $\quad g(x) = (x^3 - 2x)^6$

e) $f(x) = \left(2x + \frac{1}{x}\right)^3$, $\quad g(x) = \left(x^2 - \frac{1}{x^2}\right)^4$

f) $f(x) = (\sqrt{x} + 3x)^4$, $\quad g(x) = x^3\sqrt{3x - 19}$

a) $$f(x) = \frac{1}{\sqrt[3]{x}} + \sqrt{2x+1} = x^{-\frac{1}{3}} + (2x+1)^{\frac{1}{2}}$$

$$f'(x) = -\frac{1}{3x^{4/3}} + \frac{1}{\sqrt{2x+1}} = -\frac{1}{3x\sqrt[3]{x}} + \frac{1}{\sqrt{2x+1}}$$

$$g(x) = \frac{1}{x\sqrt[4]{x}} + \sqrt{x^3+2x} = x^{-\frac{5}{4}} + (x^3+2x)^{\frac{1}{2}}$$

$$g'(x) = -\frac{5}{4x^{9/4}} + \frac{3x^2+2}{2\sqrt{x^3+2x}} = -\frac{5}{4x^2\sqrt[4]{x}} + \frac{3x^2+2}{2\sqrt{x^3+2x}}$$

b) $$f(x) = (1+2\sqrt{x})(1-2\sqrt{x}) = 1-4x, \qquad f'(x) = -4$$

$$g(x) = (1+x^2)(1-x^2) = 1-x^4, \qquad g'(x) = -4x^3$$

c) $f(x) = \sqrt{x}\,\ln(x)$, $\quad f'(x) = \dfrac{1}{2\sqrt{x}}\left(\ln x + 2\right)$

$g(x) = (1 - x^2)\,\ln(x - 1)$, $\quad g'(x) = -2\,x\ln(x - 1) - x - 1$

d) $f(x) = (4\,x + 7)^3 = 12\,(4\,x + 7)^2$

$g(x) = (x^3 - 2\,x)^6 = 6\,(3\,x^2 - 2)\,(x^3 - 2\,x)^5$

e) $f(x) = \left(2\,x + \dfrac{1}{x}\right)^3$, $\quad f'(x) = 3\left(2\,x + \dfrac{1}{x}\right)^2\left(2 - \dfrac{1}{x^2}\right)$

$g(x) = \left(x^2 - \dfrac{1}{x^2}\right)^4$, $\quad g'(x) = 4\left(x^2 - \dfrac{1}{x^2}\right)^3\left(2\,x + \dfrac{2}{x^3}\right)$

f) $f(x) = (\sqrt{x} + 3\,x)^4$, $\quad f'(x) = 4(\sqrt{x} + 3\,x)^3\left(\dfrac{1}{2\sqrt{x}} + 3\right)$

$g(x) = x^3\,\sqrt{3\,x - 19}$, $\quad g'(x) = 3\,x^2\,\sqrt{3\,x - 19} + \dfrac{3}{2}\,\dfrac{x^3}{\sqrt{3\,x - 19}}$

a) $f(x) = \sin(2x - 7)$, $g(x) = \cos(3x^2 + 6x)$

b) $f(x) = \sin(\sqrt{3x - 5})$, $g(x) = \cos(\sqrt{x} + 2x)$

c) $f(x) = \sqrt{\sin x + 2}$, $g(x) = \sqrt{\sin(3x) - 6}$

d) $f(x) = \dfrac{1}{\sin(2x)}$, $g(x) = \dfrac{3}{\sin(x^2 + 1)}$

e) $f(x) = \dfrac{1}{\sin(x^2)}$, $g(x) = \dfrac{1}{\sin(\sqrt{x})}$

f) $f(x) = \sin^2(2x - 1)$, $g(x) = \cos^3(x^2 + 4x)$

g) $f(x) = \sqrt{\sin^2 x + 4}$, $g(x) = \sqrt{\cos^2(3x) + 1}$

a) $f(x) = \sin(2x - 7)$, $\quad f'(x) = 2\cos(2x - 7)$

$g(x) = \cos(3x^2 + 6x)$, $\quad g'(x) = -6(x+1)\sin(3x^2 + 6x)$

b) $f(x) = \sin(\sqrt{3x - 5})$, $\quad f'(x) = \dfrac{3}{2}\dfrac{\cos(\sqrt{3x-5})}{\sqrt{3x-5}}$

$g(x) = \cos(\sqrt{x} + 2x)$, $\quad g'(x) = -\left(\dfrac{1}{2\sqrt{x}} + 2\right)\sin(\sqrt{x} + 2x)$

c) $f(x) = \sqrt{\sin x + 2}$, $\quad f'(x) = \dfrac{1}{2}\dfrac{\cos x}{\sqrt{\sin x + 2}}$

$g(x) = \sqrt{\sin(3x) - 6}$, $\quad g'(x) = \dfrac{3}{2}\dfrac{\cos(3x)}{\sqrt{\sin(3x) - 6}}$

d) $f(x) = \dfrac{1}{\sin(2x)}$, $\quad f'(x) = -\dfrac{2\cos(2x)}{\sin^2(2x)}$

$g(x) = \dfrac{3}{\sin(x^2+1)}$, $\quad g'(x) = -\dfrac{6x\cos(x^2+1)}{\sin^2(x^2+1)}$

e) $f(x) = \dfrac{1}{\sin(x^2)}$, $\quad f'(x) = -\dfrac{2x\cos(x^2)}{\sin^2(x^2)}$

$g(x) = \dfrac{1}{\sin(\sqrt{x})}$, $\quad g'(x) = -\dfrac{1}{2}\,\dfrac{\cos(\sqrt{x})}{\sqrt{x}\cdot\sin^2(\sqrt{x})}$

f) $f'(x) = 4\sin(2x-1)\cos(2x-1)$

$g'(x) = -6(x+2)\cos^2(x^2+4x)\sin(x^2+4x)$

g) $f(x) = \dfrac{\sin x\ \cos x}{\sqrt{\sin^2 x + 4}} = \dfrac{\sin(2x)}{2\sqrt{\sin^2 x + 4}}$

$g(x) = -\dfrac{3\cos(3x)\ \sin(3x)}{\sqrt{\cos^2(3x)+1}} = -\dfrac{3\sin(6x)}{2\sqrt{\cos^2(3x)+1}}$

Exercise 3: Determine the derivatives of the following functions:

a) $f(x) = x \cdot \ln x$, $g(x) = \sqrt{x} \cdot \ln x$, $h(x) = (x^2 + x - 2) \cdot \ln x$

b) $f(x) = \ln(5x)$, $g(x) = x \cdot \ln(3x)$, $h(x) = x \cdot \ln(3x + 1)$

e) $f(x) = \ln(x+4)^2$, $g(x) = \ln(x-2)^5$, $h(x) = \ln(x^2 + 3x - 2)^3$

Exercise 4: Determine the derivatives at $x = 0$

$$f(x) = \left(\frac{x-1}{x+1}\right)^3, \qquad g(x) = \left(\frac{x-2}{x^2+2}\right)^4 \qquad h(x) = \left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^2.$$

a) $f(x) = x \cdot \ln x$, $\quad f'(x) = 1 + \ln x$

$g(x) = \sqrt{x} \cdot \ln x$, $\quad g'(x) = \frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}} = \frac{2 + \ln x}{2\sqrt{x}}$

$h(x) = (x^2 + x - 2) \cdot \ln x$, $\quad h'(x) = x + 1 - \frac{2}{x} + (1 + 2x) \cdot \ln x$

b) $f(x) = \ln(5x)$, $\quad f'(x) = \frac{1}{x}$

$g(x) = x \cdot \ln(3x)$, $\quad g'(x) = \ln(3x) + 1$

$h(x) = x \cdot \ln(3x + 1)$, $\quad h'(x) = \ln(3x + 1) + \frac{3x}{3x + 1}$

Solution 3 (cont):

e) $f(x) = \ln (x+4)^2, \quad f'(x) = \dfrac{2}{x+4}$

$g(x) = \ln (x-2)^5, \quad g'(x) = \dfrac{5}{x-2}$

$h(x) = \ln (x^2+3x-2)^3, \quad h'(x) = \dfrac{3(2x+3)}{x^2+3x-2}$

Solution 4:

$$f(x) = \left(\frac{x-1}{x+1}\right)^3, \qquad f'(x) = 6\,\frac{(x-1)^2}{(x+1)^4}, \qquad f'|_{x=0} = 6,$$

$$g(x) = \left(\frac{x-2}{x^2+2}\right)^4, \qquad g'(x) = -4\,\frac{(x-2)^3\,(x^2-4x-2)}{(x^2+2)^5}, \qquad g'|_{x=0} = -2,$$

$$h(x) = \left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^2, \qquad h'(x) = \frac{2\,(\sqrt{x}-1)}{\sqrt{x}(1+\sqrt{x})^3}.$$

$a)$	$f(x) = e^{3x}$,	$g(x) = e^{3x^2}$,	$h(x) = e^{3x - x^3}$,
$b)$	$f(x) = x^2 e^{-2x}$,	$g(x) = e^{\sin x}$,	$h(x) = e^{\cos(5x)}$,

$$a)\quad f'(x) = \left(e^{3x}\right)' = 3\,e^{3x}\,, \qquad g'(x) = \left(e^{3x^2}\right) = 6\,x\,e^{3x^2}\,,$$

$$h'(x) = \left(e^{3x - x^3}\right)' = 3\,(1 - x^2)\,e^{3x - x^3}\,,$$

$$b)\quad f'(x) = \left(x^2\,e^{-2x}\right)' = 2x\,(1 - x)\,e^{-2x}\,, \qquad g'(x) = \left(e^{\sin x}\right)' = \cos x\,e^{\sin x}\,,$$

$$h'(x) = \left(e^{\cos(5x)}\right)' = -5\,\sin(5x)\,e^{\cos(5x)}\,,$$

a) $f(x) = \ln(3x + 1)$, $g(x) = \ln(9x + 3)$, $h(x) = \ln\left(\frac{1}{3x + 1}\right)$,

b) $f(x) = \ln(x^2 + 2)^2$, $g(x) = \ln(\sqrt{x^2 + 2})$, $h(x) = \ln\left(\frac{1}{\sqrt{x^2 + 2}}\right)$,

c) $f(x) = \ln\left(\frac{x + 2}{x - 3}\right)$, $g(x) = \ln\left(\frac{x^2 - 9}{x^2}\right)$, $h(x) = \ln\left(\frac{16 - x^2}{(x + 2)(x + 4)}\right)$,

d) $f(x) = \ln\left(\sqrt{\frac{x}{x - 3}}\right)$, $g(x) = \ln\left(\frac{1}{\sqrt{x^2 - 3}}\right)$, $h(x) = \ln\left(\sqrt{\frac{4 - x^2}{2 + x^2}}\right)$,

e) $f(x) = \ln\left(\frac{1 + \sqrt{x}}{1 - \sqrt{x}}\right)$, $g(x) = \ln\left(\frac{\sqrt{x - 1} - 1}{\sqrt{x + 1}}\right)$.

a) $f(x) = \ln(3x + 1), \qquad f'(x) = \dfrac{3}{3x + 1},$

$$g(x) = \ln(9x + 3) = \ln(3) + f(x), \qquad g'(x) = f'(x) = \frac{3}{3x + 1},$$

$$h(x) = \ln\left(\frac{1}{3x + 1}\right) = -\ln(3x + 1) = -f(x),$$

$$h'(x) = -f'(x) = -\frac{3}{3x + 1}.$$

b) $f(x) = \ln(x^2 + 2)^2 = 2\ln(x^2 + 2), \qquad f'(x) = \dfrac{4x}{x^2 + 2}$

$$g(x) = \ln(\sqrt{x^2 + 2}) = \frac{1}{2}\ln(x^2 + 2) = \frac{1}{4}f(x), \qquad g'(x) = \frac{1}{4}f'(x) = \frac{x}{x^2 + 2}$$

$$h(x) = \ln\left(\frac{1}{\sqrt{x^2 + 2}}\right) = -\ln(\sqrt{x^2 + 2}) = -g(x), \qquad h'(x) = -g'(x) = -\frac{x}{x^2 + 2}$$

c) $f(x) = \ln\left(\frac{x+2}{x-3}\right) = \ln(x+2) - \ln(x-3)$

$$f'(x) = \frac{1}{x+2} - \frac{1}{x-3} = -\frac{5}{(x+2)(x-3)} = \frac{5}{-x^2+x+6}$$

$$g(x) = \ln\left(\frac{x^2-9}{x^2}\right) = \ln(x^2-9) - \ln(x^2) = \ln(x^2-9) - 2\ln x$$

$$g'(x) = \frac{2x}{x^2-9} - \frac{2}{x} = \frac{18}{x(x^2-9)} = \frac{18}{x^3-9x}$$

$$h(x) = \ln\left(\frac{16-x^2}{(x+2)(x+4)}\right) = \ln(16-x^2) - \ln((x+2)(x+4))$$

$$= \ln(4-x) + \ln(4+x) - \ln(x+2) - \ln(x+4) = \ln(4-x) - \ln(x+2)$$

$$h'(x) = -\frac{1}{4-x} - \frac{1}{x+2} = \frac{1}{x-4} - \frac{1}{x+2} = \frac{6}{(x-4)(x+2)}$$

$$d)\quad f(x) = \ln\left(\sqrt{\frac{x}{x-3}}\right) = \frac{1}{2}\ln\left(\frac{x}{x-3}\right) = \frac{1}{2}\left(\ln(x) - \ln(x-3)\right)$$

$$f'(x) = \frac{1}{2}\left(\frac{1}{x} - \frac{1}{x-3}\right) = -\frac{3}{2}\,\frac{1}{x(x-3)}$$

$$g(x) = \ln\left(\frac{1}{\sqrt{x^2-3}}\right) = -\ln\left(\sqrt{x^2-3}\right) = -\frac{1}{2}\ln\left(x^2-3\right)$$

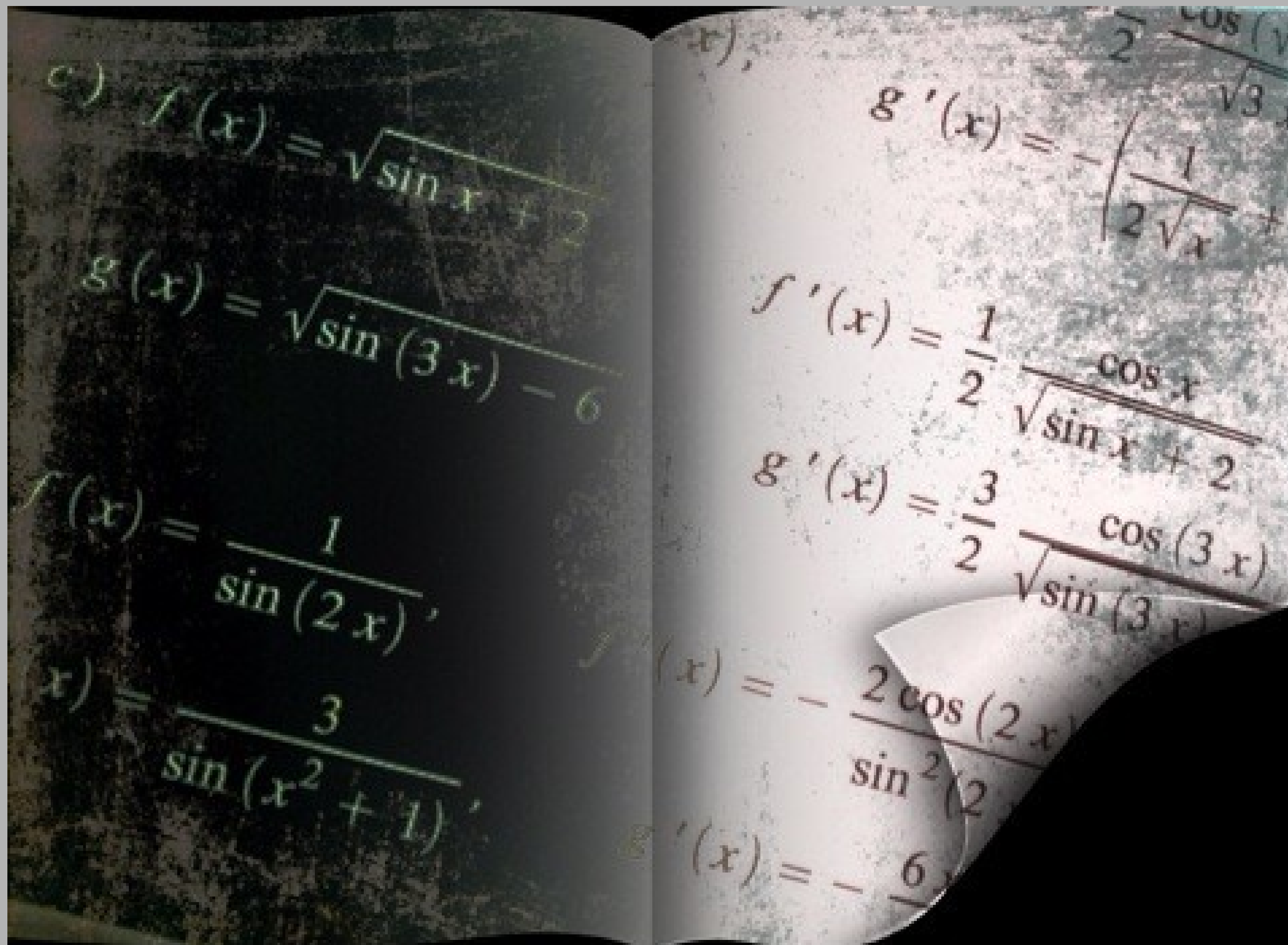
$$g'(x) = -\frac{1}{2}\,\frac{2x}{x^2-3} = -\frac{x}{x^2-3}$$

$$h(x) = \ln\left(\sqrt{\frac{4-x^2}{2+x^2}}\right) = \frac{1}{2}\ln\left(\frac{4-x^2}{2+x^2}\right) = \frac{1}{2}\left(\ln(4-x^2) - \ln(2+x^2)\right)$$

$$h'(x) = \frac{-12x}{(x^2-4)(2+x^2)} = \frac{-12x}{x^4-2x^2-8}$$

$$f(x) = \ln\left(\frac{1 + \sqrt{x}}{1 - \sqrt{x}}\right), \qquad f'(x) = \frac{1}{(1 - x)\sqrt{x}}$$

$$g(x) = \ln\left(\frac{\sqrt{x - 1} - 1}{\sqrt{x + 1}}\right), \qquad g'(x) = \frac{\sqrt{x - 1} + 2}{2\left(\sqrt{x - 1} - 1\right)(x + 1)\sqrt{x - 1}}$$


c) f(x) = √sin x + 2
g(x) = √sin (3 x) − 6
(x) = 1 / sin (2 x),
x) = 3 / sin (x² + 1),
g′(x) = −
f′(x) = 1/2 cos x / √sin x + 2
g′(x) = 3/2 cos (3 x)