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Different Representations of functions

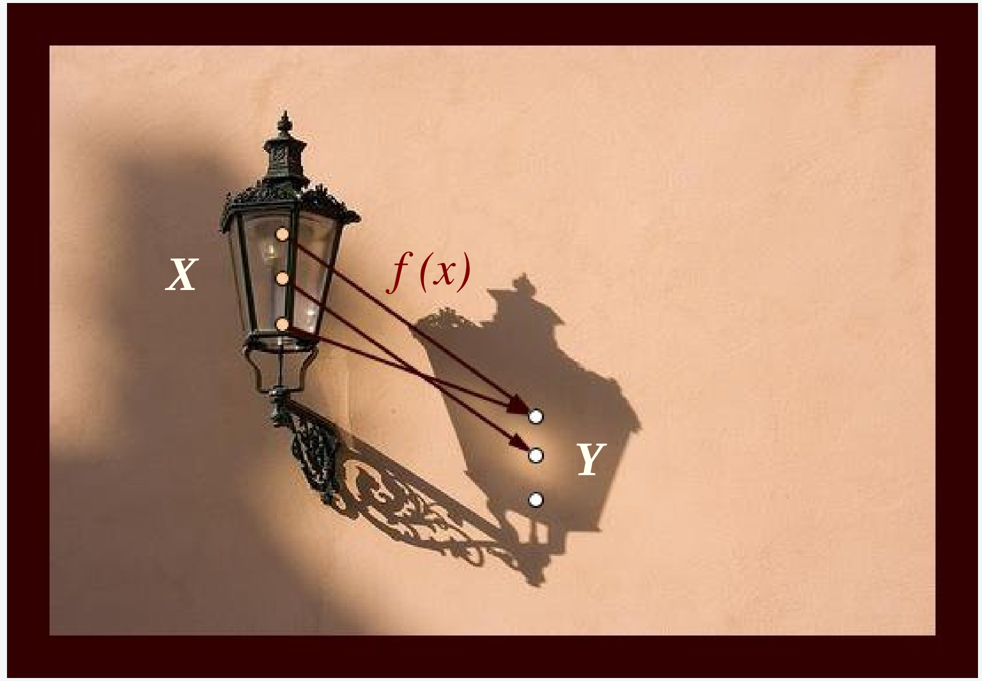

Fig. 1: Concept of a function f (x): assignment of elements of a set to those of another set

Often it occurs, that different variables are connected by an equation, which is not of the type $y = f(x)$. One may then ask, what are the conditions, that such an equation defines a function by mapping one variable to another? We say in such cases, that the function is defined implicitly by the equation.

A function is declared implicitly, if the specifying function is not solved for one of the variables x and y. For example

$$2x - y + 1 = 0$$

is an implicit linear function of y.

However, the equation

$$y^2 - x - 2 = 0$$

is not equivalent to an equation $y = f(x)$, because solving this equation for the dependent variable y does not lead to a unique assignment :

$$y = \pm\sqrt{x + 2}$$

We will study this example in more detail.

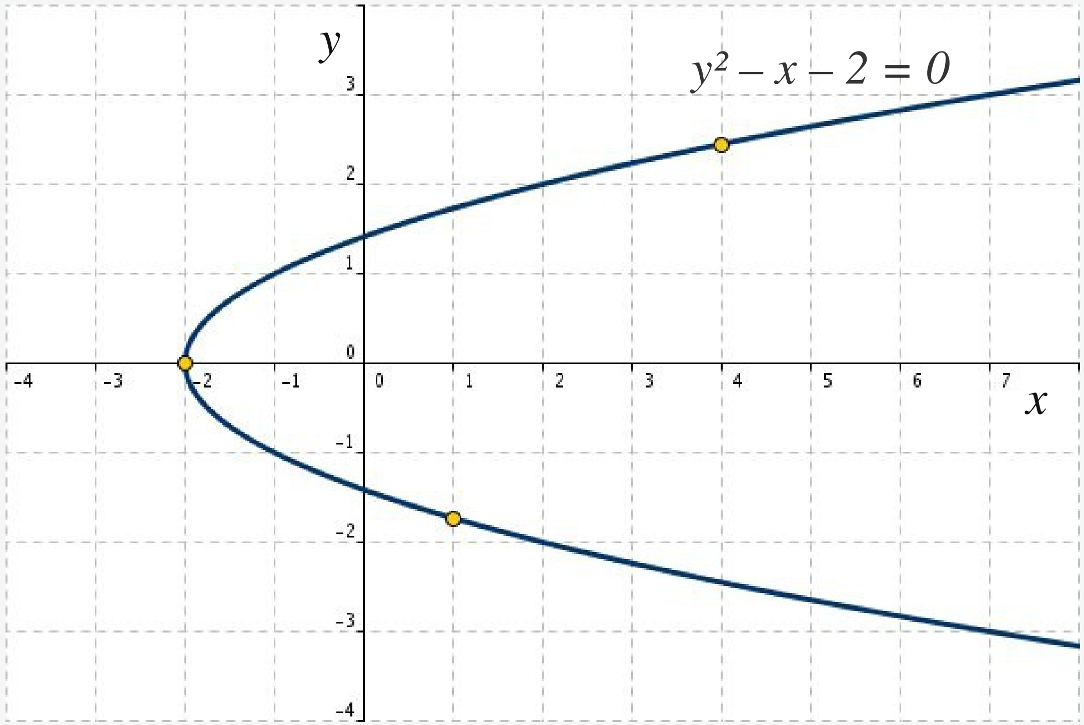

Fig. 2-1: Graphical representation of the function which is given implicitly by the equation $F(x, y) = y^2 - x - 2 = 0$.

The equation $y^2 - x - 2 = 0$ defines a subset of all the points of the x,y-plane which satisfy this equation. Examples are the points $(-2, 0)$, $(4, \sqrt{6})$ oder $(1, -\sqrt{3})$.

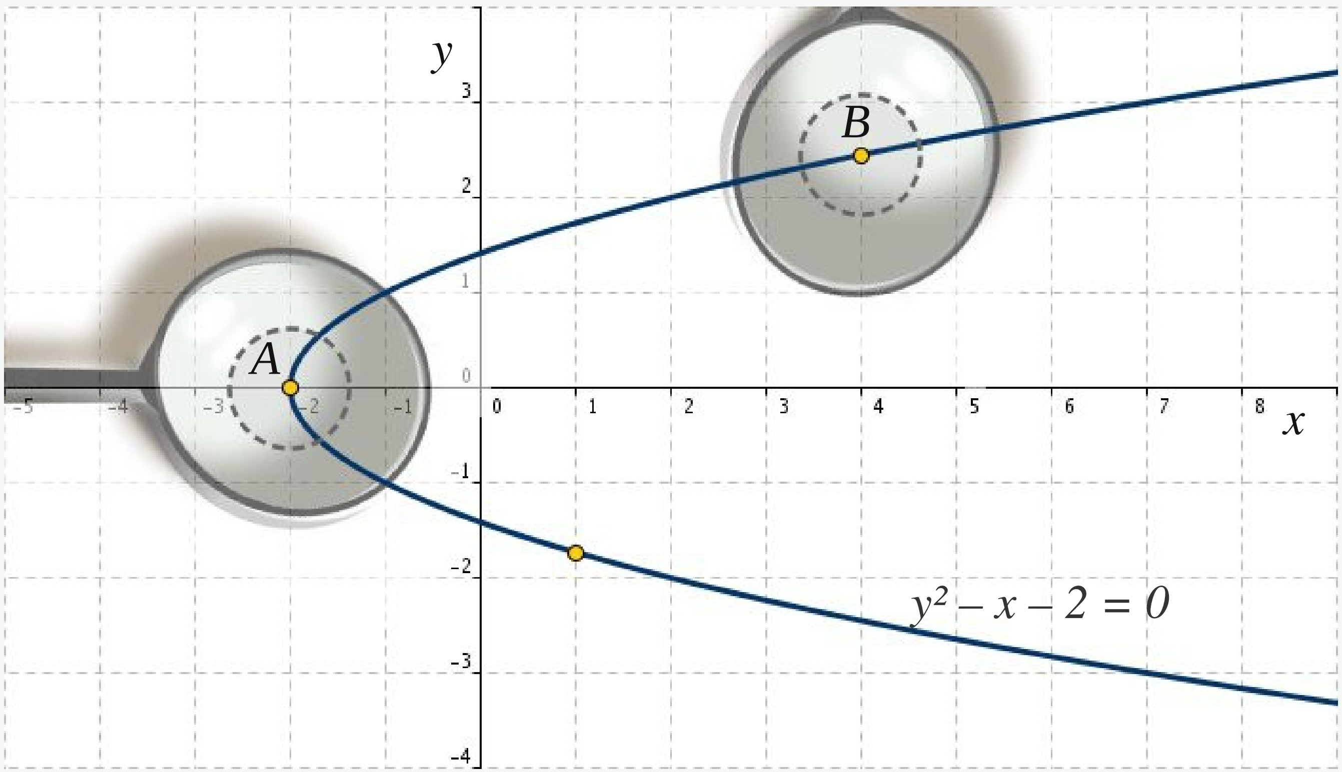

Fig. 2-2: The two marked regions of the graph of the implicitly given function $F(x, y) = y^2 - x - 2 = 0$ will be enlarged.

In the following the regions with centers A and B will be enlarged and analysed.

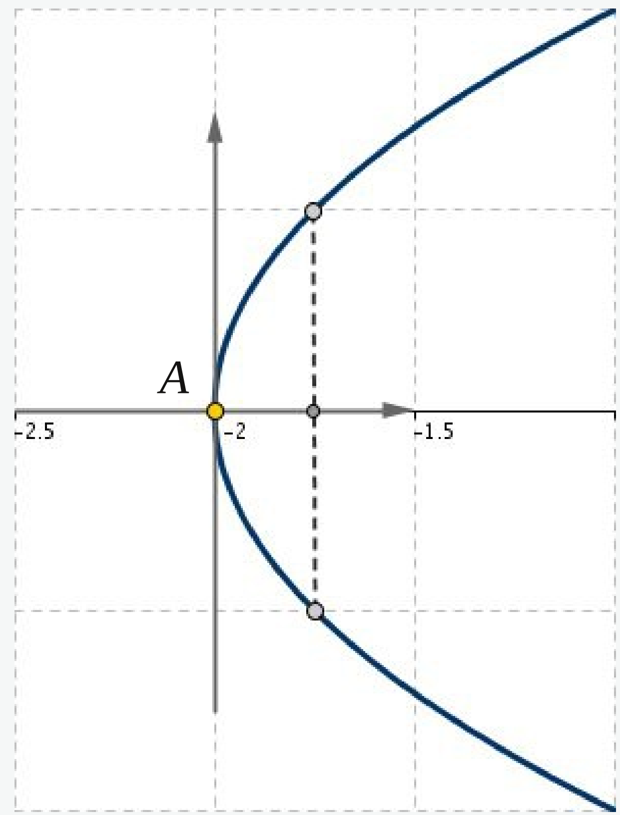

Fig. 2-3: The enlarged region of $F(x, y) = y^2 - x - 2 = 0$ in the neighbourhood of point A.

We consider the neighbourhood of point $A = (-2, 0)$. Clearly, here we deal not with a function $y = f(x)$. Each value of x is assigned to two different values of y.

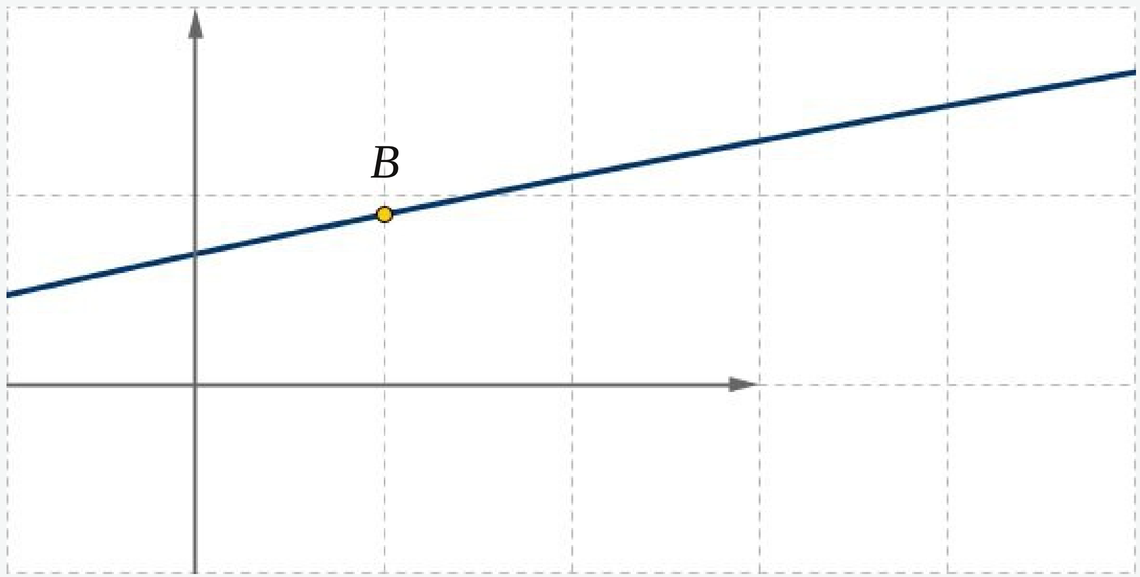

Fig. 2-4: The enlarged region of $F(x, y) = y^2 - x - 2 = 0$ in the neighbourhood of point B.

Looking at the neighbourhood of the point $B = (4, \sqrt{6})$, one might think to see a graph of a function $y = f(x)$.

Implicitly declared functions and maps are generally given by equations of the type

$$F(x, y) = 0$$

Examples:

Circle: $F(x, y) = x^2 + y^2 - 4 = 0$

Ellipse: $F(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$

Lemniscate: $F(x, y) = (x^2 + y^2)^2 - 2a^2(x^2 - y^2) = 0$

Jakob Bernoulli
(1655-1705)

The curve of the lemniscate is known as symbol for infinity. But the term "lemniscate" usually does not refer to infinity, but to geometrically defined curves with that name. Best known among these is the lemniscate of Bernoulli.

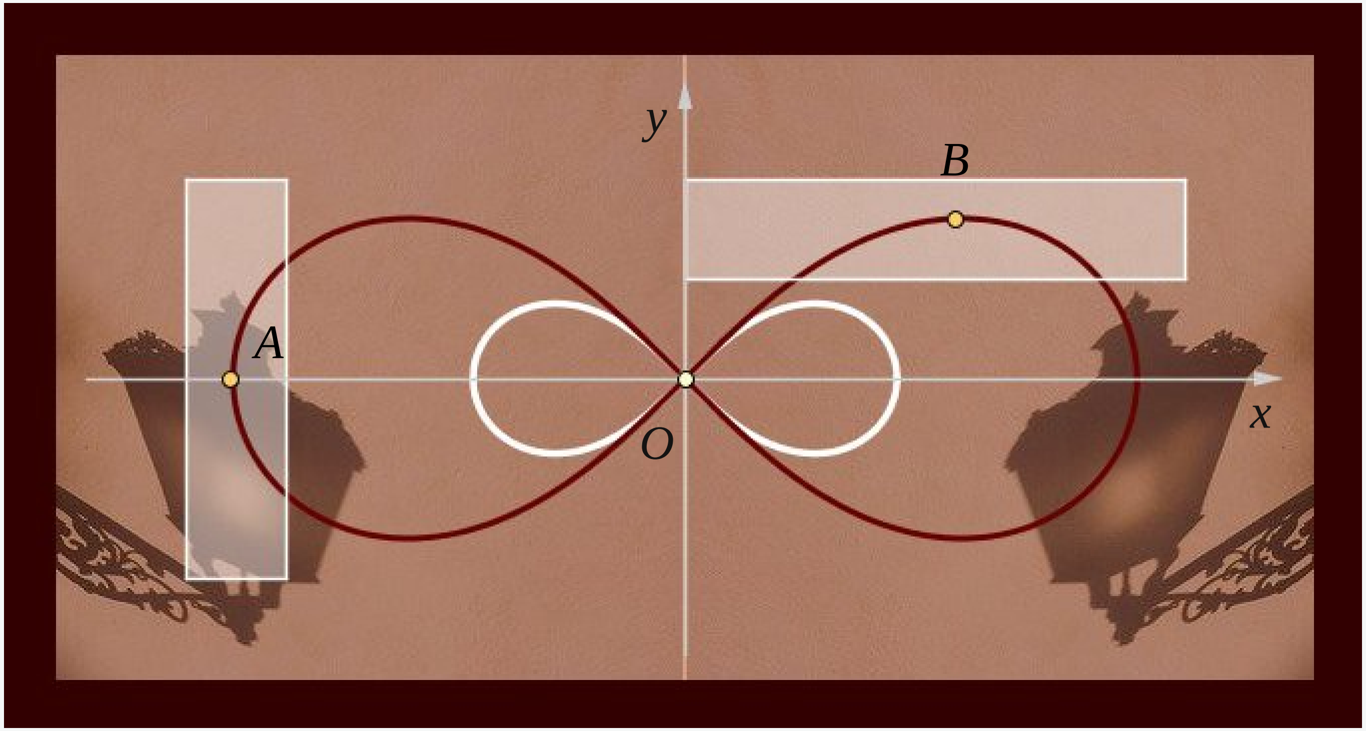

Fig. 3: Representation of a lemniscate

The lemniscate of Bernoulli is a special plane, algebraic curve of 4th order described by the equations:

Cartesian coordianates: $(x^2 + y^2)^2 = 2a^2 (x^2 - y^2)$

Parametric equation:

$$x = a \cos t \sqrt{2 \cos(2t)}, \quad y = a \sin t \sqrt{2 \cos(2t)}$$

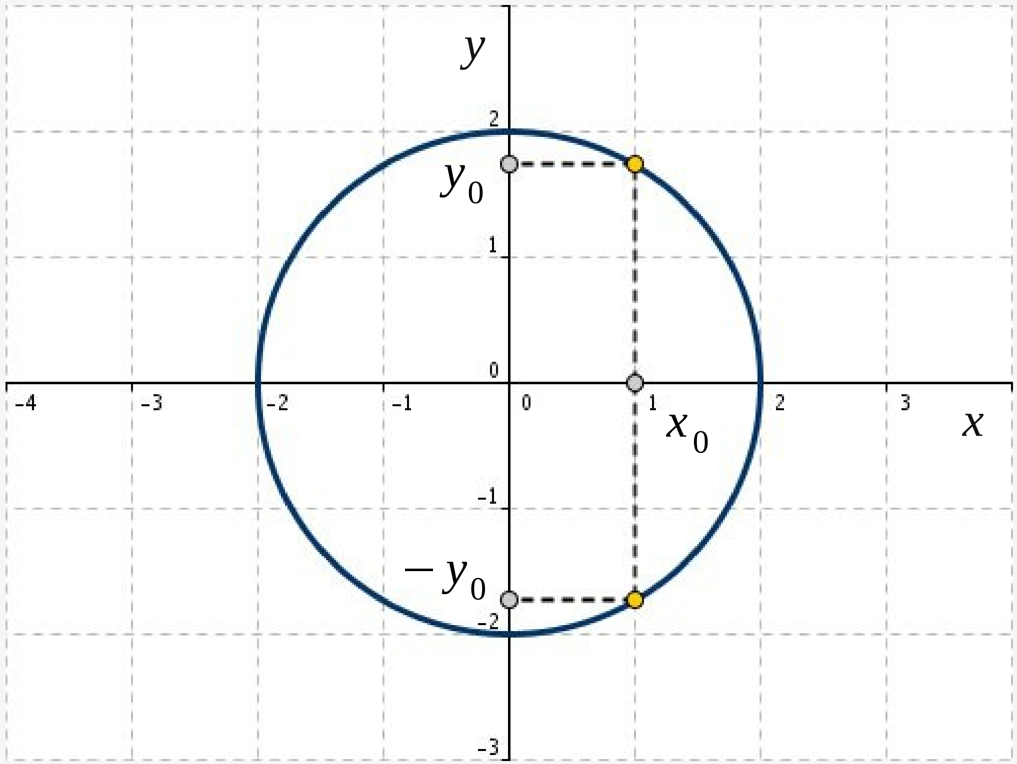

Fig. 4-1: Representation of a circle with a radius of 2

One observes that the assignment $x \rightarrow y$ is not unique. The above equation of a circle implicitly defines two functions as shown below.

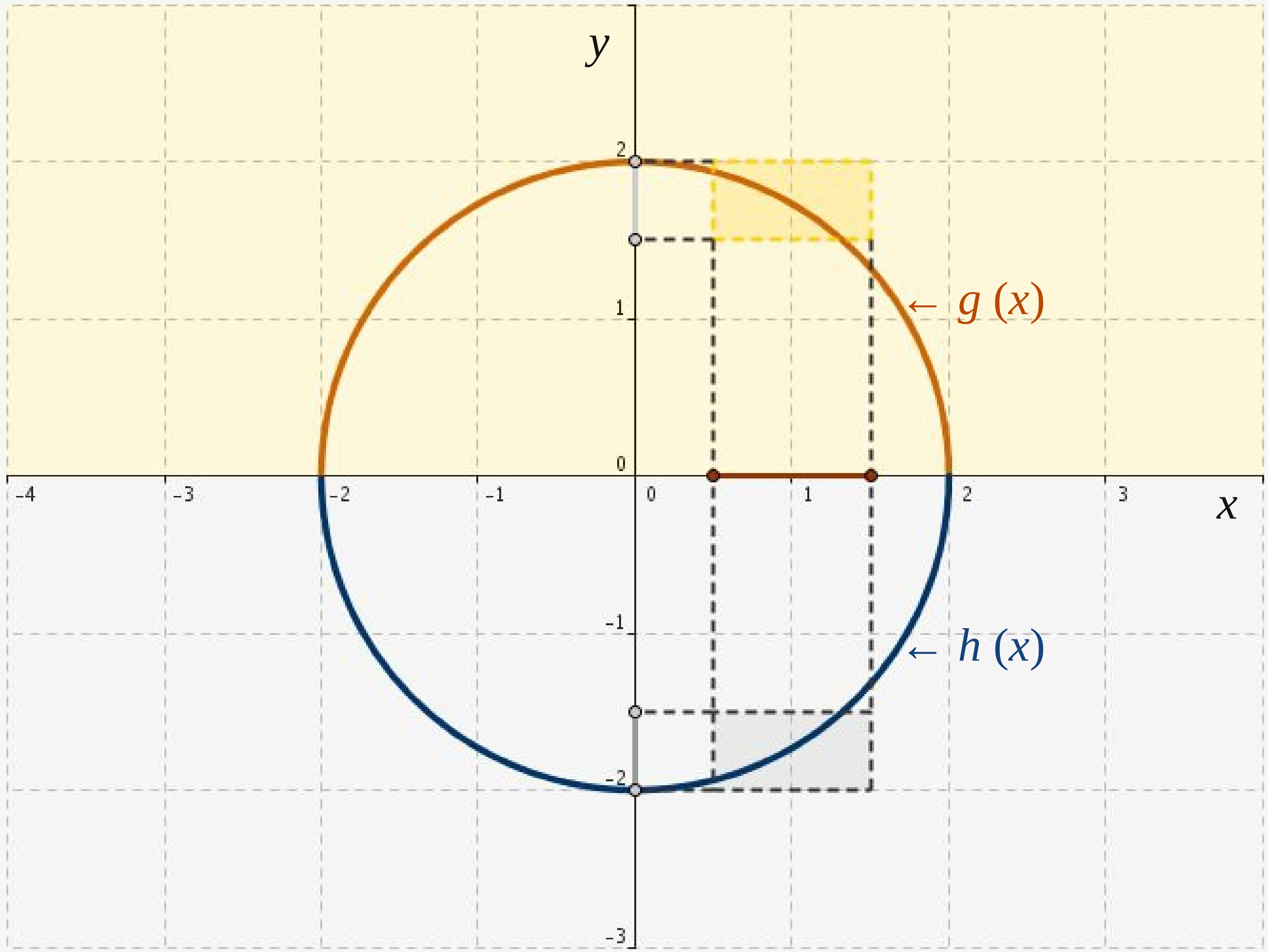

Fig. 4-2: Representation of a circle by two root functions g (x) and h (x)

$$g(x) = \sqrt{4 - x^2}: \quad [-2, 2] \to [0, 2]$$

$$h(x) = -\sqrt{4 - x^2}: \quad (-2, 2) \to [-2, 0)$$

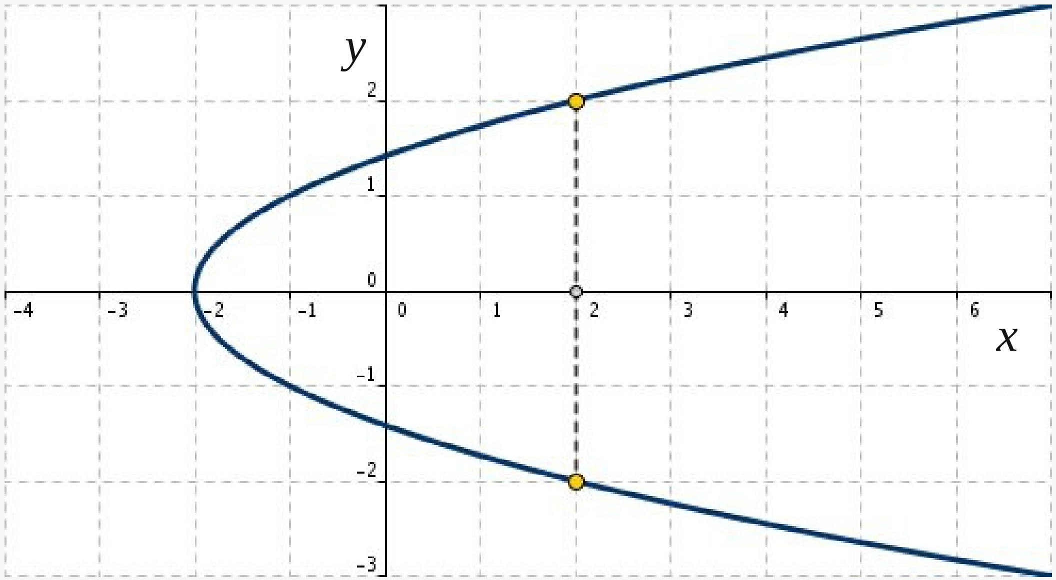

Fig. 5-1: Representation of the function $y^2 = x + 2$

Again, one observes that the assignment $x \to y$ is not unique. The equation $y^2 = x + 2$ implicitly defines two functions as shown below.

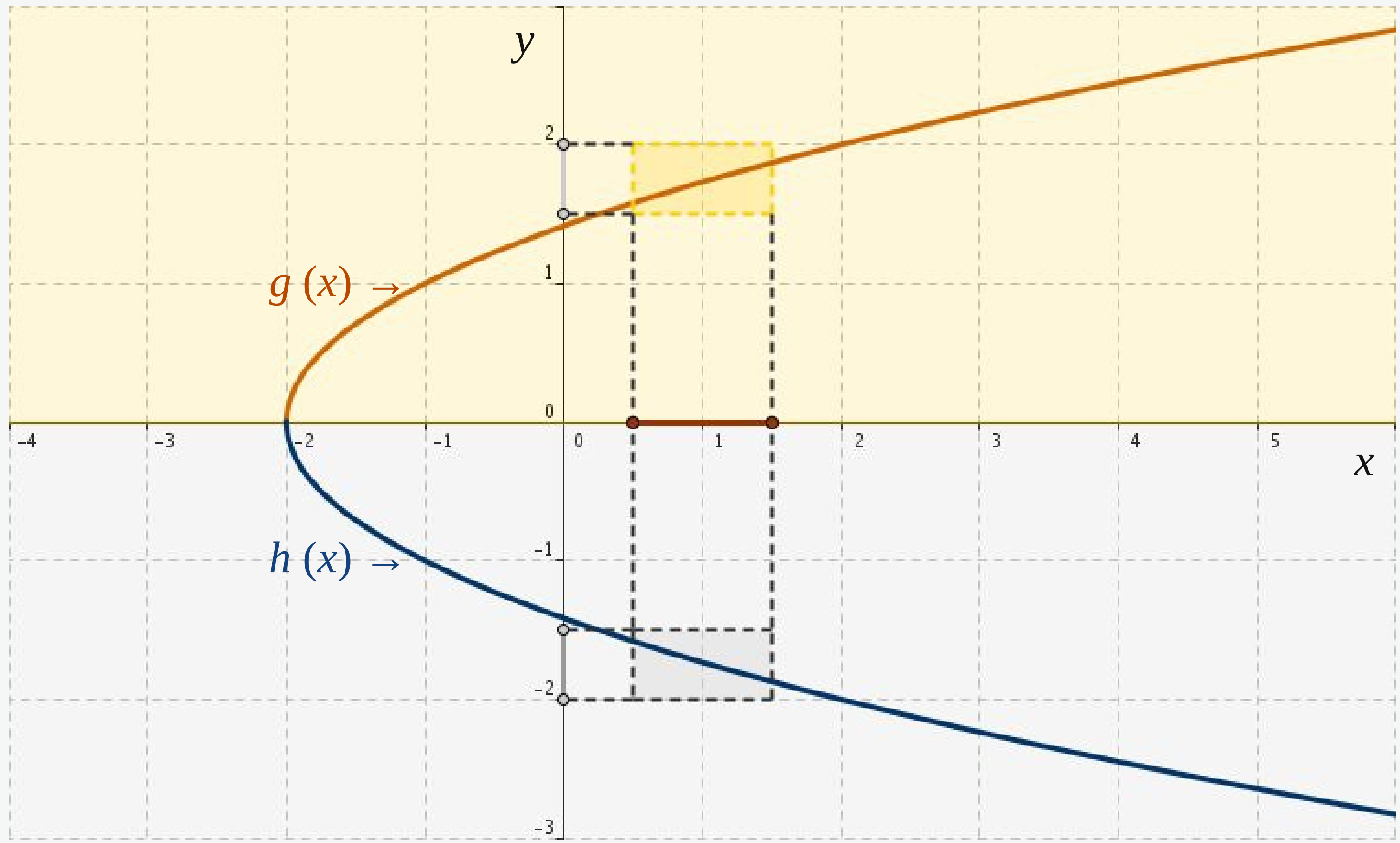

Fig. 5-2: Representation of the equation $y^2 = x + 2$ by two root functions $g(x)$ and $h(x)$

$$g(x) = \sqrt{x+2}: \quad [-2, \infty) \to [0, \infty)$$

$$h(x) = -\sqrt{x+2}: \quad (-2, \infty) \to (-\infty, 0)$$

Explicit Representation

If possible, functions are declared explicitly, that is, the function output is on one side of the equation, and all other terms are on the other side. This is called an explicitly defined function.

$$y = f(x)$$

y is isolated at one side of the equation.

Examples:

$$y = x^3 + 3x - 7$$

$$y = 3 \sin x + 2$$

$$y = x e^x$$

Transform the following implicitly given functions into an explicit representation, if possible.

$a)\quad F(x, y) = 2x + 3y - 6 = 0$

$b)\quad x^4 = -x^2 \cdot \ln \sqrt{y}$

$c)\quad x^2 - 6y + 8 = 0$

$d)\quad e^x = x y$

$g)\quad \ln y = x^2$

$$a)\quad y = -\frac{2}{3}\,x + 2$$

$$b)\quad y = e^{-2\,x^2}$$

$$-x^2 = \ln\sqrt{y} \quad\Leftrightarrow\quad -x^2 = \frac{1}{2}\ln y \quad\Leftrightarrow$$

$$-2\,x^2 = \ln y \quad\Rightarrow\quad y = e^{-2\,x^2}$$

$$c)\quad y = \frac{x^2}{6} + \frac{4}{3}$$

$$d)\quad y = \frac{e^x}{x}, \qquad x \neq 0$$

$$e)\quad y = e^{x^2}$$