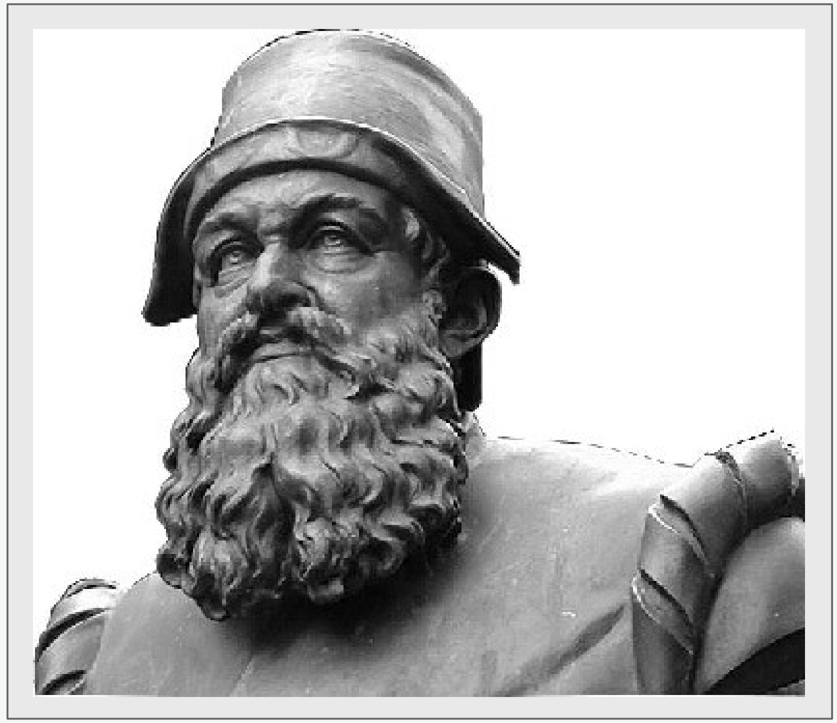
Fractions



Adam Ries (1492-1559), a German arithmetician

Adam Ries is generally seen as "father of modern arithmetic". In his time calculations with fractions were considered as very difficult and very few were able to do such calculations.

To reduce a fraction, we divide numerator and denominator by the same term which is unequal zero.

Numerator and denominator are factorised as much as possible to prepare the reduction. Factors which then appear in the numerator and also in the denominator are removed by the reduction.

$$\frac{6\,x\,y}{3\,y} = \frac{2\,x \cdot 3\,y}{3\,y} = 2\,x$$

$$\frac{a^2 - b^2}{a\,(a + b)} = \frac{(a - b)\,(a + b)}{a\,(a + b)} = \frac{a - b}{a}$$

Please note, that a reduced fraction may have a different domain than the original fraction. In the second example above the left side is not defined for $a + b = 0$ in contrast to the right side.

Reduce the fractions below as much as possible:

a) $\dfrac{a^2 b^3 c}{a b^2 c}$

b) $\dfrac{3 + x}{9 - x^2}$

c) $\dfrac{x y + x^2}{a x - x^3}$

d) $\dfrac{a^2 + 2 a b + b^2}{a c + b c}$

e) $\dfrac{x y}{x^2 - 2 x}$

f) $\dfrac{a^2 + 2 a b}{a b c}$

a) $\dfrac{a^2 b^3 c}{a b^2 c} = a b$

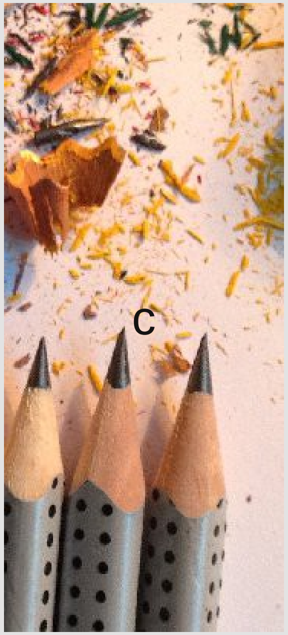
b) $\dfrac{3 + x}{9 - x^2} = \dfrac{3 + x}{(3 + x)(3 - x)} = \dfrac{1}{3 - x}$

c) $\dfrac{x y + x^2}{a x - x^3} = \dfrac{x (y + x)}{x (a - x^2)} = \dfrac{y + x}{a - x^2}$

d) $\dfrac{a^2 + 2 a b + b^2}{a c + b c} = \dfrac{(a + b)^2}{c (a + b)} = \dfrac{a + b}{c}$

e) $\dfrac{x y}{x^2 - 2 x} = \dfrac{y}{x - 2}$

f) $\dfrac{a^2 + 2 a b}{a b c} = \dfrac{a + 2 b}{b c}$



Reduce the fractions below as much as possible:

a) $\dfrac{x^2 - y^2}{y + x}$

b) $\dfrac{(7x - 3y)^2}{49x^2 - 9y^2}$

c) $\dfrac{7x^3 - x^2 y}{7xy^2 - y^3}$

d) $\dfrac{(x - y)^2}{7xy - 7x^2}$

e) $\dfrac{x^2 - 6x + 9}{x^2 - 9}$

f) $\dfrac{x^2 - 8x + 16}{2x^2 - 32}$

a) $\dfrac{x^2 - y^2}{y + x} = x - y$

b) $\dfrac{(7x - 3y)^2}{49x^2 - 9y^2} = \dfrac{(7x - 3y)^2}{(7x)^2 - (3y)^2} = \dfrac{7x - 3y}{7x + 3y}$

c) $\dfrac{7x^3 - x^2 y}{7xy^2 - y^3} = \dfrac{x^2}{y^2}$, d) $\dfrac{(x - y)^2}{7xy - 7x^2} = \dfrac{y - x}{7x}$

c) $\dfrac{x^2 - 6x + 9}{x^2 - 9} = \dfrac{(x - 3)^2}{x^2 - 3^2} = \dfrac{(x - 3)^2}{(x - 3)(x + 3)} = \dfrac{x - 3}{x + 3}$

f) $\dfrac{x^2 - 8x + 16}{2x^2 - 32} = \dfrac{(x - 4)^2}{2(x^2 - 16)} = \dfrac{(x - 4)^2}{2(x^2 - 4^2)} =$

$= \dfrac{(x - 4)^2}{2(x - 4)(x + 4)} = \dfrac{x - 4}{2(x + 4)}$

Fractions are multiplied by multiplying the numerators and the denominators.

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a\,c}{b\,d}$$

A fraction is divided by another fraction by multiplying the first fraction by the reciprocal of the second.

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a\,d}{b\,c}$$

Fractions with the same denominator are added (subtracted) by division of the sum (difference) of the numerators by the common denominator.

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}, \qquad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

Fractions with different denominators must be converted to fractions with similar denominators. Then having a common denominator they can be added or subtracted as described above.

$$\frac{a}{b} + \frac{c}{d} = \frac{a\,d}{b\,d} + \frac{b\,c}{b\,d} = \frac{a\,d + b\,c}{b\,d}$$

The least common denominator is the least common multiple of the two denominators.

$$\frac{5a+b}{5a-b} - \frac{20ab}{25a^2-b^2} - \frac{5a-b}{5a+b} = \rightarrow$$

<u>Common denominator:</u> $25a^2 - b^2 = (5a+b)(5a-b)$

$5a+b$ is the converting factor of the fraction $\frac{5a+b}{5a-b}$

$5a-b$ is the converting factor of the fraction $\frac{5a-b}{5a+b}$

$$\rightarrow = \frac{(5a+b)^2}{(5a-b)(5a+b)} - \frac{20ab}{(5a+b)(5a-b)} - \frac{(5a-b)^2}{(5a+b)(5a-b)} =$$

common denominator

$$= \frac{(5a+b)^2 - 20ab - (5a-b)^2}{(5a-b)(5a+b)} = \frac{0}{(5a-b)(5a+b)} = 0$$

Add the fractions. Simplify the results as much as possible.

a) $\dfrac{a}{x} - \dfrac{a}{x - y}$

b) $\dfrac{x}{x + y} + \dfrac{y}{x - y}$

c) $\dfrac{x}{y - x} + \dfrac{2x}{x + y} + \dfrac{y}{x - y}$

d) $\dfrac{1}{y + 1} + \dfrac{1}{y + 2} + \dfrac{1}{y + 3}$

e) $\dfrac{x - y}{x + y} + \dfrac{x + y}{x - y}$

f) $\dfrac{1 - x}{1 + x} - \dfrac{1 + x}{1 - x} + \dfrac{4x}{1 - x^2}$

a) $$\frac{a}{x} - \frac{a}{x-y} = -\frac{a\,y}{x\,(x-y)}$$

b) $$\frac{x}{x+y} + \frac{y}{x-y} = \frac{x^2+y^2}{(x+y)(x-y)} = \frac{x^2+y^2}{x^2-y^2}$$

c) $$\frac{x}{y-x} + \frac{2\,x}{x+y} + \frac{y}{x-y} = \frac{x-y}{x+y}$$

d) $$\frac{1}{y+1} + \frac{1}{y+2} + \frac{1}{y+3} = \frac{3\,y^2+12\,y+11}{(y+1)(y+2)(y+3)}$$

e) $$\frac{x-y}{x+y} + \frac{x+y}{x-y} = \frac{2\,(x^2+y^2)}{x^2-y^2}$$

f) $$\frac{1-x}{1+x} - \frac{1+x}{1-x} + \frac{4\,x}{1-x^2} = \frac{1-x}{1+x} - \frac{1+x}{1-x} + \frac{4\,x}{(1-x)\,(1+x)} =$$

$$= \frac{1-x}{1+x} - \frac{1+x}{1-x} + \frac{4\,x}{(1-x)\,(1+x)} =$$

$$= \frac{(1-x)^2-(1+x)^2+4\,x}{(1-x)\,(1+x)} = 0$$

Simplify as much as possible

a) $\frac{1}{a} + \frac{2}{b} + \frac{1}{c}$

b) $\frac{1}{a\,b} + \frac{1}{a\,c} + \frac{1}{b\,c}$

c) $\frac{b - 4\,a}{a + b} + \frac{2\,a^2 + 5\,a\,b + 3\,b^2}{(a + b)^2} - 1$

d) $\frac{a}{b^2\,c} + \frac{c}{a\,b^2} - \frac{2}{b^2}$

e) $\frac{3}{a - 1} + \frac{6}{1 - a^2} - \frac{5}{a + 1}$

a) $\frac{1}{a} + \frac{2}{b} + \frac{1}{c} = \frac{bc + 2ac + ab}{abc}$

b) $\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} = \frac{a + b + c}{abc}$

c) $\frac{b - 4a}{a + b} + \frac{2a^2 + 5ab + 3b^2}{(a + b)^2} - 1 = -\frac{3(a - b)}{a + b}$

d) $\frac{a}{b^2 c} + \frac{c}{ab^2} - \frac{2}{b^2} = \frac{(a - c)^2}{ab^2 c}$

e) $\frac{3}{a - 1} + \frac{6}{1 - a^2} - \frac{5}{a + 1} = -\frac{2}{a + 1}$

Simplify as much as possible

a) $\dfrac{a}{x^2 - a^2} - \dfrac{x + a}{x - a} + \dfrac{x - a}{x + a}$

b) $\dfrac{a}{x^2 - a^2} - \dfrac{2x}{x - a} - \dfrac{a - x}{x + a}$

c) $\dfrac{1}{x} - \dfrac{1}{y} + \dfrac{x - y}{x y} - \dfrac{(x + y)^2}{x y}$

d) $\dfrac{1 + x}{1 - x} - \dfrac{1 - x}{1 + x} + \dfrac{2x}{1 - x^2}$

e) $\dfrac{4x + 1}{x - 1} - \dfrac{3x - 2}{x + 2} - \dfrac{x(x + 14)}{(x - 1)(x + 2)}$

f) $\dfrac{9x + 5}{x^2 - 1} - \dfrac{7x + 4}{x^2 - x} - \dfrac{2(x - 2)}{x^2 + x} + 1$

a) $$\frac{a}{x^2 - a^2} - \frac{x + a}{x - a} + \frac{x - a}{x + a} = \frac{a(4x - 1)}{(x + a)(x - a)}$$

b) $$\frac{a}{x^2 - a^2} - \frac{2x}{x - a} - \frac{a - x}{x + a} = \frac{a + a^2 - 4ax - x^2}{(x + a)(x - a)}$$

c) $$\frac{1}{x} - \frac{1}{y} + \frac{x - y}{xy} - \frac{(x + y)^2}{xy} = -\frac{(x + y)^2}{xy}$$

d) $$\frac{1 + x}{1 - x} - \frac{1 - x}{1 + x} + \frac{2x}{1 - x^2} = \frac{6x}{1 - x^2}$$

e) $$\frac{4x + 1}{x - 1} - \frac{3x - 2}{x + 2} - \frac{x(x + 14)}{(x - 1)(x + 2)} = 0$$

f) $$\frac{9x + 5}{x^2 - 1} - \frac{7x + 4}{x^2 - x} - \frac{2(x - 2)}{x^2 + x} + 1 = \frac{x^3 - x - 8}{x^3 - x}$$

Remove the roots in the denominators of the following fractions:

Example:

$$\frac{1-\sqrt{2}}{1+\sqrt{2}} = \frac{\left(1-\sqrt{2}\right)^2}{\left(1+\sqrt{2}\right)\left(1-\sqrt{2}\right)} = \frac{1-2\sqrt{2}+\left(\sqrt{2}\right)^2}{1-\left(\sqrt{2}\right)^2} =$$

$$= \frac{3-2\sqrt{2}}{-1} = 2\sqrt{2}-3$$

a) $\dfrac{2+\sqrt{2}}{\sqrt{2}}$

b) $\dfrac{a+b}{\sqrt{a}-\sqrt{b}}$ $\quad (a\,,\ b \geqslant 0, \quad a \neq b)$

c) $\dfrac{1+\sqrt{3}}{2\sqrt{3}+3\sqrt{2}}$, $\quad d$) $\dfrac{1+\sqrt{2}-\sqrt{3}}{1-\sqrt{2}+\sqrt{6}}$

e) $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{8}}$

$$a)\quad \frac{2+\sqrt{2}}{\sqrt{2}} = \frac{(2+\sqrt{2})\sqrt{2}}{2} = 1+\sqrt{2}$$

$$b)\quad \frac{a+b}{\sqrt{a}-\sqrt{b}} = \frac{(a+b)(\sqrt{a}+\sqrt{b})}{a-b}$$

$$c)\quad \frac{1+\sqrt{3}}{2\sqrt{3}+3\sqrt{2}} = \frac{(1+\sqrt{3})(2\sqrt{3}-3\sqrt{2})}{(2\sqrt{3}+3\sqrt{2})(2\sqrt{3}-3\sqrt{2})} =$$

$$= -1+\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{3}+\frac{\sqrt{6}}{2}$$

$$d)\quad \frac{1+\sqrt{2}-\sqrt{3}}{1-\sqrt{2}+\sqrt{6}} = 15-11\sqrt{2}+9\sqrt{3}-6\sqrt{6}$$

Hint:

$$\frac{1+\sqrt{2}-\sqrt{3}}{1-\sqrt{2}+\sqrt{6}} = \frac{(1+\sqrt{2}-\sqrt{3})\,(1-\sqrt{2}-\sqrt{6})}{(1-\sqrt{2})^2-6}$$

$$e)\quad \frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{8}} = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}+\sqrt{3}-2\sqrt{2}} = 1$$

Remove the roots in the denominators of the following fractions:

$$a)\quad \frac{2\sqrt{3}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$$

$$b)\quad \frac{1}{2+\sqrt{2}+\sqrt{3}+\sqrt{6}}$$

$$a)\ \frac{2\sqrt{3}}{\sqrt{2}+\sqrt{3}+\sqrt{5}} = \frac{2\sqrt{3}\,(\sqrt{2}+\sqrt{3}-\sqrt{5})}{(\sqrt{2}+\sqrt{3}+\sqrt{5})\,(\sqrt{2}+\sqrt{3}-\sqrt{5})} =$$

$$= \frac{2\sqrt{3}\,(\sqrt{2}+\sqrt{3}-\sqrt{5})}{(\sqrt{2}+\sqrt{3})^2-(\sqrt{5})^2} = \frac{2\sqrt{3}\,(\sqrt{2}+\sqrt{3}-\sqrt{5})}{(\sqrt{2})^2+2\sqrt{2}\cdot\sqrt{3}+(\sqrt{3})^2-(\sqrt{5})^2} =$$

$$= \frac{1}{\sqrt{2}}\,(\sqrt{2}+\sqrt{3}-\sqrt{5}) = \frac{1}{2}\,(2+\sqrt{6}-\sqrt{10})$$

$$b)\ \frac{1}{2+\sqrt{2}+\sqrt{3}+\sqrt{6}} = \frac{2+\sqrt{2}-(\sqrt{3}+\sqrt{6})}{(2+\sqrt{2}+\sqrt{3}+\sqrt{6})\,(2+\sqrt{2}-(\sqrt{3}+\sqrt{6}))} =$$

$$= \sqrt{6}+\sqrt{2}-\sqrt{3}-2$$

Remove the roots in the denominators of the following fractions:

$$a)\quad \frac{\sqrt{\sqrt{a}+\sqrt{b}}}{\sqrt{\sqrt{a}-\sqrt{b}}}$$

$$b)\quad \frac{\sqrt{\sqrt{a}-\sqrt{b}}}{\sqrt{\sqrt{a}+\sqrt{b}}}$$

$$c)\quad \frac{\sqrt{2\sqrt{3}+\sqrt{2}}}{\sqrt{2\sqrt{3}-\sqrt{2}}}$$

$$a)\ \frac{\sqrt{\sqrt{a}+\sqrt{b}}}{\sqrt{\sqrt{a}-\sqrt{b}}} = \frac{\sqrt{\sqrt{a}+\sqrt{b}}}{\sqrt{\sqrt{a}-\sqrt{b}}} \cdot \frac{\sqrt{\sqrt{a}-\sqrt{b}}}{\sqrt{\sqrt{a}-\sqrt{b}}} = \frac{\sqrt{a-b}}{\sqrt{a}-\sqrt{b}} =$$

$$= \frac{\sqrt{a-b}}{\sqrt{a}-\sqrt{b}} \cdot \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{a-b}\cdot(\sqrt{a}+\sqrt{b})}{a-b}$$

$$b)\ \frac{\sqrt{\sqrt{a}-\sqrt{b}}}{\sqrt{\sqrt{a}+\sqrt{b}}} = \frac{\sqrt{a-b}\cdot(\sqrt{a}-\sqrt{b})}{a-b}$$

$$c)\ \frac{\sqrt{2\sqrt{3}+\sqrt{2}}}{\sqrt{2\sqrt{3}-\sqrt{2}}} = \frac{\sqrt{5}\,(\sqrt{6}+1)}{5}$$