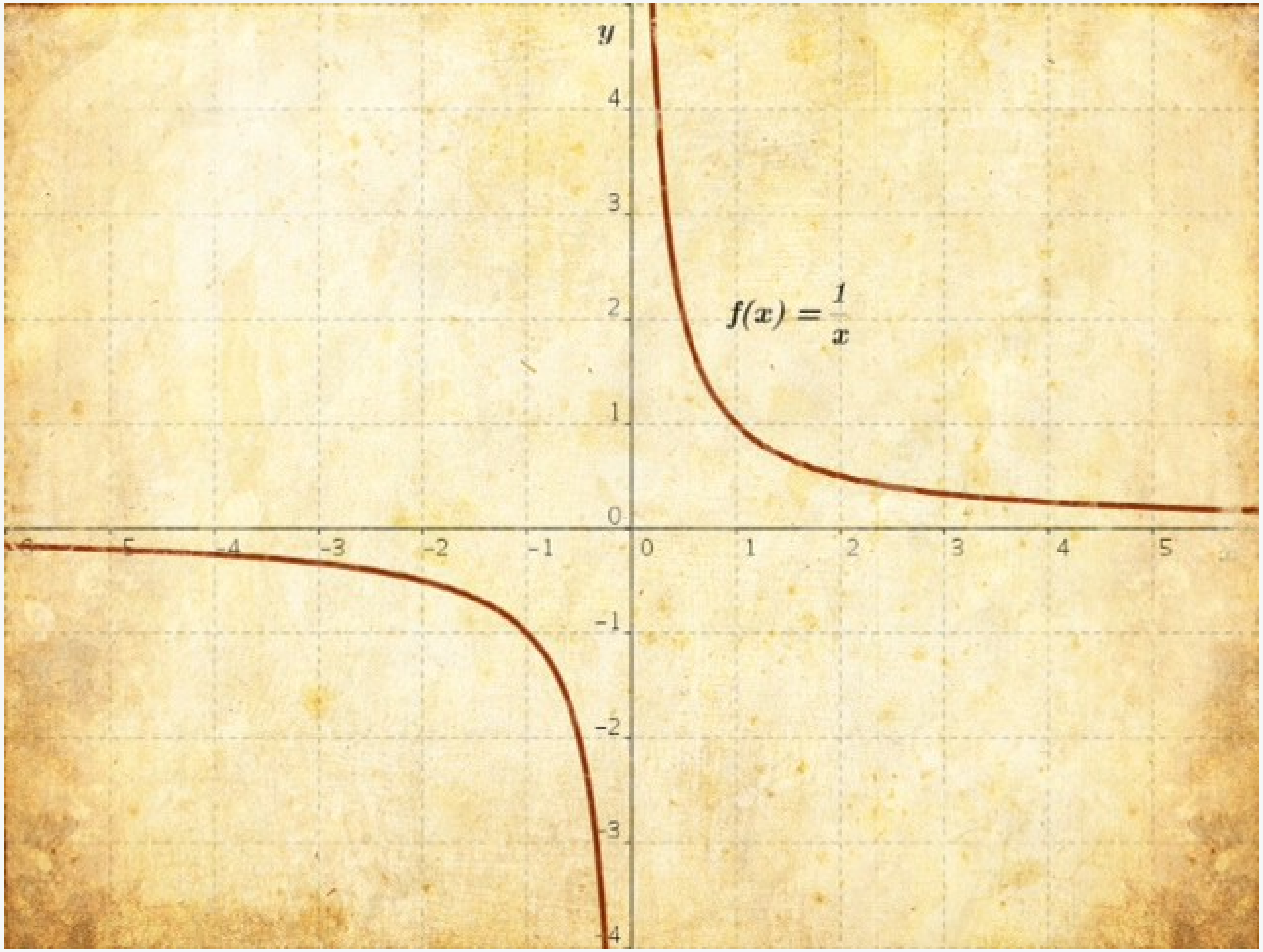
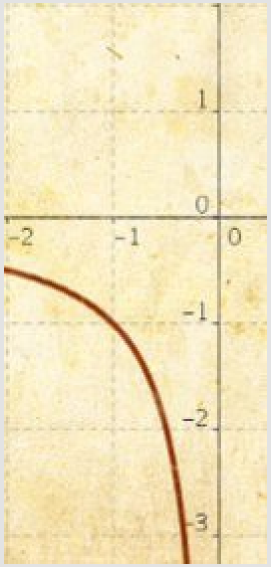


Equations with Fractions



Definition:

A fractional equation is an equation with rational functions where the unknown appears at least once in the denominator.

Which of the following equations are fractional equations?

$$a)\quad \frac{2+x}{7} = 2$$

$$b)\quad \frac{2+x}{7+x} = 2$$

$$c)\quad \frac{1-x^2}{1-x} = 2x$$

$a)\ \dfrac{2+x}{7} = 2$ – Not a fractional equation. The unknown is in the numerator only.

$b)\ \dfrac{2+x}{7+x} = 2$ – a fractional equation

$c)\ \dfrac{1-x^2}{1-x} = 2x$ – a fractional equation

Exercise 1:

The following fractional equations are to be solved

$$a)\ \frac{3}{x-2} = 1$$

$$b)\ x + \frac{1}{x} - \frac{1}{x} = 0$$

$$c)\ \frac{x}{x-1} = \frac{1}{x-1}$$

a) $\dfrac{3}{x-2} = 1, \qquad \mathbb{R} \setminus \{2\}$

$$\frac{3}{x-2} = 1 \quad | \quad \times \; (x-2)$$

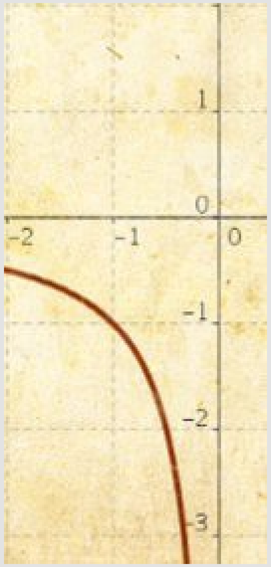
$$3 = x - 2 \quad \Rightarrow \quad x = 5$$

b) $x + \dfrac{1}{x} - \dfrac{1}{x} = 0, \qquad \mathbb{R} \setminus \{0\}$

There is no solution, because the equation is not defined at $x = 0$

c) $\dfrac{x}{x-1} = \dfrac{1}{x-1}, \qquad \mathbb{R} \setminus \{1\}$

This fractional equation has no solution, because $x = 1$ is not part of the domain of the equation.



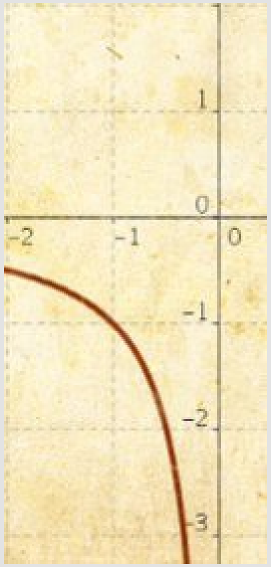
Typical Problems:

- Determination of the domain of a fractional equation
- Solution of a fractional equation and exclusion of false solutions.

Solution of fractional equations:

First the fractional equation is multiplied with the common denominator of all fractions. This leads to an equation without the unknown in the denominator which can be solved with standard methods.

Then one has to check, whether the obtained solutions of this transformed equation happen to be zeros of a denominator of the fractional equation. In this case we have found a pseudo-solution which is not element of the set of solutions of the original fractional equation.



Determine the solutions of the following fractional equations:

Exercise 2: $x + \frac{2}{x} = 3$

Exercise 3: $\frac{x - 1}{x + 3} = \frac{2x}{2x - 1}$

Exercise 4: $\frac{x^2 - 1}{x - 1} = 2x$

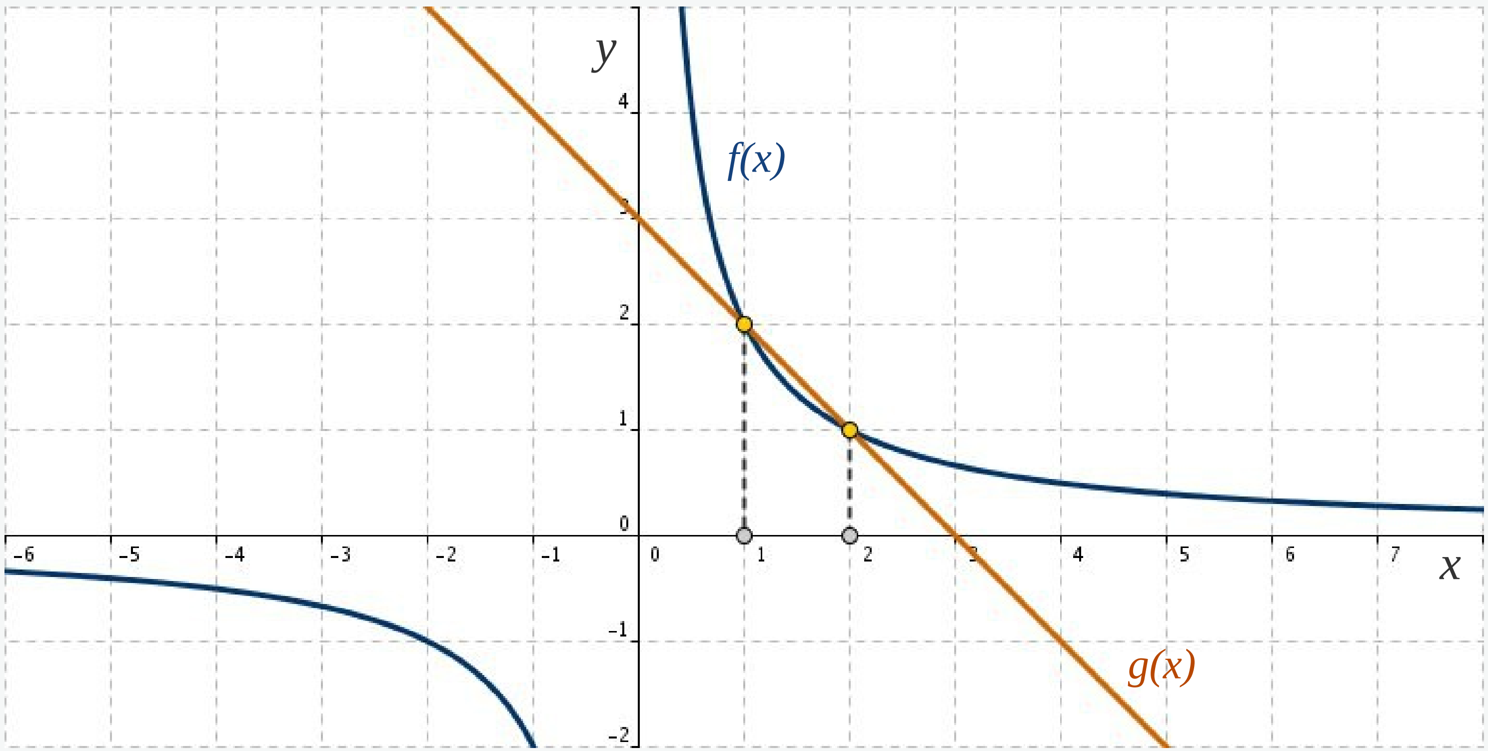


Fig. 1: Functions f (x) = 2/x and g (x) = 3 - x

$$\mathbb{R} \setminus \{ 0 \} : \quad x + \frac{2}{x} = 3 \quad | \quad (\times \quad x)$$

$$x^2 - 3x + 2 = 0 \quad \Rightarrow \quad x_1 = 1, \quad x_2 = 2$$

Fractional equations: Solution 3

$$E: \qquad \frac{x-1}{x+3} = \frac{2x}{2x-1}, \qquad \mathbb{R} \setminus \left\{-3, \ \frac{1}{2}\right\}$$

Multiplication with the common denominator $(x+3)(2x-1)$ leads to the equation

$$(x-1)(2x-1) = 2x\,(x+3)$$

Expanding the equation and collection of the terms with x leads to a linear equation with the solution

$$x = \frac{1}{9} \in D(E)$$

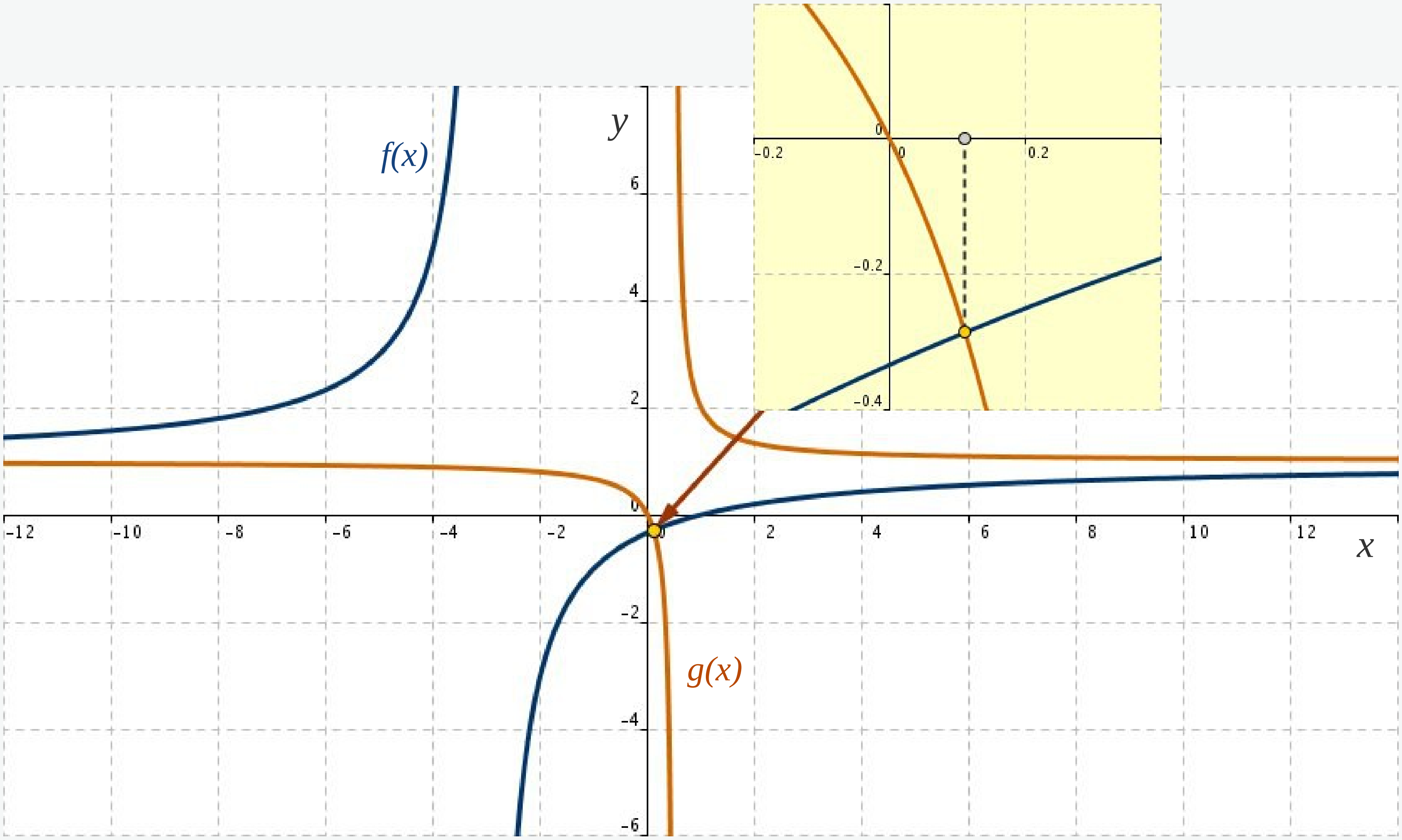


Fig. 2: Functions $f(x)$ and $g(x)$

$$f(x) = \frac{x-1}{x+3}, \qquad g(x) = \frac{2x}{2x-1}$$

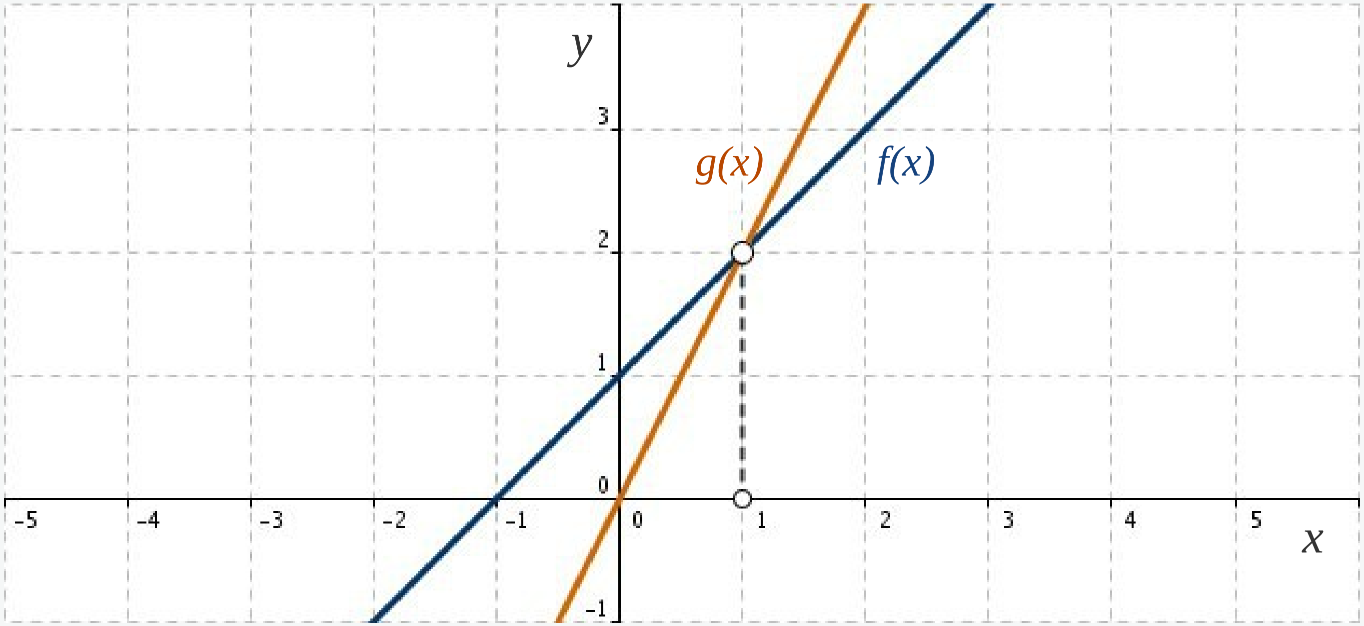


Fig. 3: Functions $f(x)$ and $g(x)$

$$f(x) = x + 1 \quad (x = 1 \notin \mathbb{R}), \qquad g(x) = 2x$$

$$\frac{x^2 - 1}{x - 1} = 2x, \qquad \mathbb{R} \setminus \{1\}$$

$$\frac{(x-1)(x+1)}{x-1} = 2x, \qquad x + 1 = 2x \quad \Rightarrow \quad x = 1$$

There is no solution of the fractional equation, because the equation is not defined at $x = 1$.

Exercise 5: $\dfrac{3}{x-2} + 5\left(x+1\right) = \dfrac{x+1}{x-2}$

Exercise 6: $\dfrac{x+1}{x-2} - \dfrac{x-3}{x+2} = \dfrac{12}{x^2-4}$

Exercise 7: $\dfrac{12x}{x^2+x-6} - 2 = \dfrac{x+1}{x-2}$

1. $E: \quad \frac{3}{x-2} + 5\,(x+1) = \frac{x+1}{x-2}$

2. Domain of the equation $\quad D(E) = \mathbb{R} \setminus \{2\}$

3. Multiplication by: $\quad x - 2$

   E and $\tilde{E}$ are not equivalent:

4. $\tilde{E}: \quad 3 + 5(x+1)\,(x-2) = x + 1$

5. $D(\tilde{E}) = \mathbb{R}, \quad D(E) \neq D(\tilde{E}), \quad D(E) \subset D(\tilde{E})$

6. $5x^2 - 6x - 8 = 0$ (expansion of $\tilde{E}$)

7. Solution of a quadratic equation $\quad ax^2 + bx + c = 0$

   $x_1 = 2, \quad x_2 = -\frac{4}{5} \quad S(\tilde{E}) = \left\{-\frac{4}{5}, 2\right\}$

8. check whether $S(\tilde{E}) = \left\{-\frac{4}{5}, 2\right\} \in D(E) \qquad 2 \notin D(E)$

9. $S(E) = \left\{-\frac{4}{5}\right\}$

Fractional equations: Solution 6

1. $E: \quad \frac{x+1}{x-2} - \frac{x-3}{x+2} = \frac{12}{x^2-4} \quad \Leftrightarrow$

$$\frac{x+1}{x-2} - \frac{x-3}{x+2} = \frac{12}{(x-2)(x+2)}$$

2. Domain $\quad D(E) = \mathbb{R} \setminus \{-2, 2\}$

3. Common denominator $(x-2)(x+2)$

4. Multiplication by common denominator:

   Transformation to non-equivalent equation:

   $\tilde{E}: \quad (x+1)(x+2) - (x-3)(x-2) = 12 \quad \Leftrightarrow$

   $8x - 4 = 12 \quad \Leftrightarrow \quad x = 2$

5. $x = 2 \notin D(E)$

   There is no solution

1. $E: \quad \dfrac{12x}{x^2 + x - 6} - 2 = \dfrac{x+1}{x-2}$

2. Domain:

$$x^2 + x - 6 = (x+3)(x-2) \neq 0$$

$$\frac{12x}{x^2 + x - 6} - 2 = \frac{x+1}{x-2} \quad \Leftrightarrow \quad \frac{12x}{(x+3)(x-2)} - 2 = \frac{x+1}{x-2}$$

$$D = \mathbb{R} \setminus \{-3,\ 2\}$$

3. Multiplication by: $\quad (x+3)(x-2)$

$$12x - 2(x^2 + x - 6) = (x+1)(x+3) \quad \Leftrightarrow$$

$$3x^2 - 6x - 9 = 0 \quad \Leftrightarrow \quad x^2 - 2x - 3 = 0$$

$$x_{1,2} = 1 \pm \sqrt{1+3} = 1 \pm 2, \qquad x_1 = 3, \quad x_2 = -1$$

4. Check: – non-equivalent equations !

$$x_1: \quad \frac{36}{6} - 2 = 4 \quad \Leftrightarrow \quad 4 = 4, \qquad x_2: \quad \frac{-12}{-6} - 2 = \frac{0}{-3} \quad \Leftrightarrow \quad 0 = 0$$

5. Set of solutions $\quad S = \{-1,\ 3\}$