Excavations, Israel

Exponential Equations

Exponential equations: Exercises 1, 2

Exercise 1:

Calculate using the power rules:

$$\left[3^6 \cdot \left(\frac{2}{3} \right)^6 \right] : \left(\frac{1}{3} \right)^6 ; \qquad \left[3^5 : \left(\frac{2}{3} \right)^2 \right] \cdot \left(\frac{2}{3} \right)^5$$

Exercise 2:

Simplify:

$a)$ $(e^x \cdot e^{2x}) : (-e^{3x})$

$b)$ $e^{2x} : (-e^{-2x})$

$c)$ $(e^{-x} \cdot e^{2x}) : (e^{0.5x})$

$d)$ $e^{-x+1} : e^{-x} - e : e^{-x}$

$e)$ $e^{x-1} \cdot e - 2e^x + \dfrac{1}{2e^{-x}}$

$f)$ $(e - e^{1+x}) : (1 - e^x)$

Solution 1:

$$\left[3^6 \cdot \left(\frac{2}{3}\right)^6\right] : \left(\frac{1}{3}\right)^6 = 6^6 = 46\,656$$

$$\left[3^5 \ : \left(\frac{2}{3}\right)^2\right] \cdot \left(\frac{2}{3}\right)^5 = 3^2 \cdot 2^3 = 72$$

Solution 2:

$$a)\ \left(e^x \cdot e^{2x}\right) : \left(-e^{3x}\right) = -1$$

$$b)\ e^{2x} : \left(-e^{-2x}\right) = -e^{4x}$$

$$c)\ \left(e^{-x} \cdot e^{2x}\right) : \left(e^{0.5x}\right) = e^{0.5x}$$

$$d)\ e^{-x+1} : e^{-x} - e : e^{-x} = e\left(1 - e^x\right)$$

$$e)\ e^{x-1} \cdot e - 2e^x + \frac{1}{2e^{-x}} = -\frac{e^x}{2}$$

$$f)\ \left(e - e^{1+x}\right) : \left(1 - e^x\right) = e$$

Definition:

> An equation where the unknown appears at least once in an exponent is called an exponential equation.

Example: $3^{x} + 3^{2x} + 9^{2} = 12, \qquad a^{nx+m} = k$

There is no general strategy which automatically leads to the solution of an exponential equation. This is in contrast to linear, quadratic or radical equations, also to equations with fractions. Therefore for exponential equations, clear prescriptions can only be given for certain standard cases.

Exponential equations: Exercise 3

Which of the following equations are exponential ones ?

$a)\quad 2^{x} - 6 = 0$

$b)\quad x^{2} - 3x = 8$

$c)\quad e^{x} - x^{2} = 4x$

$d)\quad e^{x^{2}} - 3e^{-x} + 12 = 0$

$e)\quad 2^{x^{2} + 2x - 5} = 1$

$f)\quad 3^{2} + 2^{e^{2}} = x^{3} - 4x - 11$

$g)\quad \left(3^{x^{2} + x}\right)^{(x - 2)} = 1$

$a)\quad 2^{x} - 6 = 0$ – exponential equation

$b)\quad x^{2} - 3x = 8$ – polynomial equation

$c)\quad e^{x} - x^{2} = 4x$ – exponential equation

$d)\quad e^{x^{2}} - 3e^{-x} + 12 = 0$ – exponential equation

$e)\quad 2^{x^{2} + 2x - 5} = 1$ – exponential equation

$f)\quad 3^{2} + 2^{e^{2}} = x^{3} - 4x - 11$ – polynomial equation

$g)\quad \left(3^{x^{2} + x}\right)^{(x - 2)} = 1$ – exponential equation

Strategies for solving exponential equations

- If the unknown is present in only one exponent, we can solve the equation by taking the logarithm.

Example: $9^x = 27$

$$x \log 9 = \log 27 \quad \Leftrightarrow \quad x \log 3^2 = \log 3^3 \quad \Leftrightarrow$$

$$2x \log 3 = 3 \log 3 \quad \Rightarrow \quad x = \frac{3 \log 3}{2 \log 3} = \frac{3}{2}$$

- If the unknown appears in several exponents with the same base, we can apply the rules for powers and then isolate the unknown at one side of the equation. This way we arrive at the situation treated above, and we can take the logarithm.

Example: $2^{x+3} + 5 \cdot 2^{x+1} - 144 = 0$

Application of the power rules:

$$2^3 \cdot 2^x + 10 \cdot 2^x - 144 = 0 \quad \Rightarrow \quad 2^x (8 + 10) = 144 \quad \Rightarrow$$

$$2^x = 8 = 2^3 \quad \Rightarrow \quad x = 3$$

- If the unknown is present in several exponents with same base, and if the exponents are multiples of one another, then the equation may be solved by a substitution and taking the logarithm subsequently.

Example: $3^{2x} - 2 \cdot 3^{x} + 1 = 0$

Substitution $u = 3^{x}$

yields the quadratic equation

$$u^{2} - 2u + 1 = 0$$

With the solution $u = 1$, i.e. $1 = 3^{x} \Rightarrow x = 0$

- Exponential equations, which can be transformed into the form

$$a^{T_1(x)} = b^{T_2(x)}$$

can be simplified by application of the following rule

$$\log(x^n) = n \cdot \log x$$

and we obtain

$$T_1(x)\ \log a = T_2(x)\ \log b$$

If $a = b$, we simply get

$$T_1(x) = T_2(x)$$

which can also be obtained by taking the logarithm of the equation given above.

Exponential equations: Exercise 4

Solve the following equations:

$a)\ 7^{1-x} = 7^{3}$

$b)\ 3^{x} = 9$

$c)\ 3^{3x+2} = 27$

$d)\ 2^{x^2-3x} = 16$

$e)\ 2^{x+1} = \frac{1}{8}$

$f)\ 4^{2x-3} = \frac{1}{16}$

$g)\ 9^{x-2} = \sqrt{9}$

$h)\ 2^{x} = -8$

Exponential equations: Solution 4

a) $7^{1-x} = 7^3, \quad 1 - x = 3, \quad x = -2$

b) $3^x = 9, \quad 3^x = 3^2, \quad x = 2$

c) $3^{3x+2} = 27, \quad 3^{3x+2} = 3^3, \quad 3x + 2 = 3, \quad x = \frac{1}{3}$

d) $2^{x^2-3x} = 16, \quad 2^{x^2-3x} = 2^4, \quad x^2 - 3x = 4, \quad x^2 - 3x - 4 = 0$

$x_1 = -1, \quad x_2 = 4$

e) $2^{x+1} = \frac{1}{8}, \quad 2^{x+1} = 2^{-3}, \quad x+1 = -3, \quad x = -4$

f) $4^{2x-3} = \frac{1}{16}, \quad 4^{2x-3} = 4^{-2}, \quad x = \frac{1}{2}$

g) $9^{x-2} = \sqrt{9}, \quad 9^{x-2} = 9^{\frac{1}{2}}, \quad x = \frac{5}{2}$

h) $2^x = -8$ – no solution

Calculate the coordinates of the intercept of the line $y = c$ and the curve given by the function $y = f(x)$:

$a)\ \ f(x) = e^{0.5x - 1}, \qquad c = 2$

$b)\ \ f(x) = e^{x - 2}, \qquad c = 1.5$

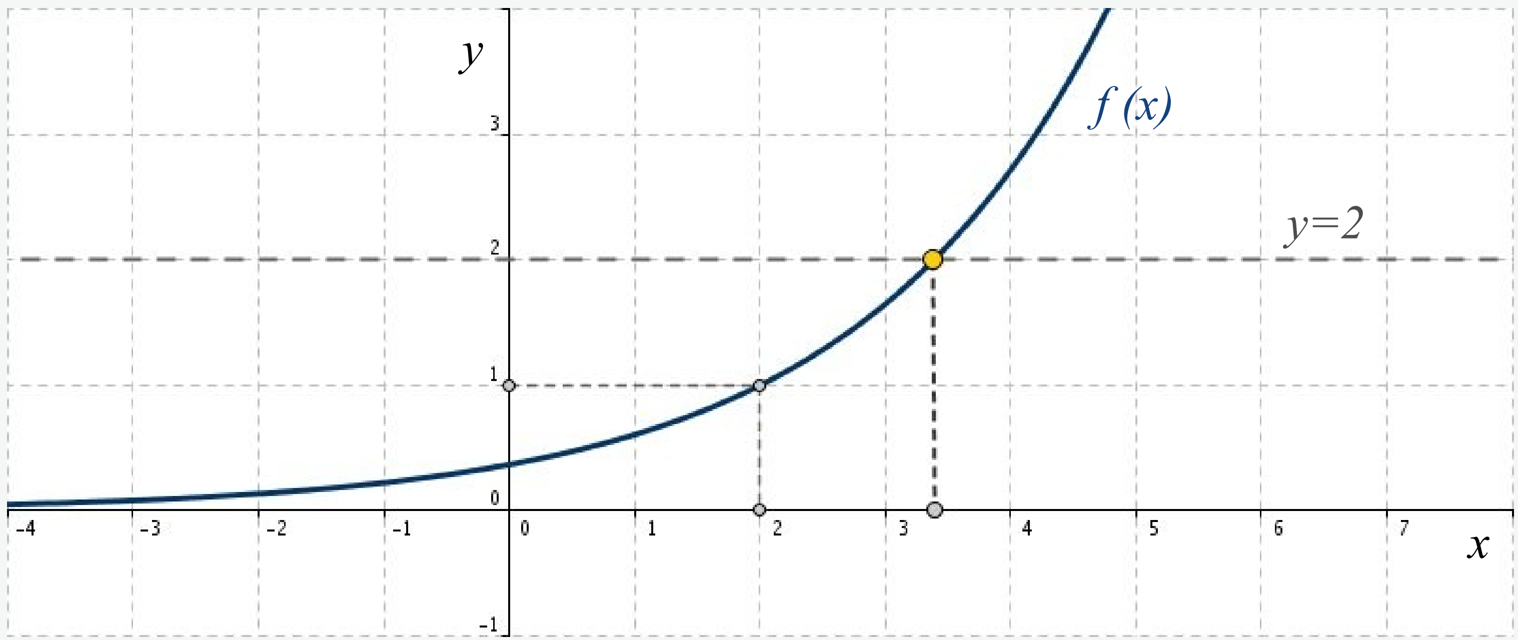

Fig. L5a: Function f (x) = exp(0.5 x - 1) and line y = 2

$$e^{0.5x-1} = 2$$

We solve this equation by taking the natural logarithm of either side of the equation.

$$\ln e^{0.5x-1} = \ln 2 \quad \Leftrightarrow \quad (0.5x - 1)\ln e = \ln 2 \quad \Leftrightarrow \quad 0.5x - 1 = \ln 2 \quad \Rightarrow$$

$$0.5x = \ln 2 + 1 \quad \Rightarrow \quad x = 2(\ln 2 + 1) \simeq 2(0.693 + 1) = 3.386$$

The line $y = 2$ and the function $y = f(x)$ intersect in the point P :

$$P = (3.386,\ 2)$$

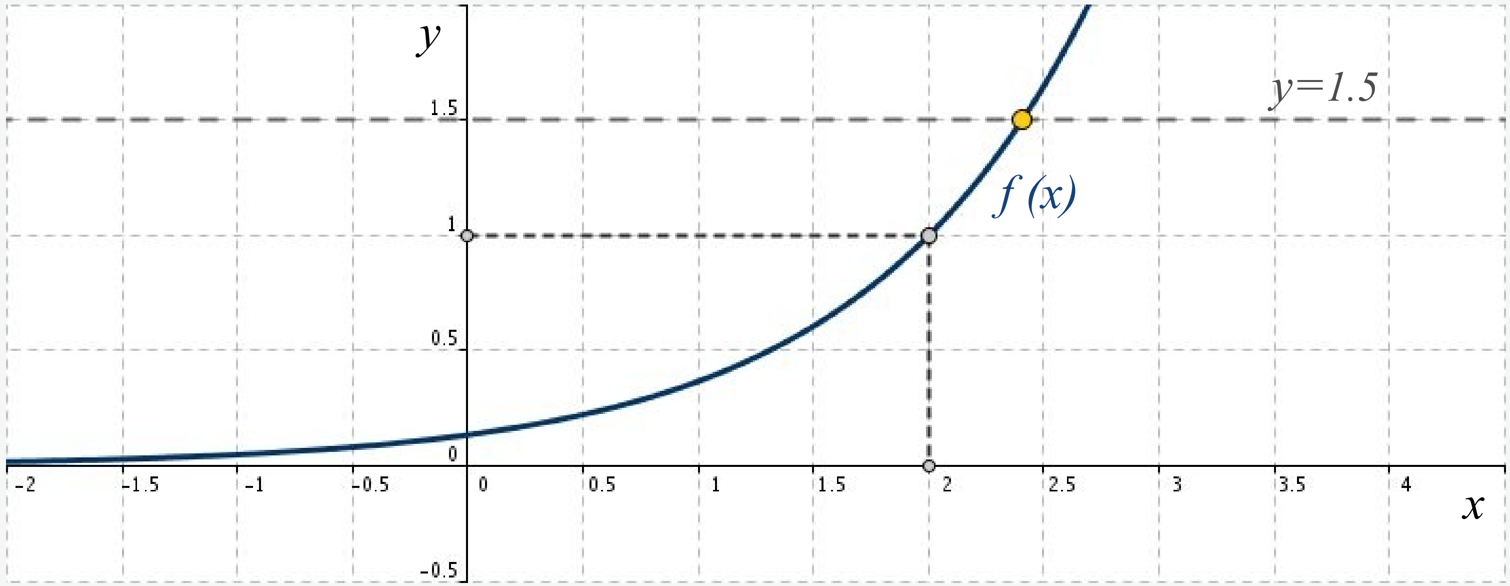

Fig. L5b: Function f (x) = exp(x - 2) and line y = 1.5

$$e^{x-2} = 1.5 \qquad | \ln$$

$$\ln e^{x-2} = \ln 1.5 \quad \Leftrightarrow \quad (x-2)\ln e = \ln 1.5 \quad \Leftrightarrow \quad x - 2 = \ln 1.5 \quad \Rightarrow$$

$$x = \ln 1.5 + 2 \simeq 2.405$$