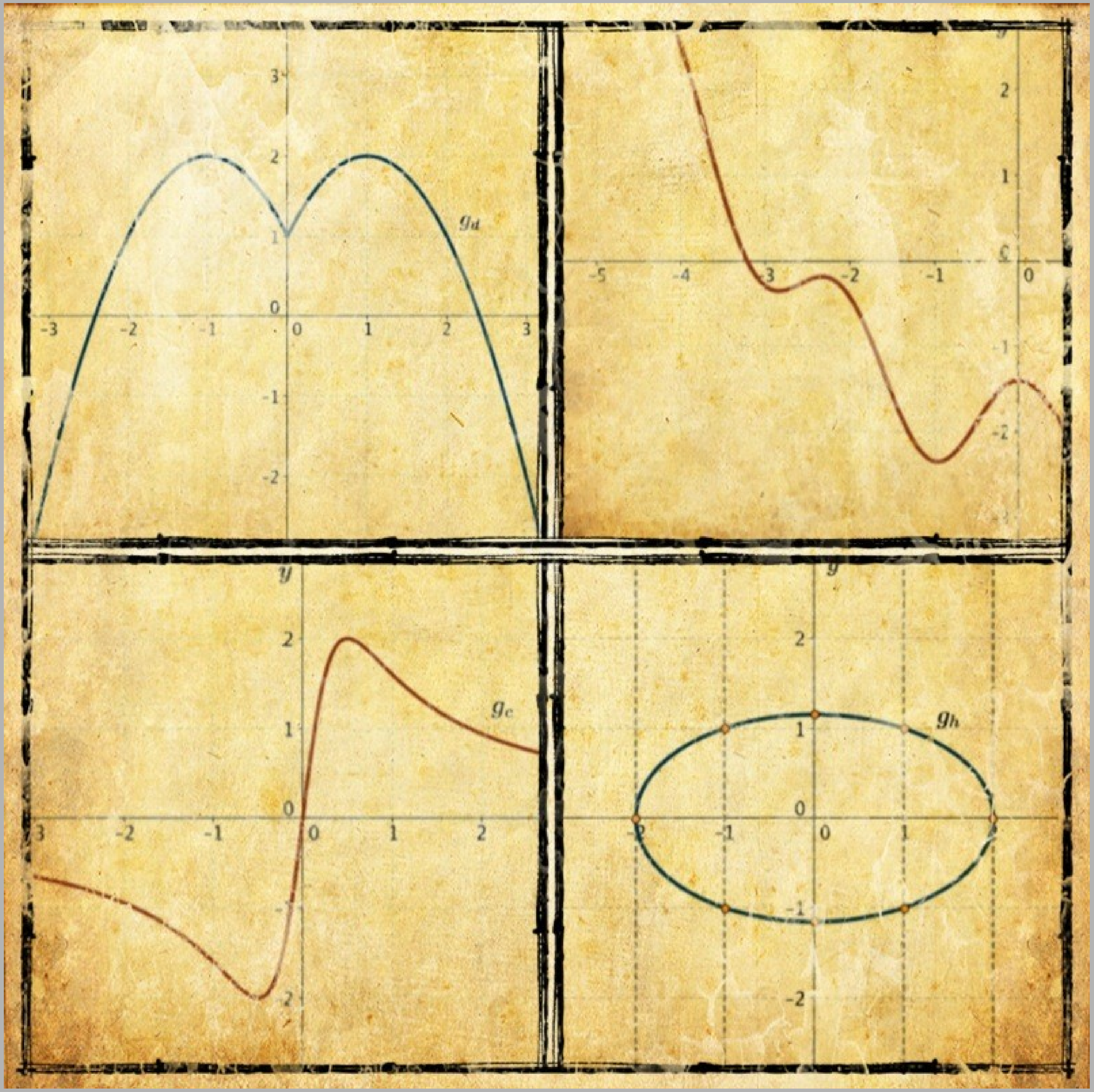
Symmetries. Even and odd functions

Humans like to admire symmetry and are attracted to it.



g_d
g_c
g_h

What should we know

- Definitions of
  - a function,
  - a relation,
  - a function domain.
- Vertical line test.

What shall we study

- Three types of curve symmetry:
  - symmetry with respect to the y-axis,
  - symmetry with respect to the x-axis,
  - symmetry with respect to the coordinate origin.
- which of the symmetry applies to functions and which to relations,
- how the symmetry with respect to the y-axis and to the coordinate origin is reflected in the function equation,
- algebraical and graphical proof of the axis or origin symmetry,
- symmetry rules for some functions: polynomials, rational, trigonometric and composed functions,
- to present a function as a sum of even and odd functions.

Exercise 1:

In Figure 1-1 three graphs, which correspond to the following equations

$$a)\ \ y = x^2, \qquad b)\ \ y = \frac{x^3}{8}, \qquad c)\ \ x = y^2$$

are given. Determine whether each graph is symmetric or not and describe the type of symmetry.

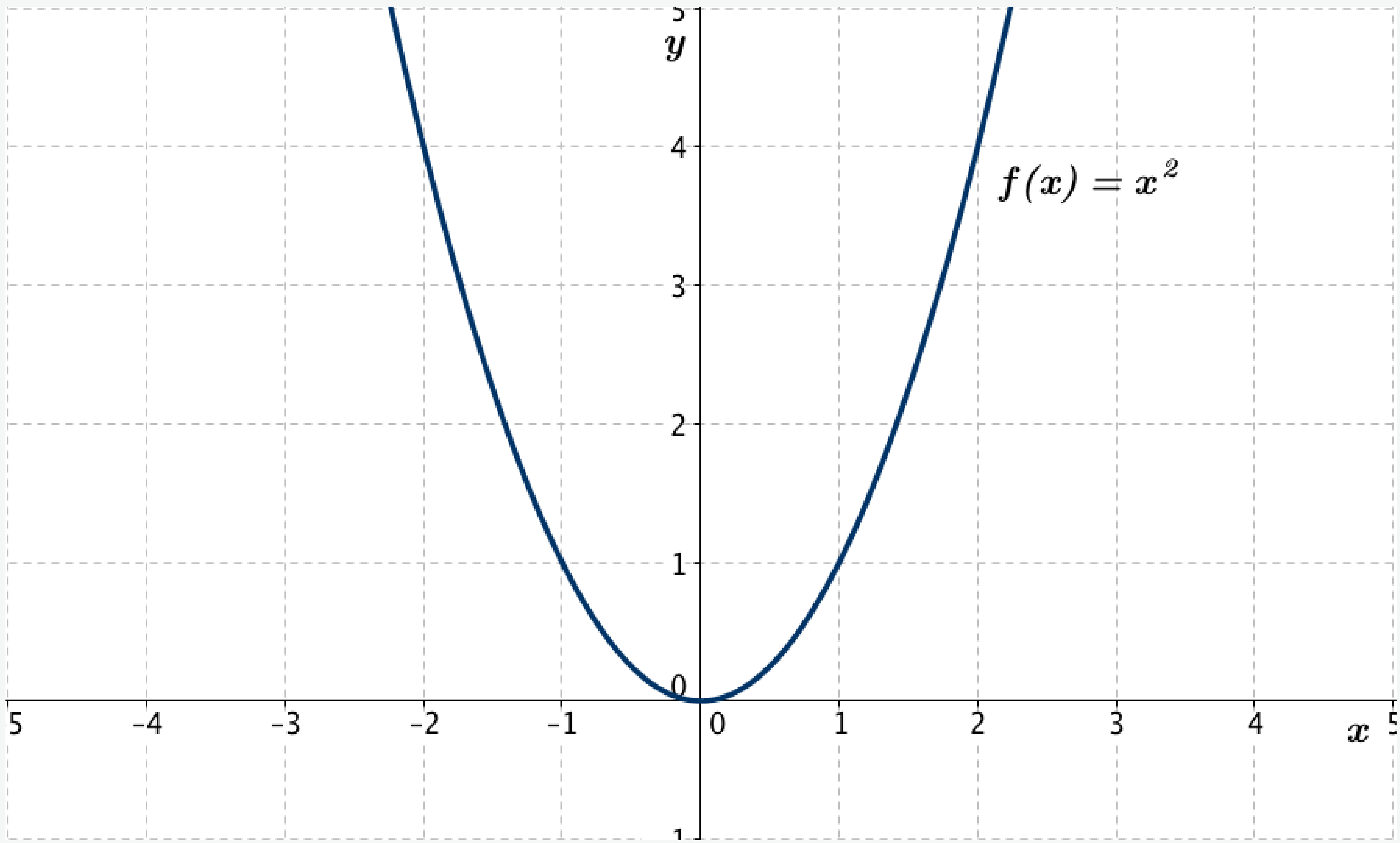


Fig. 1-1a: Graph of the function f (x) =x²

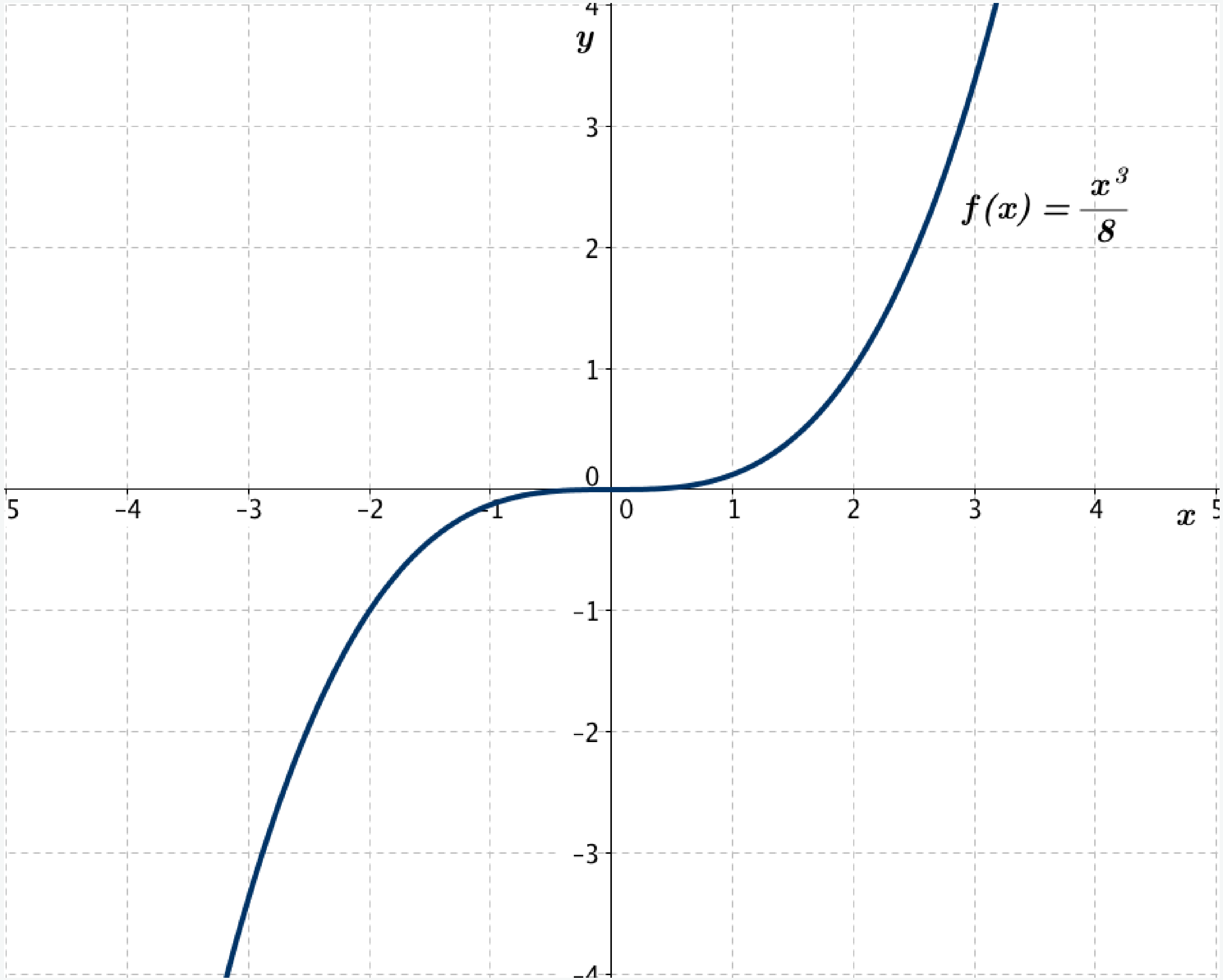


Fig. 1-1b: Graph of the function $f(x) = x^3/8$

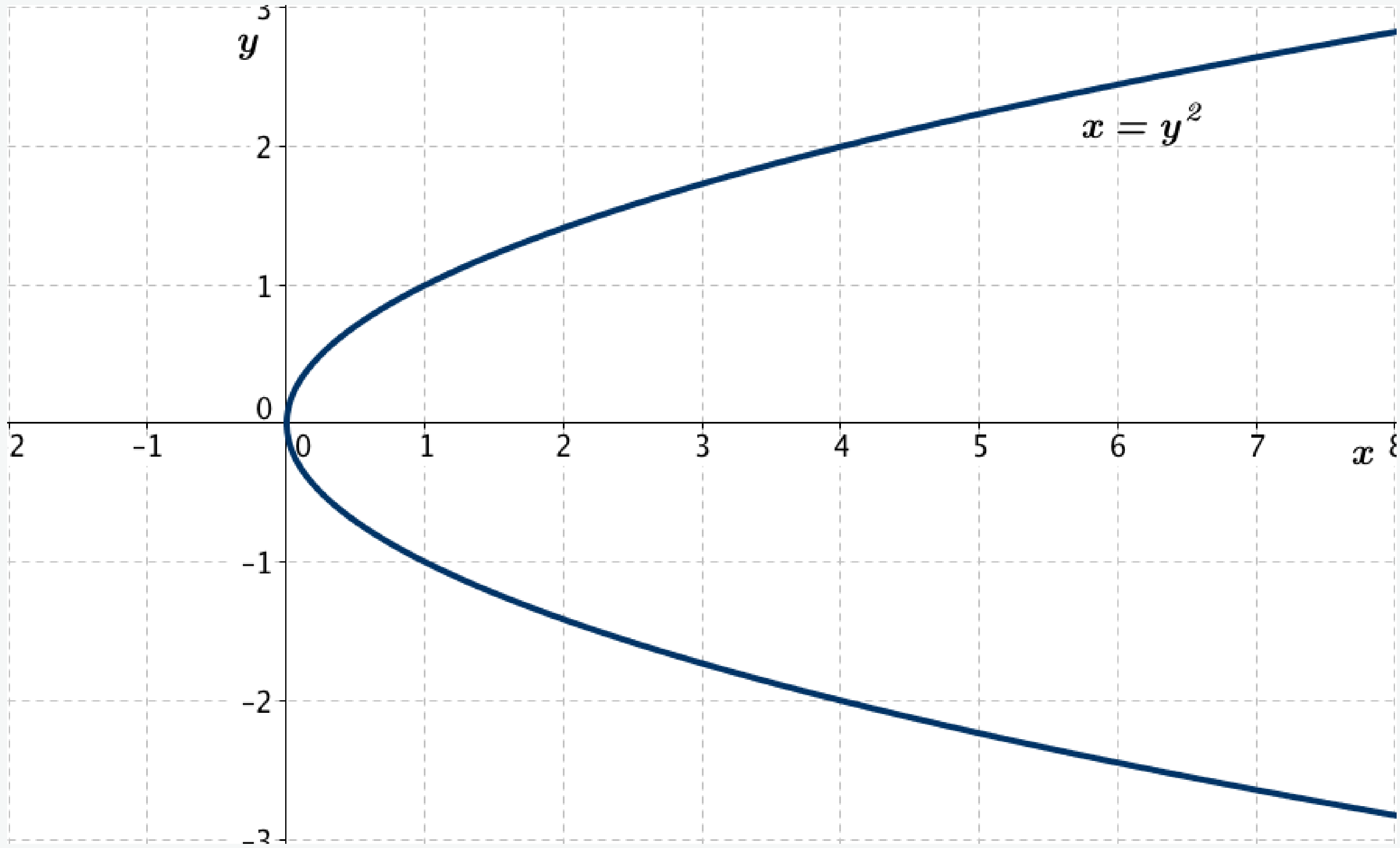


Fig. 1-1c: Graph of the equation $x = y^2$

$a)\ \ y = x^2$

The graph of the function in Fig. 1-1*a* is symmetric with respect to the *y*-axis. It means, that for each point $(x, y) = (x, f(x))$ on the graph there is the point $(-x, y) = (-x, f(x))$ on the same graph:

$$(x, f(x)) \overset{y}{\to} (-x, f(x)) = (-x, f(-x))$$

$$(1, 1) \overset{y}{\to} (-1, 1)$$

$$(2, 4) \overset{y}{\to} (-2, 4)$$

Algebraical expression of the symmetry with respect to the *y*-axis:

For all x of the function domain the symmetry with respect to the *y*-axis means algebraically:

$$f(-x) = f(x)$$

Symmetry of a graph: Solution 1a

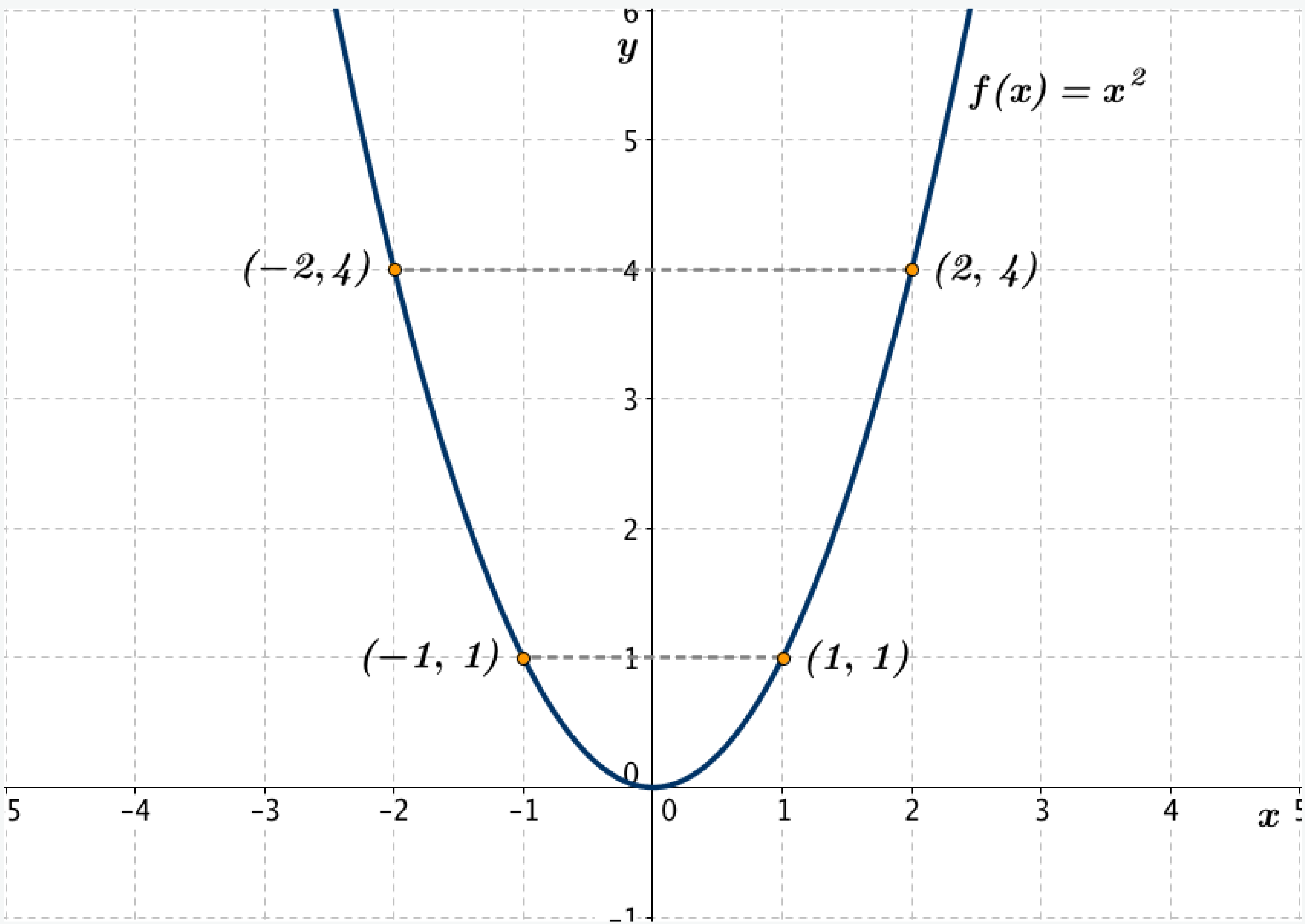


Fig. 1-2a: The graph of the function $f(x) = x^2$ is symmetric with respect to the y-axis

$b) \ y = \frac{x^3}{8}$

The graph of the function in Fig. 1-1*b* is symmetric about the origin. It means, that for each point *(x, y) = (x, f (x))* on the graph there is the point *(-x, -y)* = (*-x, - f (x))* on the same graph:

$$(x, f(x)) \overset{O}{\overbrace{\rightarrow}} (-x, -f(x)) = (-x, f(-x))$$

$$(2, 1) \overset{O}{\overbrace{\rightarrow}} (-2, -1)$$

$$(3, 3.38) \overset{O}{\overbrace{\rightarrow}} (-3, -3.38)$$

Algebraical expression of the symmetry with respect to the origin:

For all *x* of the function domain the symmetry with respect to the origin means algebraically:

$$f(-x) = -f(x)$$

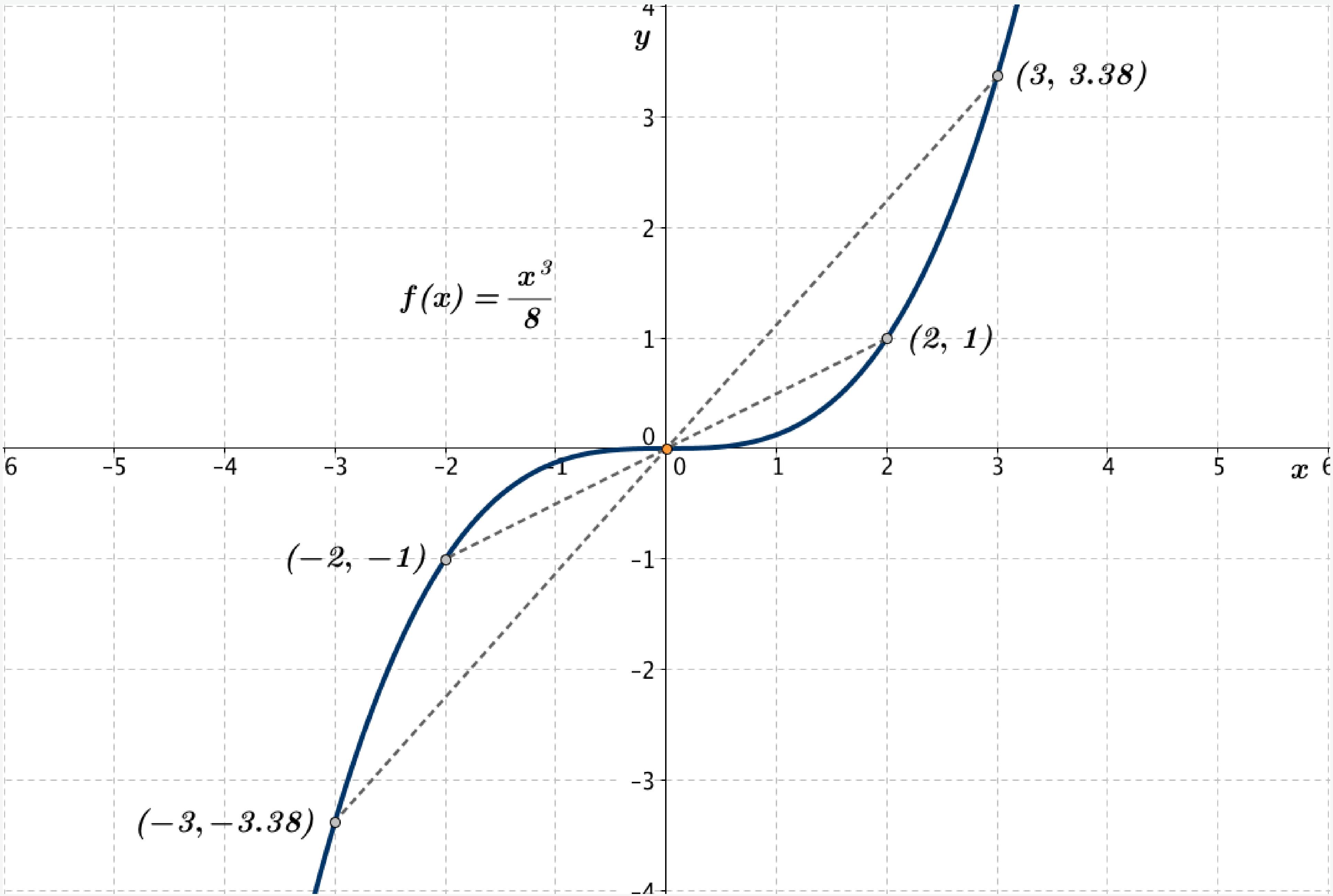


Fig. 1-2b: The graph of the function $f(x) = x^3/8$ is symmetric with respect to the point O (0, 0)

$c)\ \ x = y^2$

The graph of $x = y^2$ in Fig. 1-1c is symmetric about the x-axis. It means, that for each point $(x, y) = (x, f(x))$ on the graph there is a point $(x, -y) = (x, -f(x))$ on the same graph:

$$(x,\ f(x)) \overset{x}{\overbrace{\rightarrow}} (x, -f(x)) = (x, -f(x))$$

$$(1,\ 1) \overset{x}{\overbrace{\rightarrow}} (1, -1)$$

$$(4,\ 2) \overset{x}{\overbrace{\rightarrow}} (4, -2)$$

$x = y^2$ is not a function but a relation. The symmetry with respect to the x-axis means that one value of x can correspond to two or more values of y.

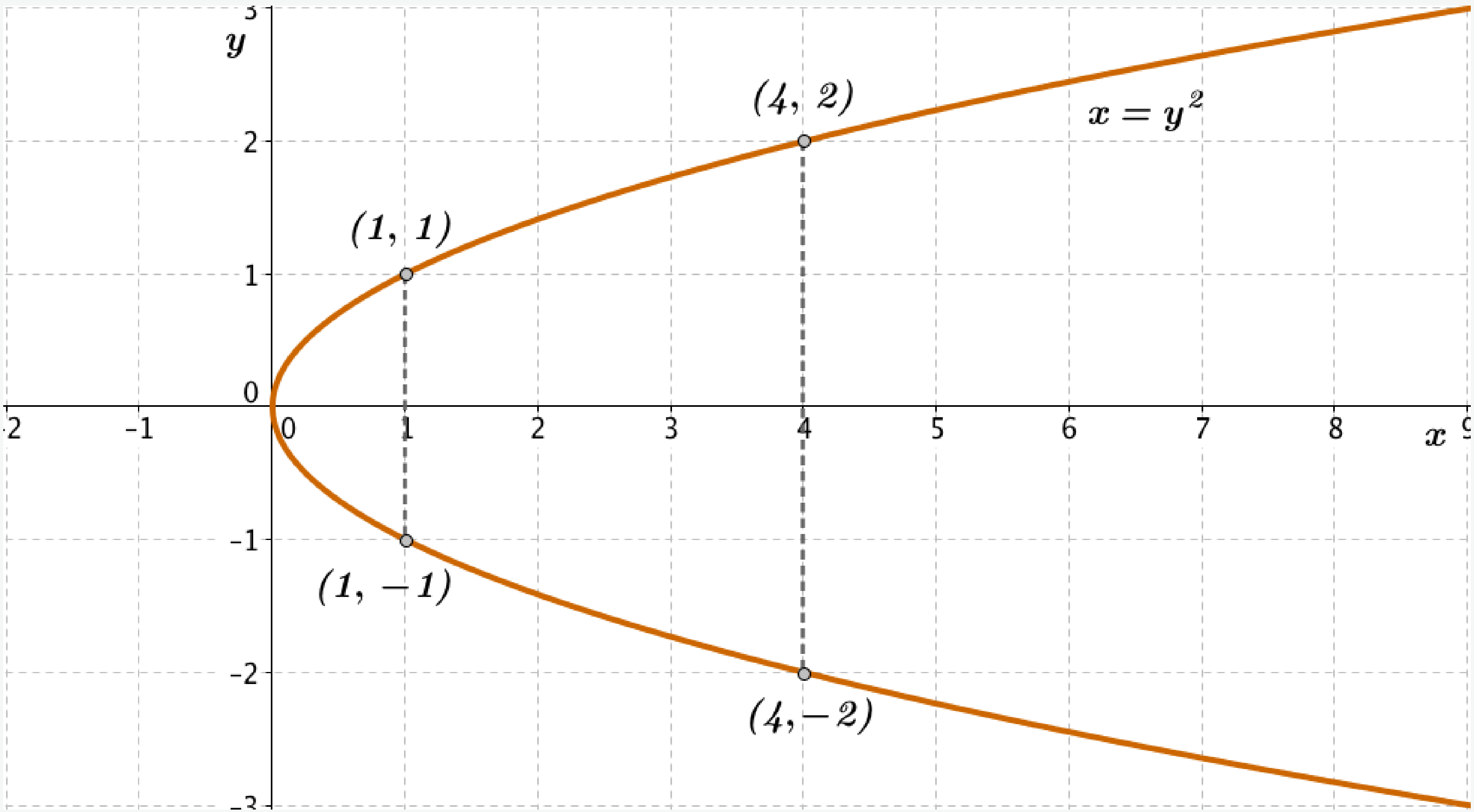


Fig. 1-2c: The graph of the relation $x = y^2$ is symmetric with respect to the x-axis

Definition:

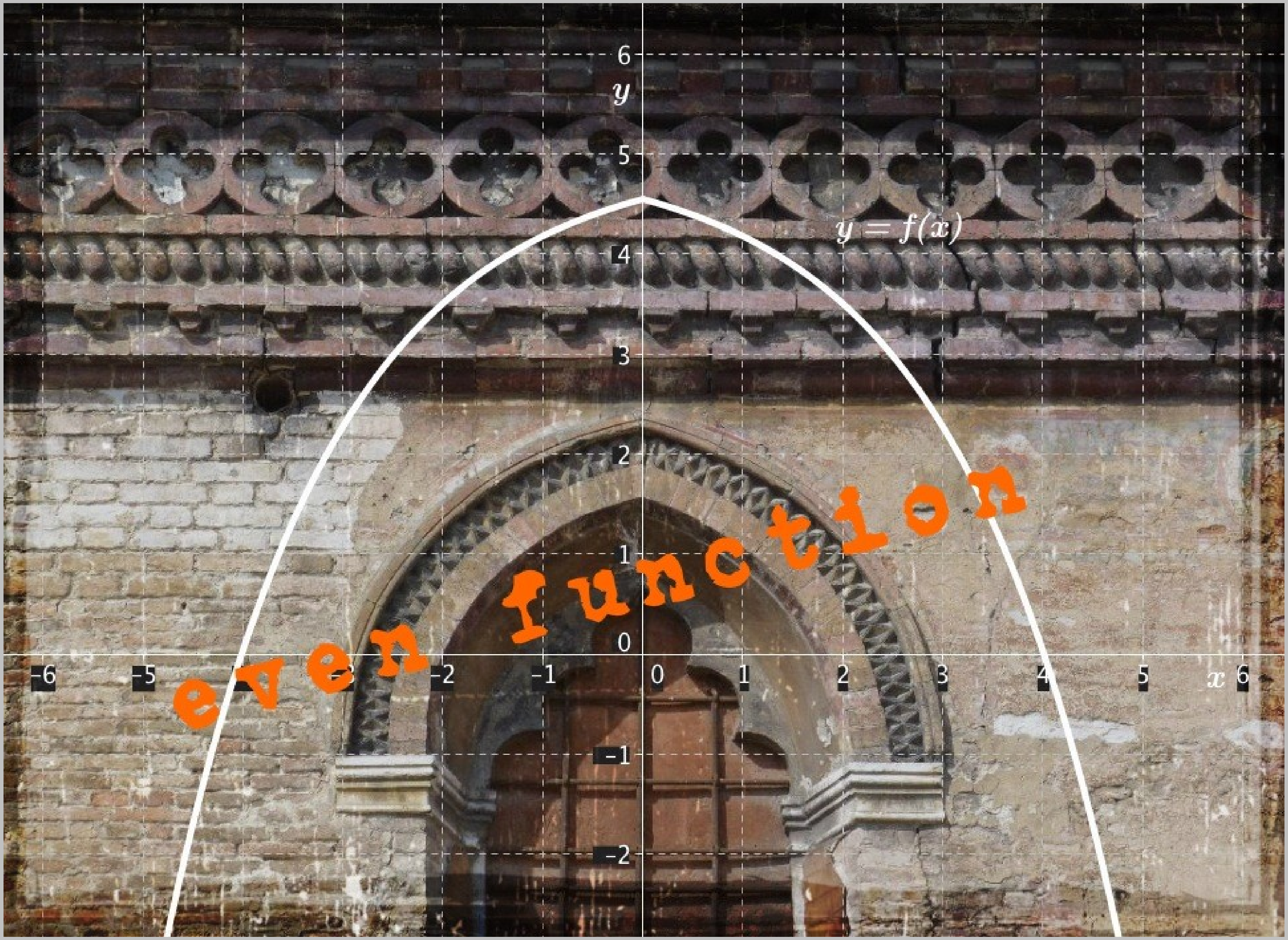
A function $y = f(x)$ with symmetric domain D is even, if the following condition holds for all x of the domain:

$$f(-x) = f(x)$$

The graph of an even function is axially symmetric with respect to the y-axis.

This means:

- The function graph remains unchanged after being reflected about the y-axis.
- If a function graph has a point $(x, f(x))$, it has also the point $(-x, f(x))$. This means that the domain of an even function is symmetric about the origin.



y
x
$y = f(x)$
even function
6
5
4
3
2
1
0
−1
−2
−6
−5
−2
−1
0
1
2
3
4
5
6

Definition:

A function $y = f(x)$ with symmetric domain D is odd, if the following condition holds for all x of the domain:

$$f(-x) = -f(x)$$

The graph of an odd function is symmetric with respect to the origin. The symmetry with respect to the origin is a rotational symmetry.

This means that:

- The graph remains unchanged after 180 degree rotation about the origin. If one rotates a right hand side of a curve by 180° about the origin, then one gets the left side of the curve.
- if a function graph has a point $(x, f(x))$, it has also the point $(-x, -f(x))$. This means that the domain of an odd function is symmetric about the origin.

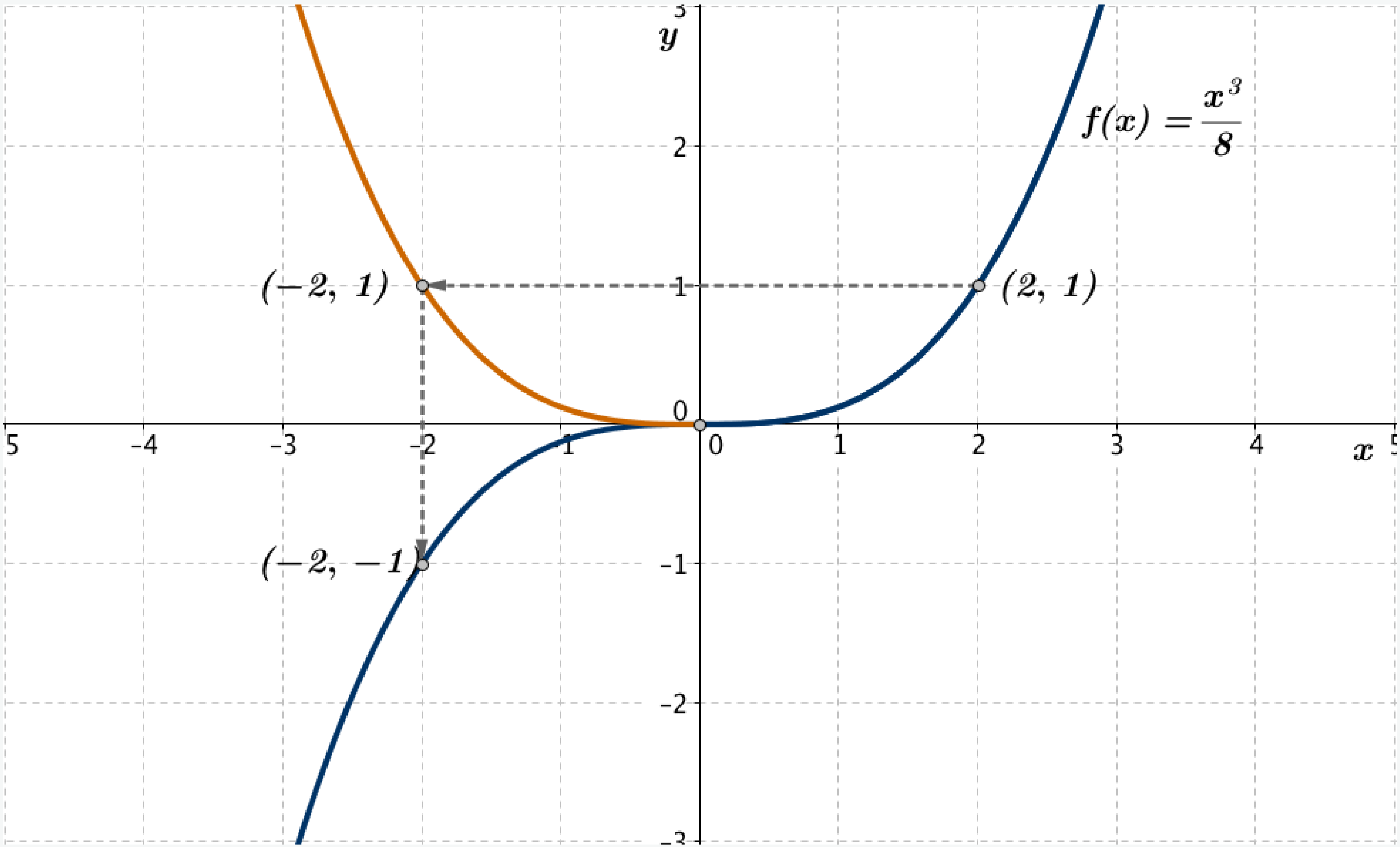
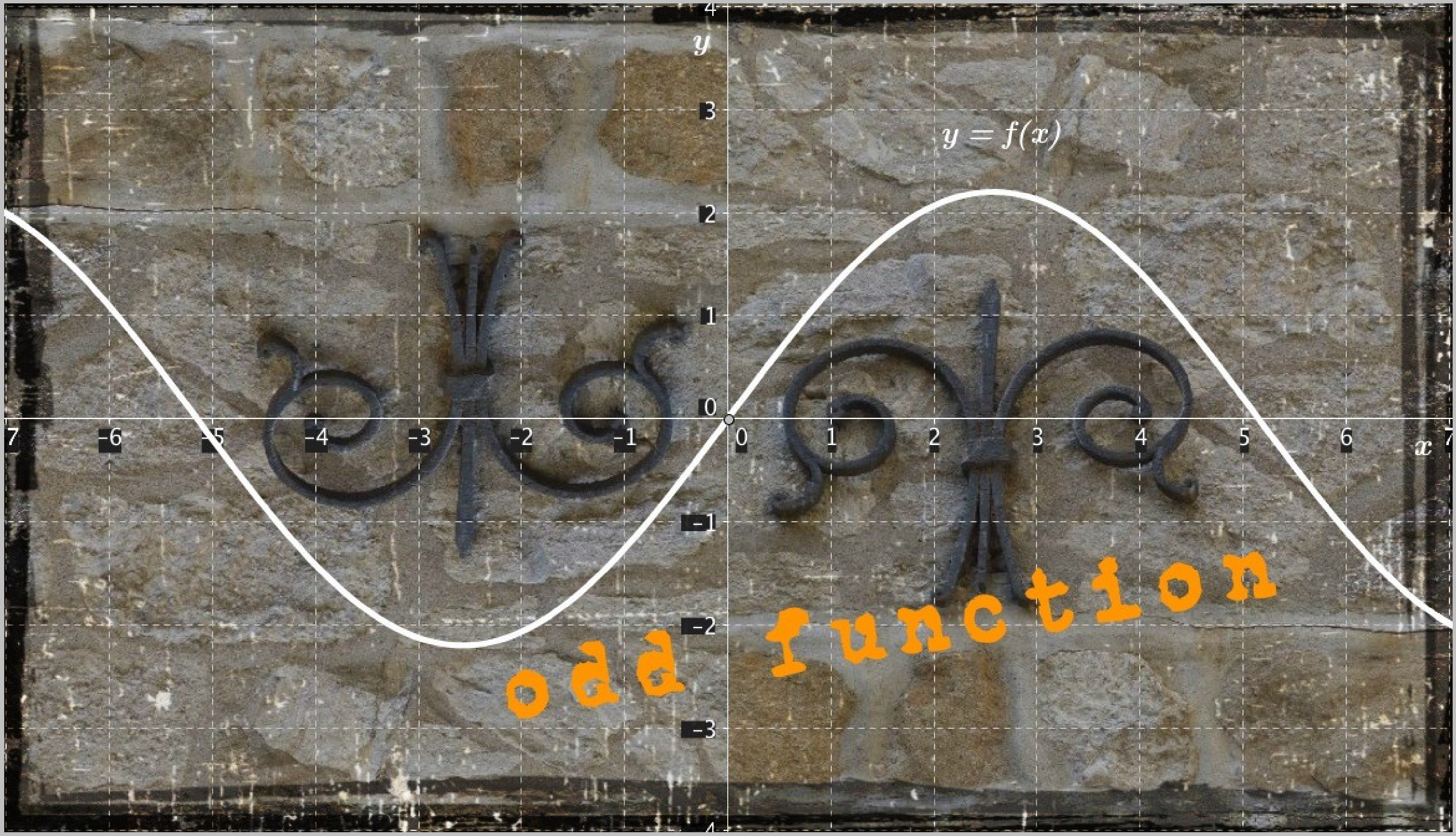


Fig. 1-3c: The graph of the odd function $y = x^3/8$ as two successive reflections of the right hand part of the graph about the y- and x-axis

The left hand part of the odd function graph can be obtained by reflecting the right hand part of the graph about the *y*-axis, followed by the reflection about the *x*-axis, as shown in Fig. 1-3c.


y
$y = f(x)$
x
odd function

Do not confuse even/odd functions and even/odd integers!

Exercise 2:

In Figures $2i$ (i stands for the letters from a till l) the graphs are given. For each graph determine

1) a symmetry with respect to the axis or to the origin,

2) which graphs describe a function,

3) which functions are even, odd or neither,

4) which graphs describe a relation.

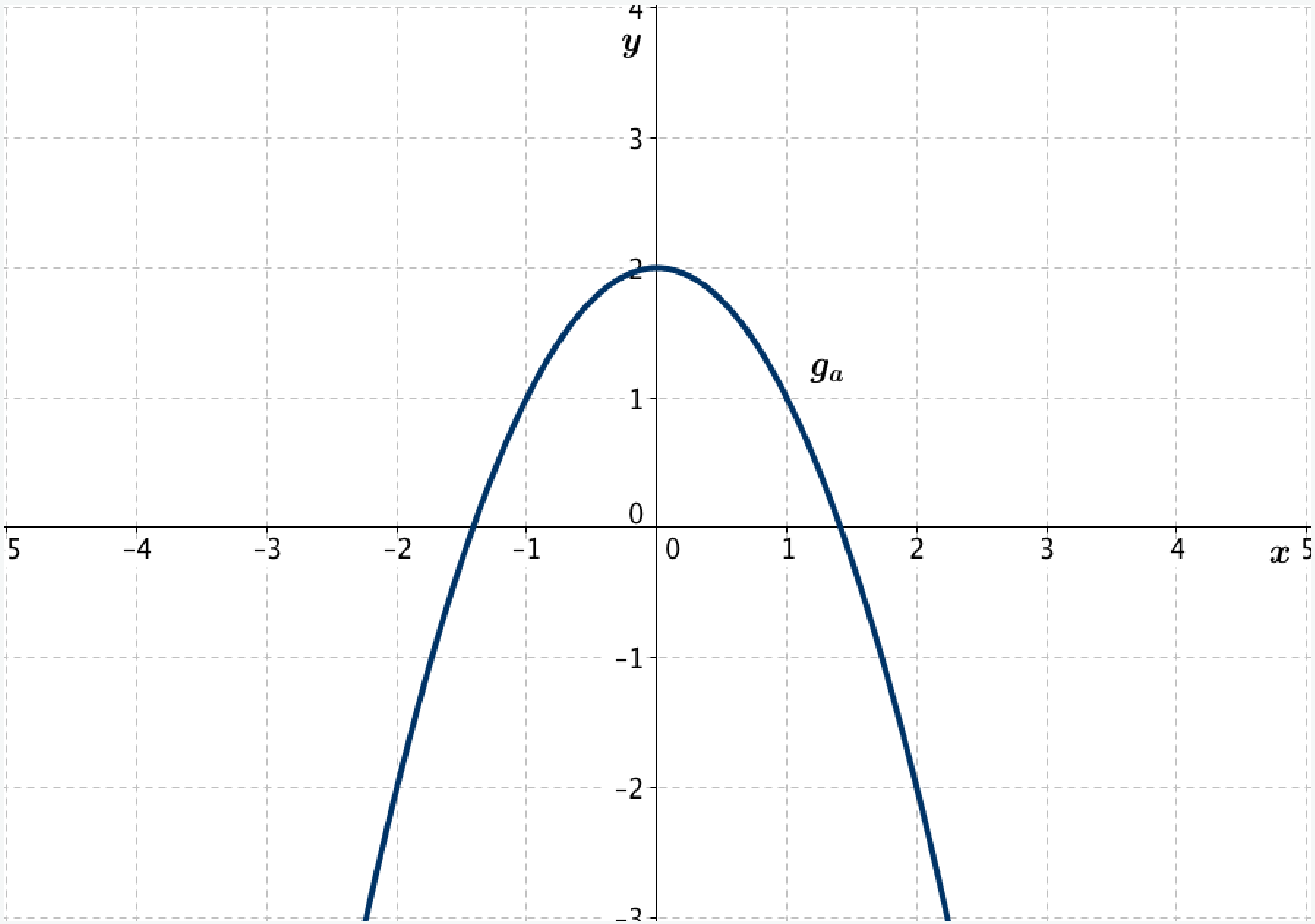


Fig. 2a: Graph a)

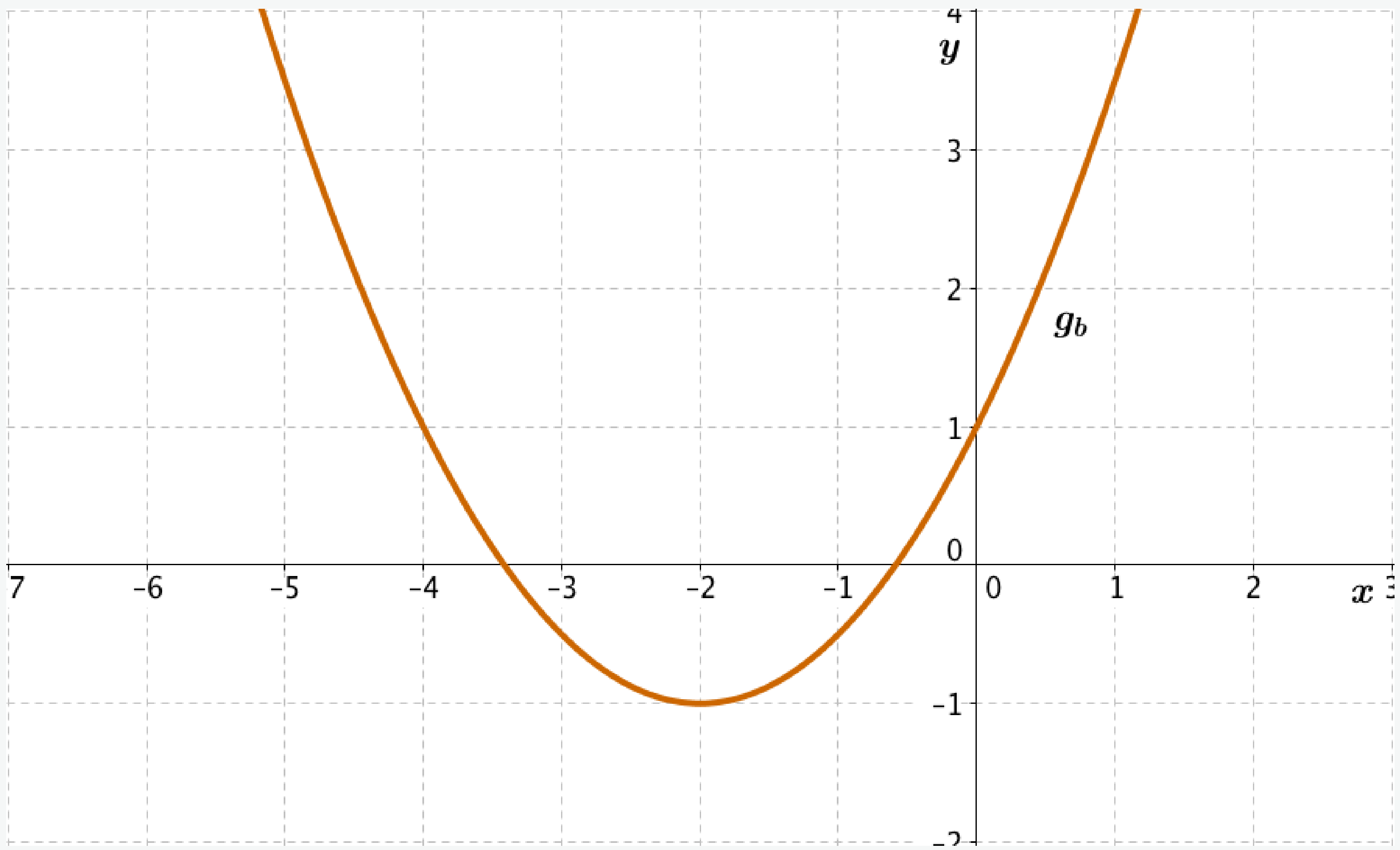


Fig. 2b: Graph b)

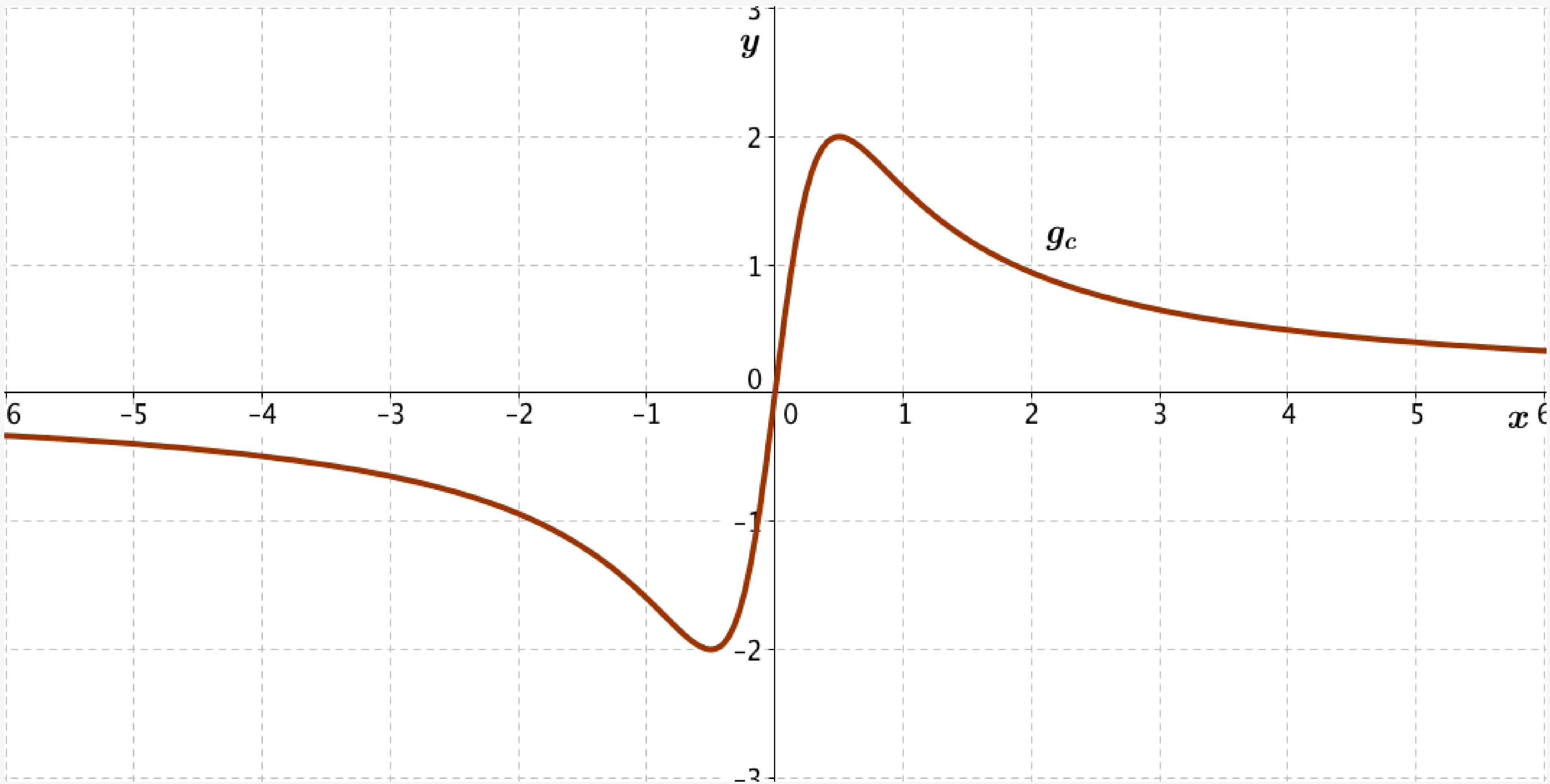


Fig. 2c: Graph c)

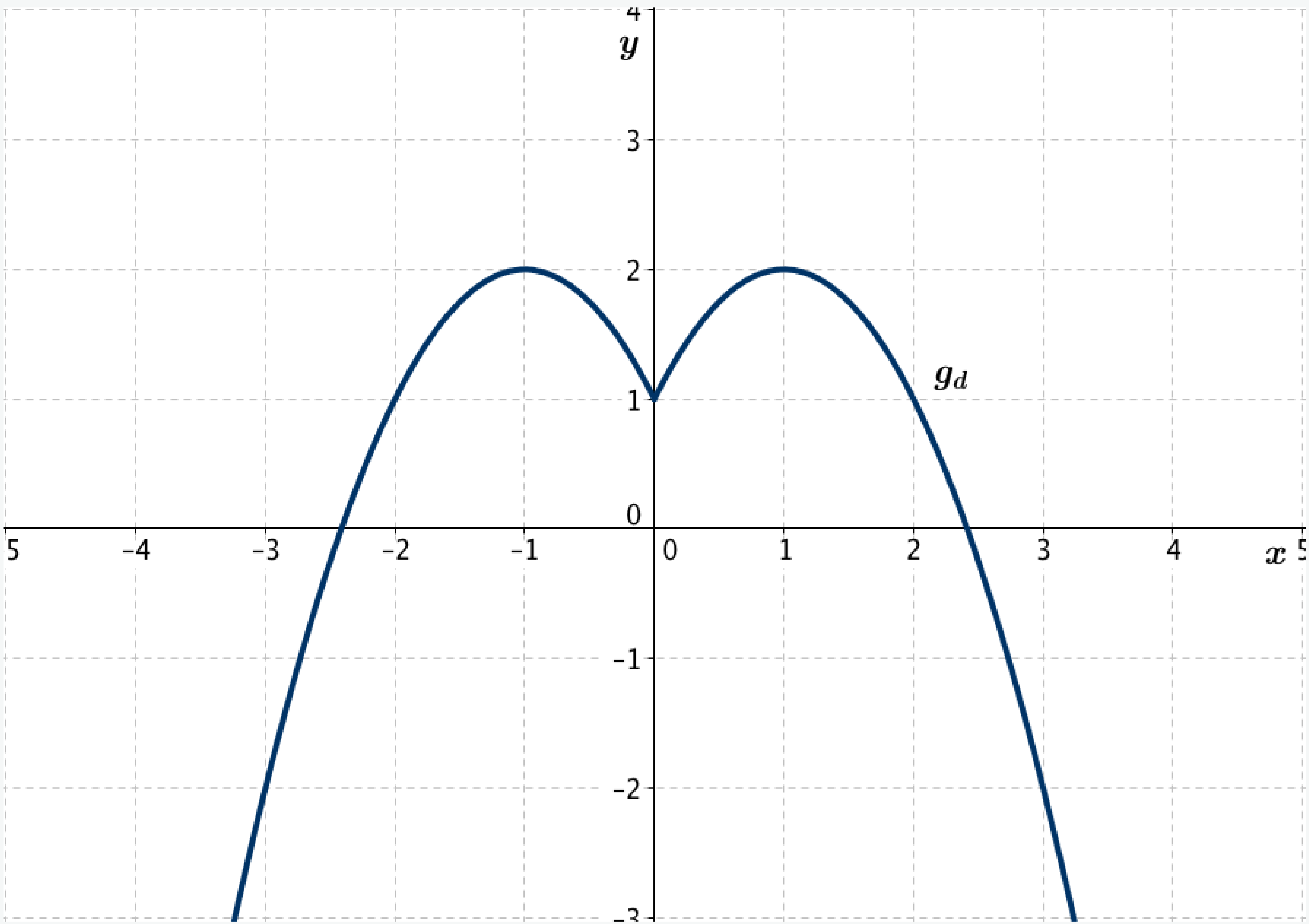


Fig. 2d: Graph d)

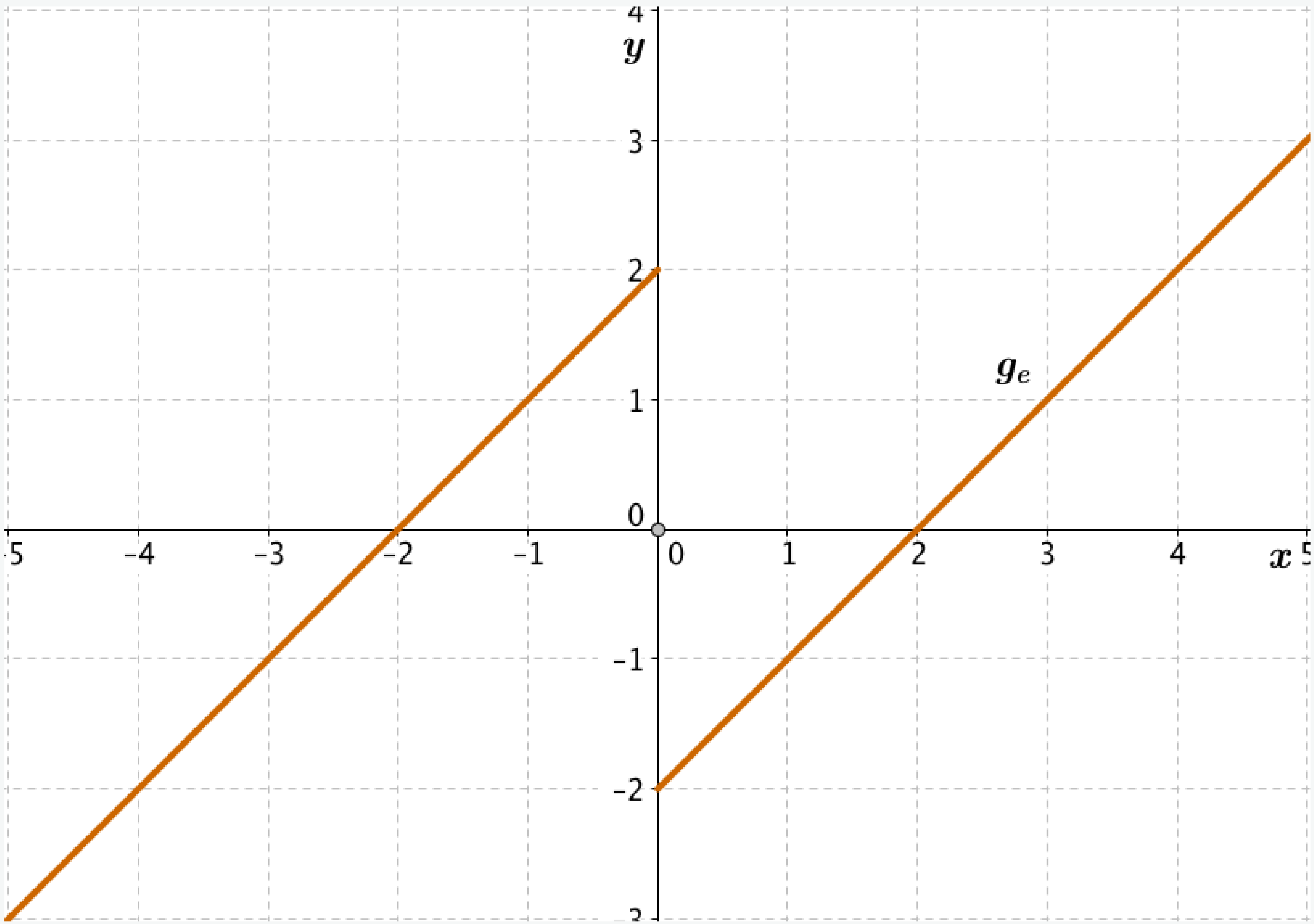


Fig. 2e: Graph e)

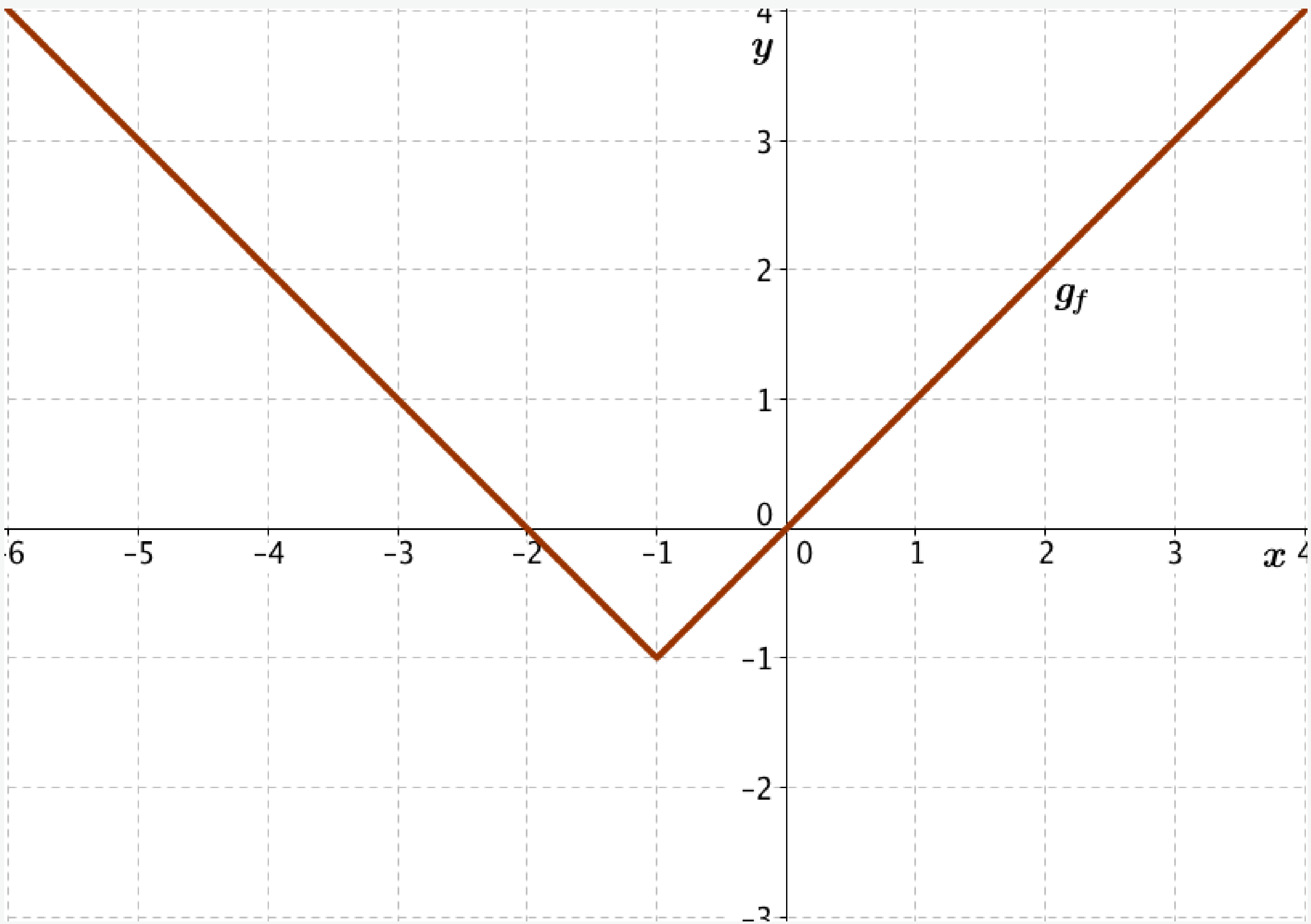


Fig. 2f: Graph f)

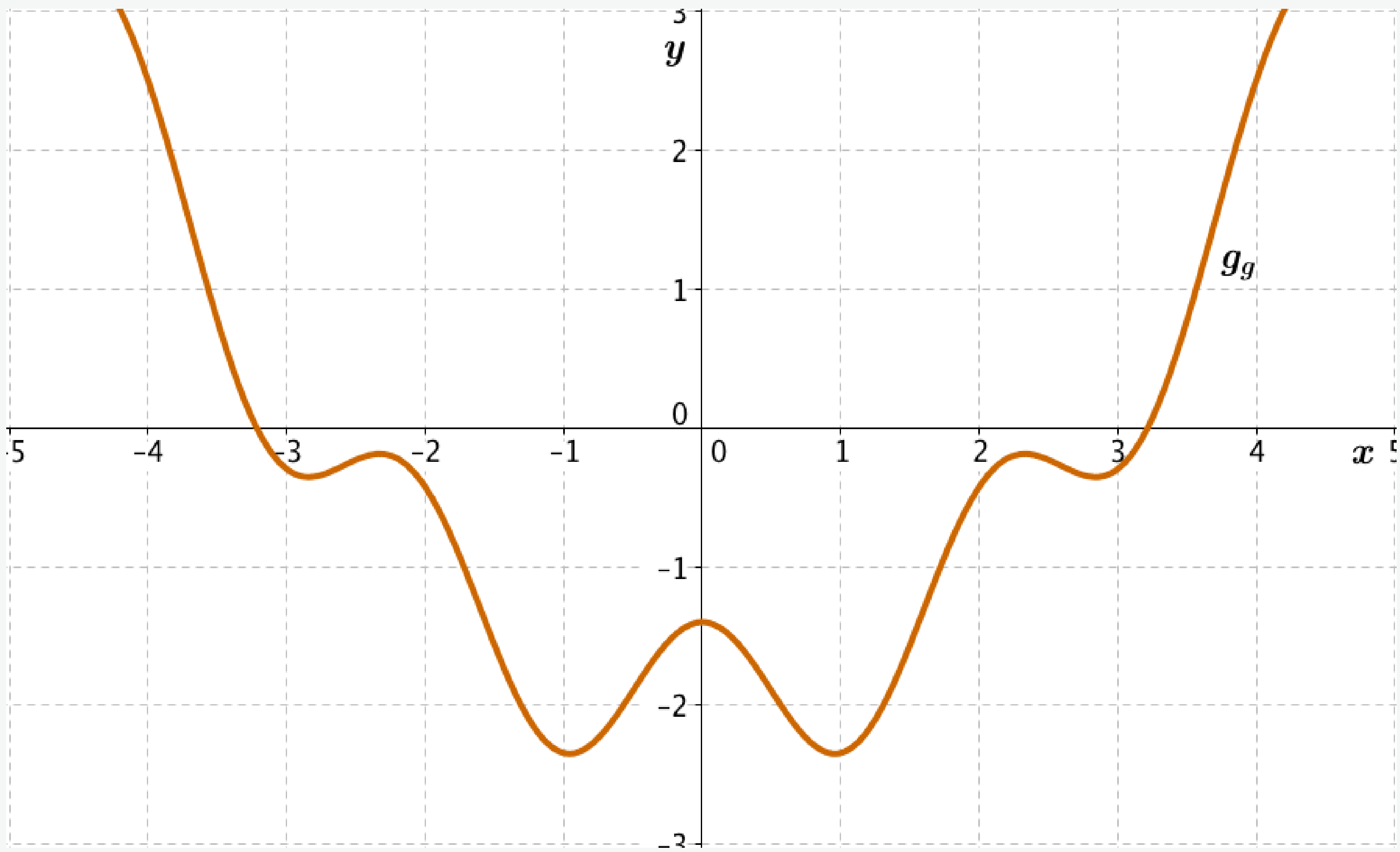


Fig. 2g: Graph g)

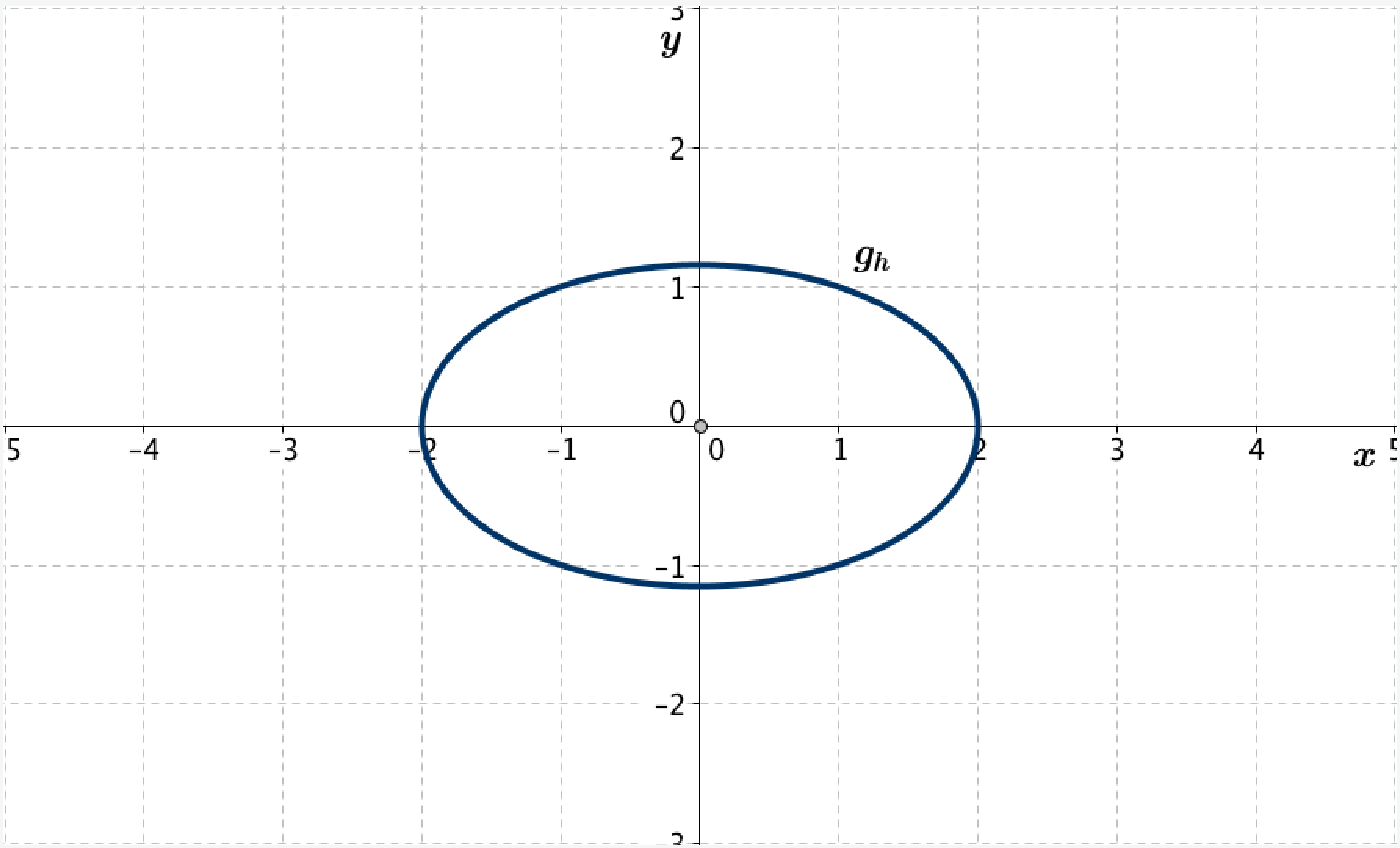


Fig. 2h: Graph h)

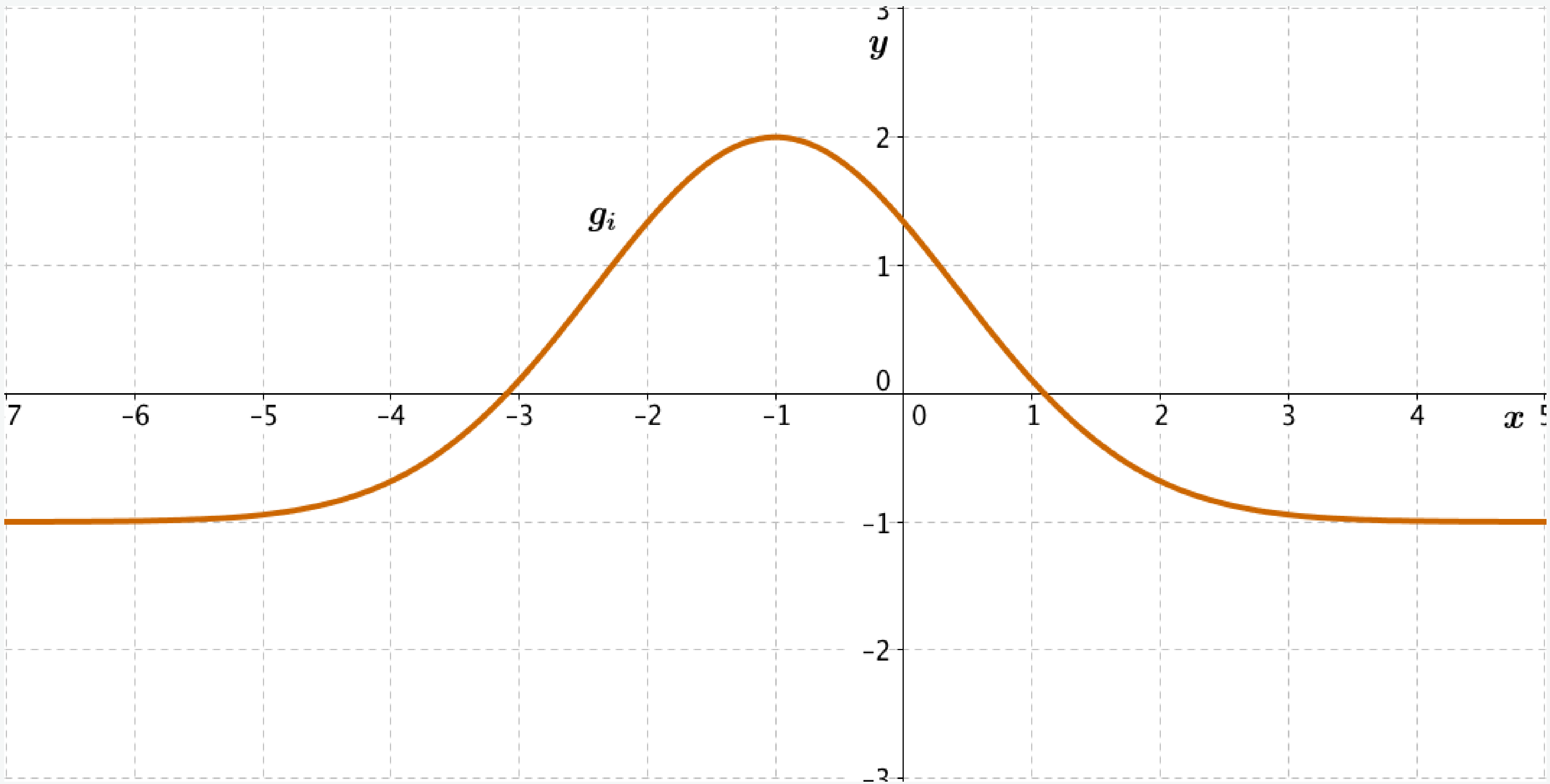


Fig. 2i: Graph i)

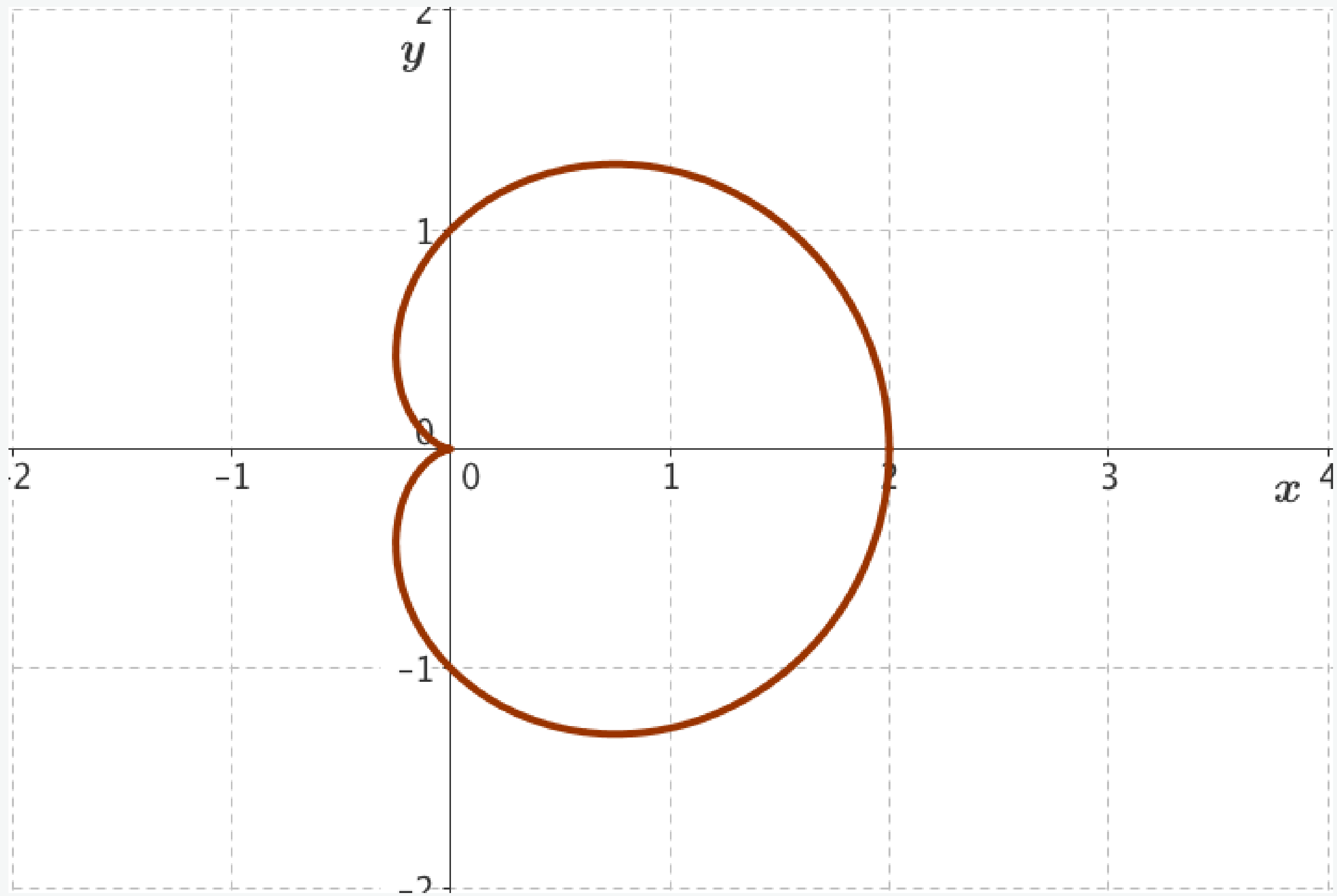


Fig. 2j: Graph j)

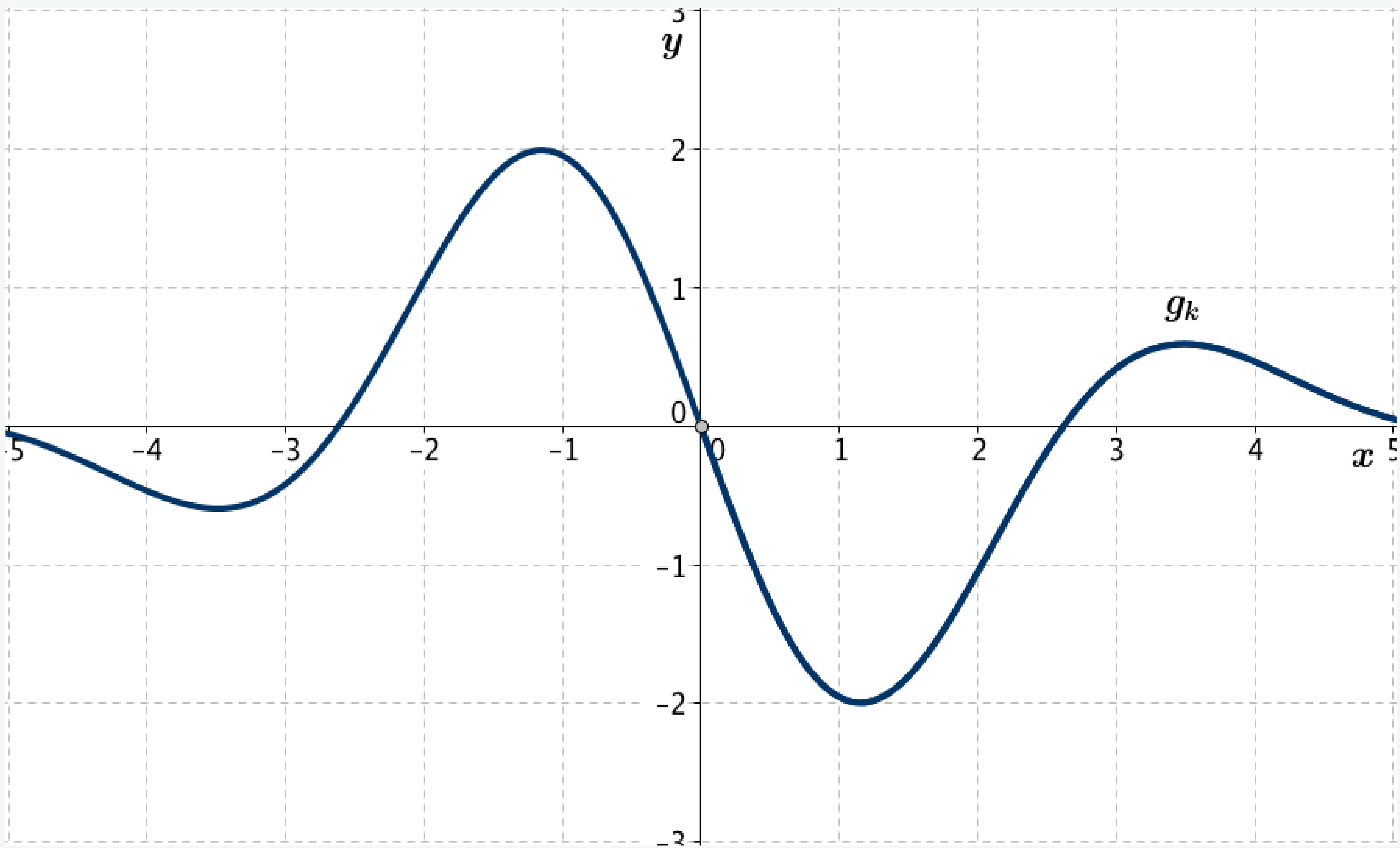


Fig. 2k: Graph k)

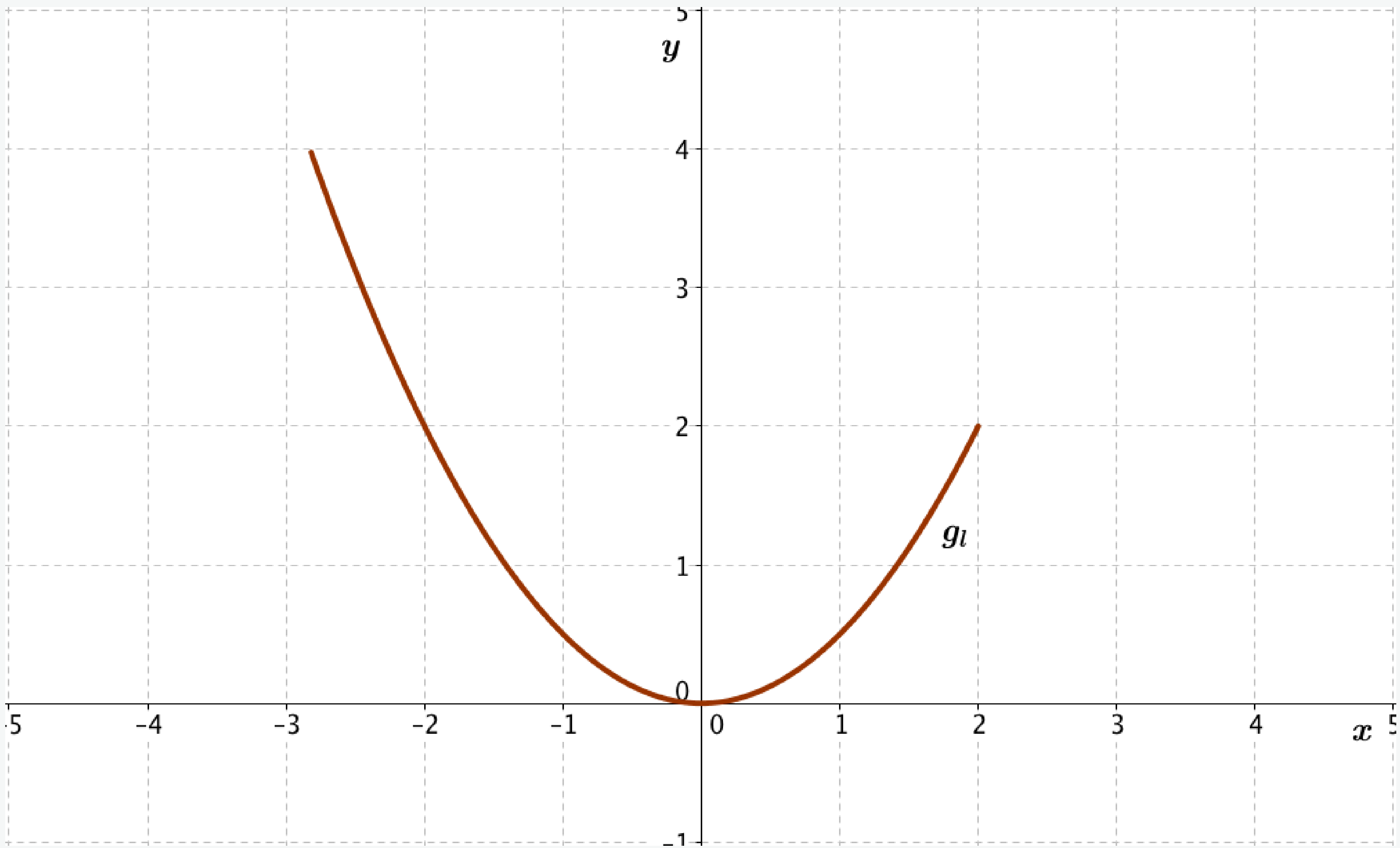


Fig. 2l: Graph l)

Symmetry of a graph: Solution 2

- The graphs *h*) and *j*) are symmetric with respect to the *x*-axis.
- The graphs *a*), *d*), *g*) and *h*) are symmetric with respect to the *y*-axis.
- The graphs *c*), *e*), *h*) and *k*) are symmetric with respect to the origin.
- The graphs of *a*), *b*), *c*), *d*), *e*), *f*) *g*), *i*), *k*) and *l*) describe functions.

  This can be shown by a vertical line test (see Figs. 3-2 and 3-5 which present this test for graphs *f*) and *h*)).
- The functions *a*), *d*) and *g*) are even (see Fig. 3-3).
- The functions *c*), *e*) and *k*) are odd (see Fig. 3-4).
- The graphs of *h*) and *j*) represent relations.

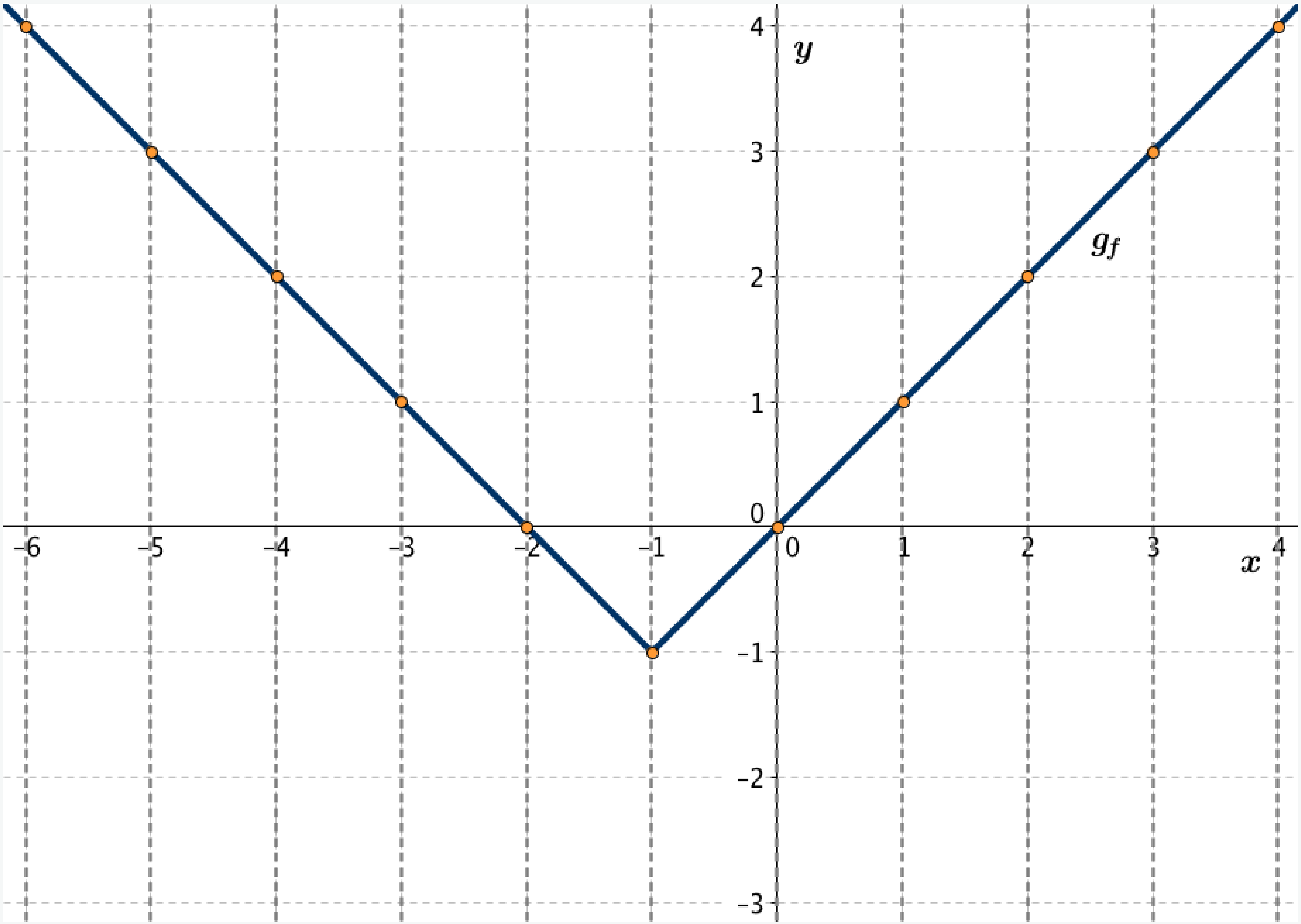


Fig. 3-2: The use of a vertical line test as a visual proof whether a graph describes a function or a relation

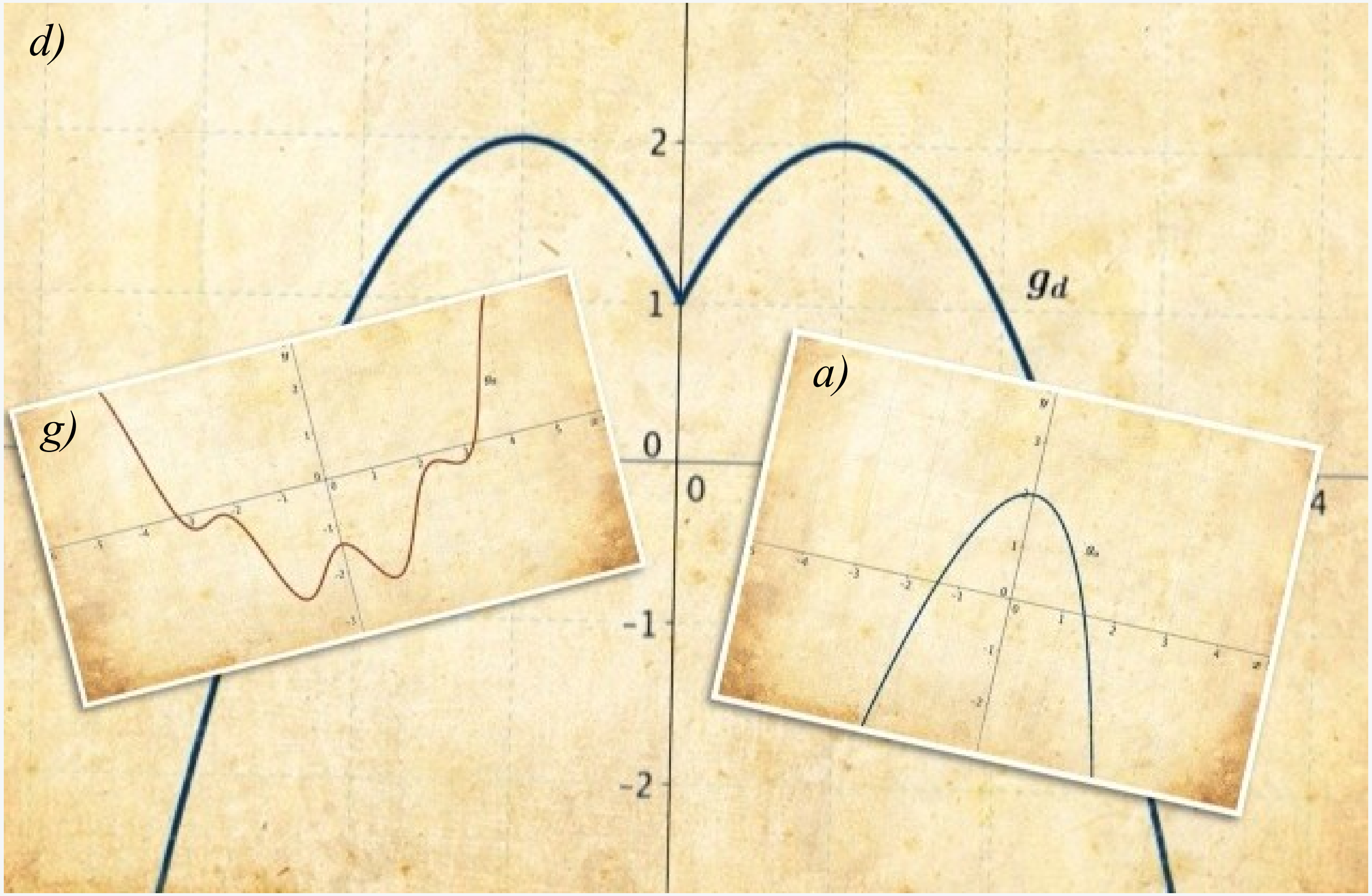


Fig. 3-3: The even functions a), d) and g)

The graphs *a*), *d*) and *g*) represent functions, which are symmetric with respect to the *y*-axis, they are even functions.

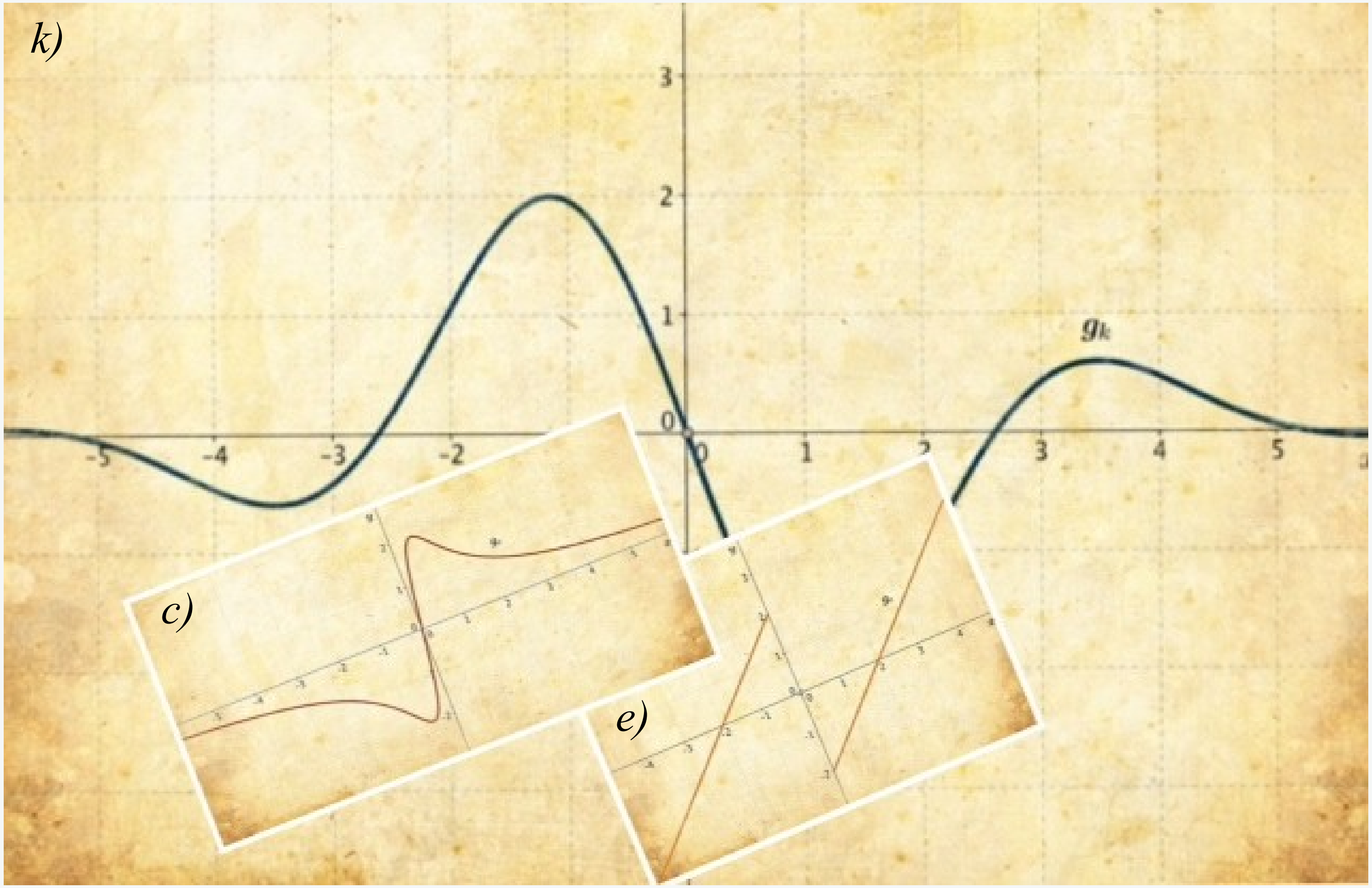


Fig. 3-4: The odd functions c), e) and k)

The graphen *c*), *e*) and *k*) represent functions, which are symmetric with respect to the origin, they are odd functions.

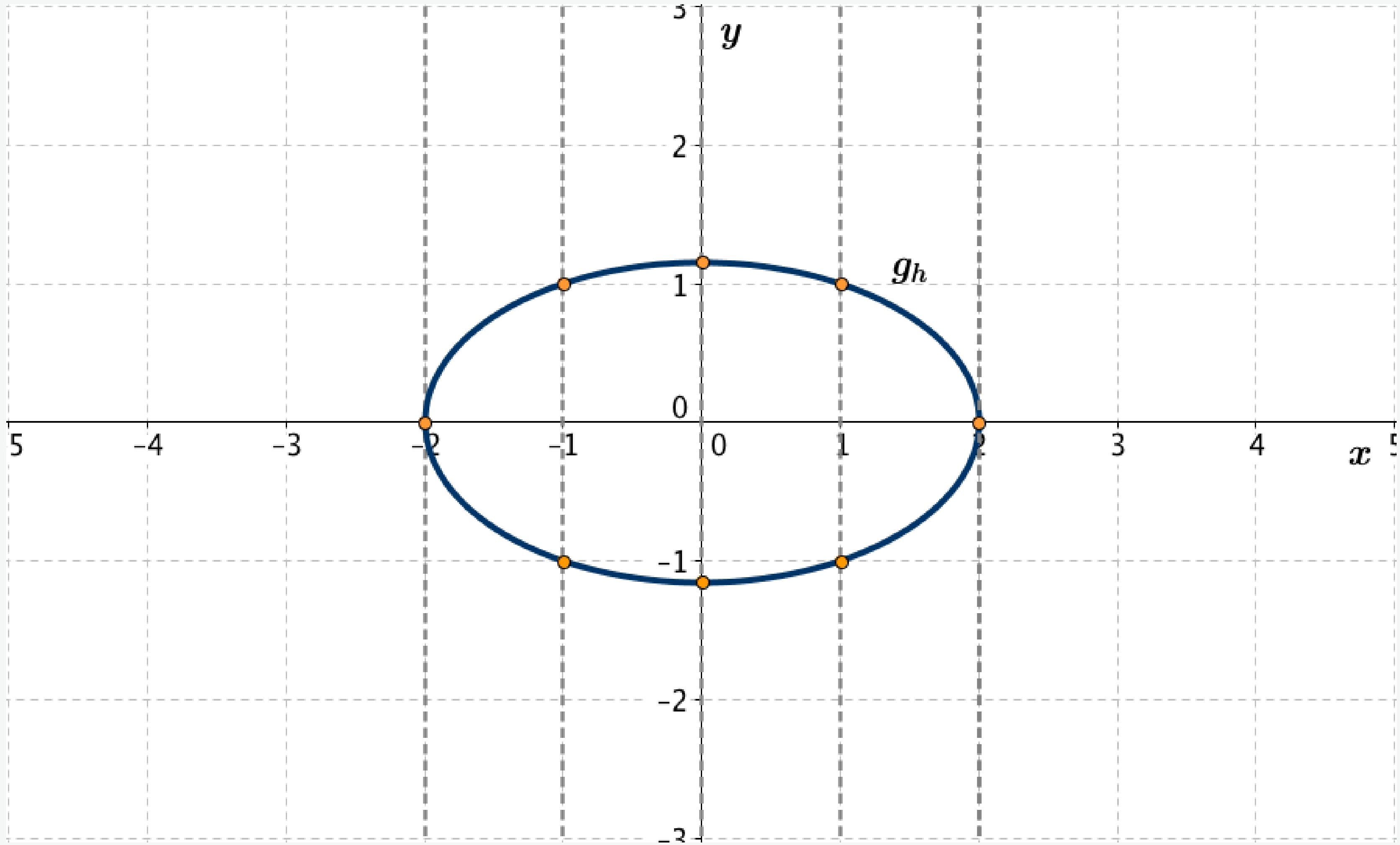
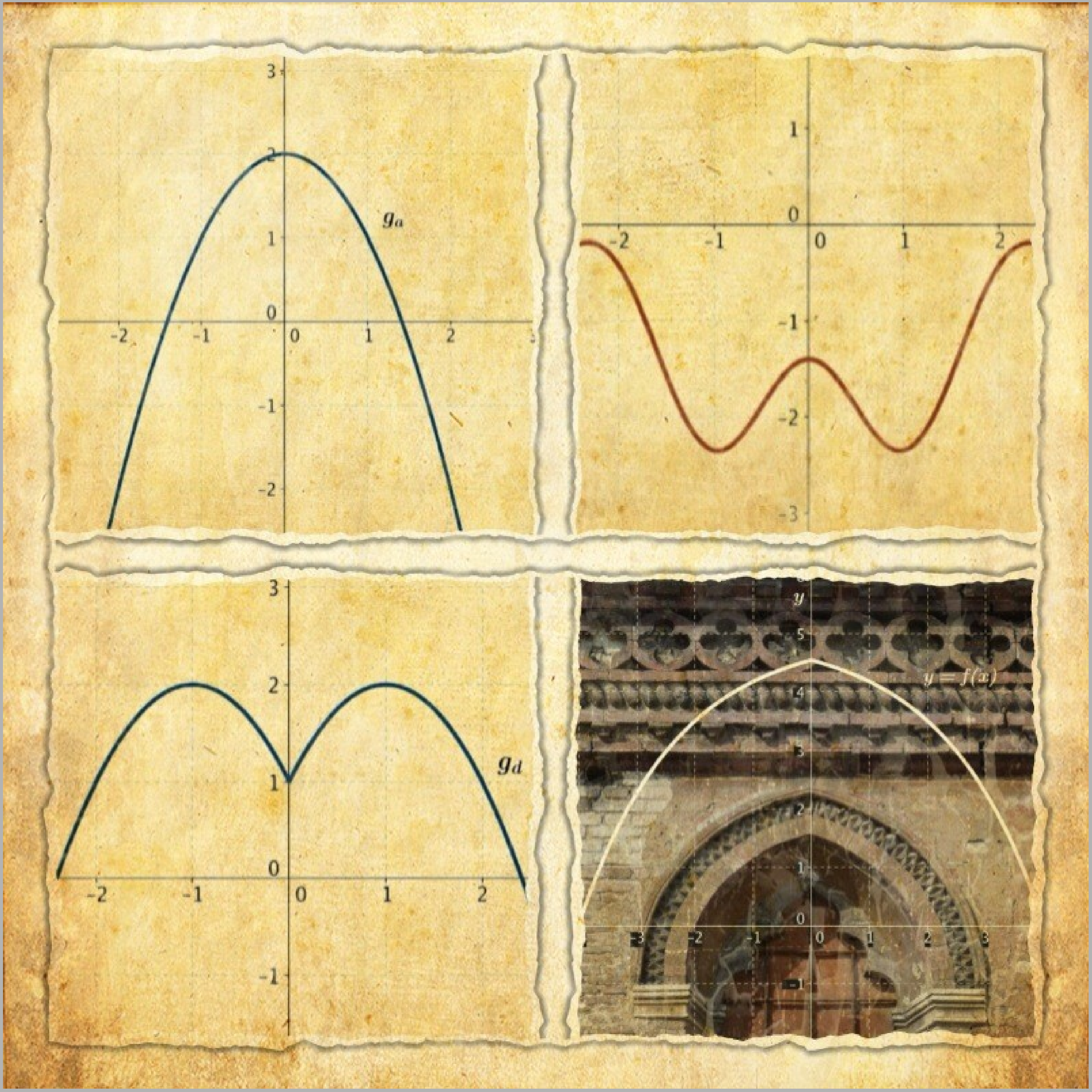
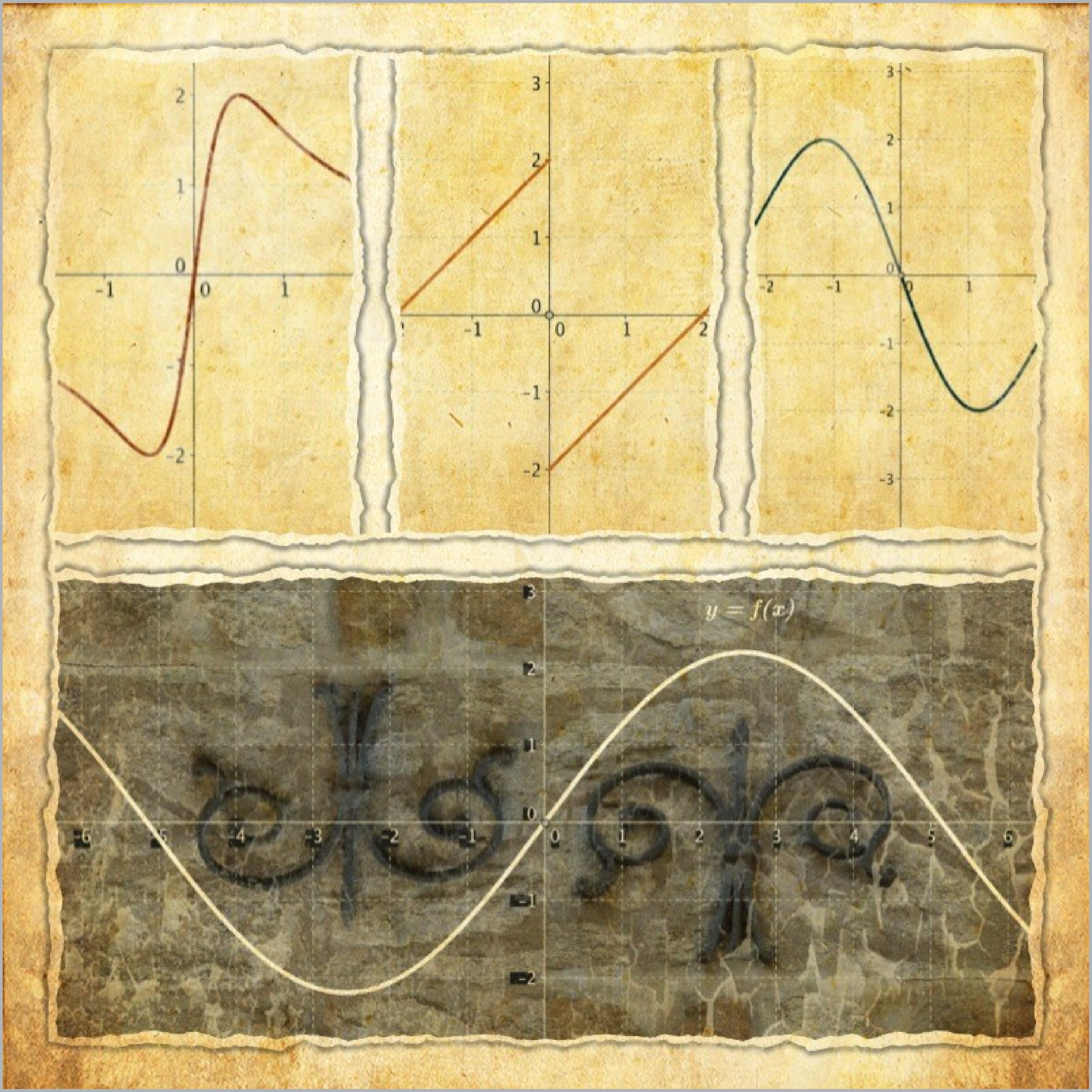
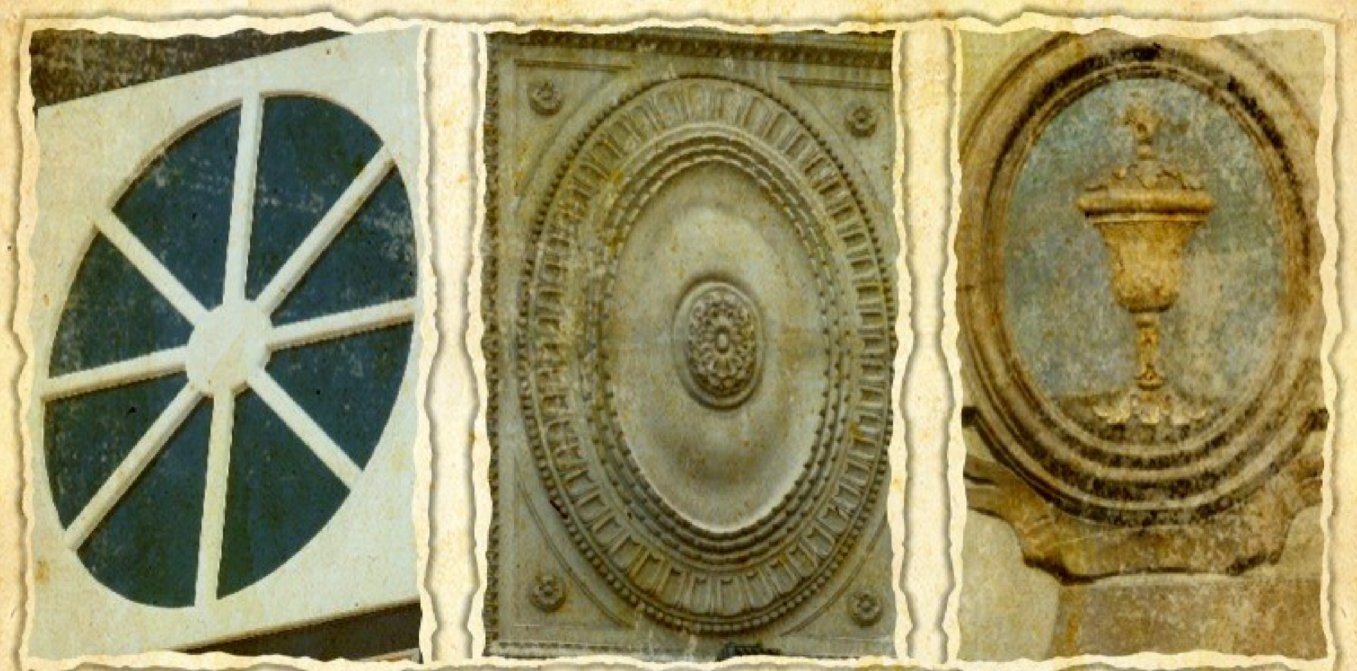
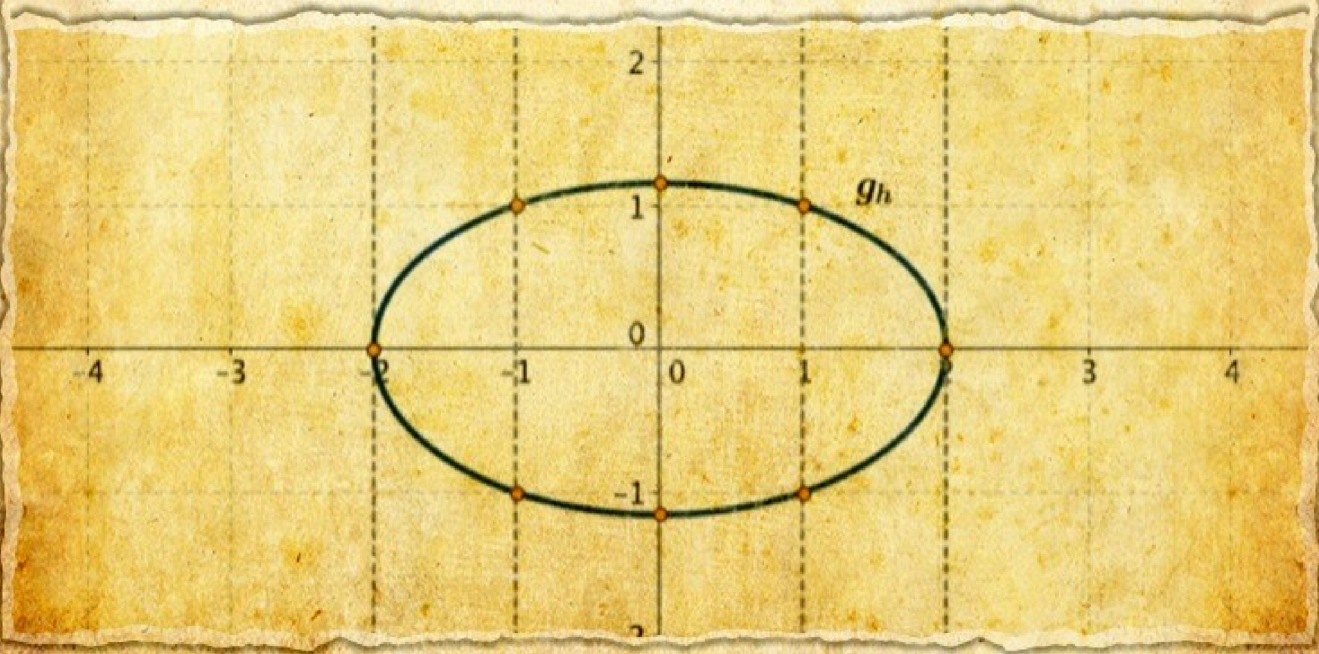


Fig. 3-5: Use of the vertical line test as a visual proof whether a graph describes a function or a relation


g_a
g_d
y
$y = f(x)$


y = f(x)





2
1
0
-1
-4
-3
-2
-1
0
1
3
4
gh