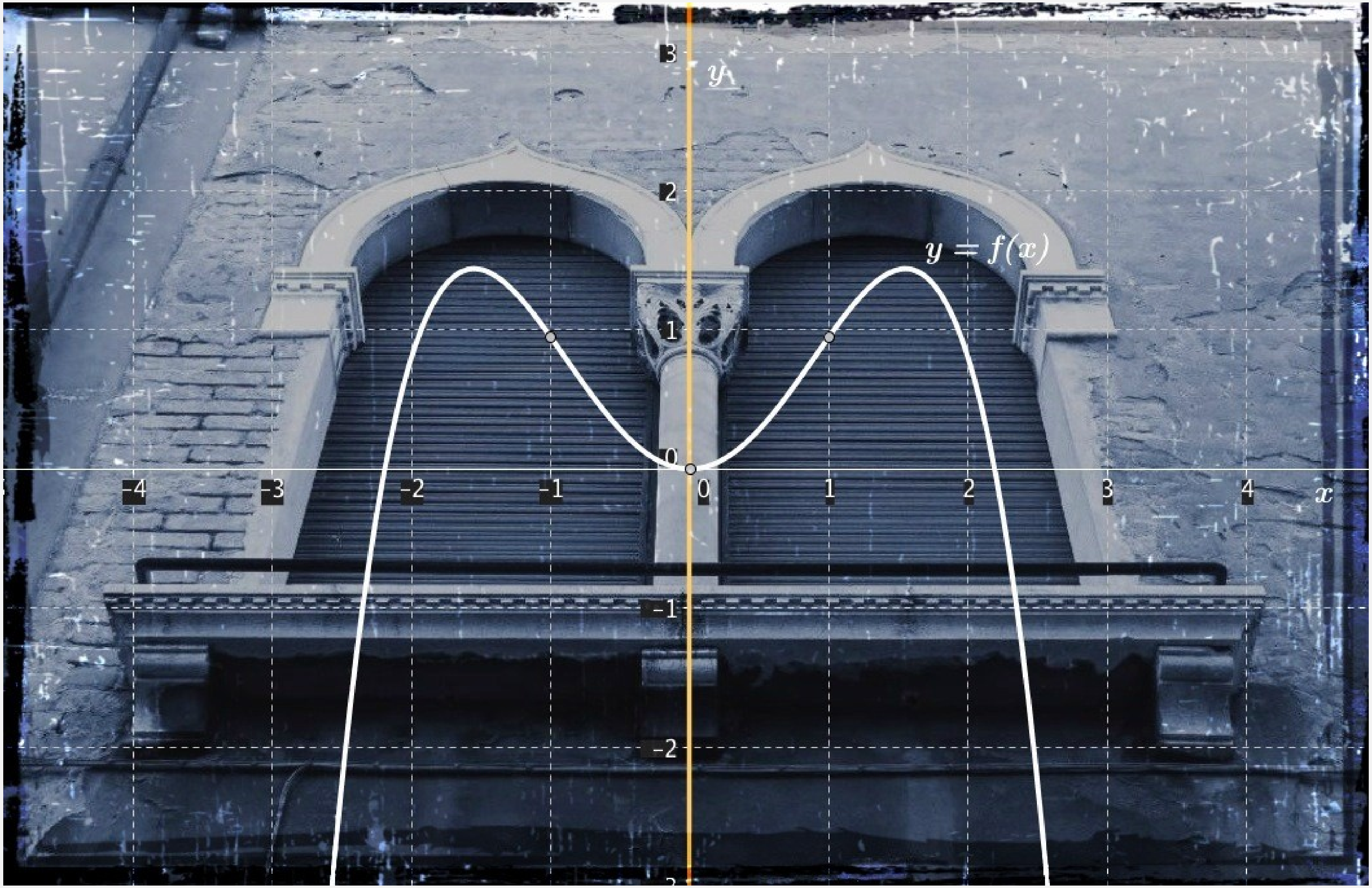

Even, odd or neither? Algebraic and graphical proof

Determine algebraically and graphically whether the functions are even, odd or neither:

$a)\ f(x) = x^4 - 2x^2,\qquad b)\ f(x) = x^3 - 4x.$

Comment:

Algebraic proof: transform x to $-x$ and compare $f(x)$ to $f(-x)$.

Graphical proof: decide on the symmetry properties by visual inspection of the function graph.

Algebraic solution *a*):

To check whether a function is even, odd or neither, we first have to find *f (-x)* and then to decide which of the following equations holds:

$(1): \quad f(-x) = f(x), \qquad (2): \quad f(-x) = -f(x)$

If equation (1) is true, it is an even function, if equation (2), it is an odd function. If neither of them holds, the function is neither even nor odd.

$$f(x) = x^4 - 2x^2$$

$$f(-x) = (-x)^4 - 2(-x)^2 = x^4 - 2x^2 = f(x)$$

Raising (-*x*) to an even power *n*, is just the same as raising $+x$ to the power *n*:

$$(-x)^2 = (-x) \cdot (-x) = x^2$$

$$(-x)^4 = (-x) \cdot (-x) \cdot (-x) \cdot (-x) = x^4$$

Graphical solution: to make a statement on the function properties, in the present case on the symmetries, by looking at the graph.

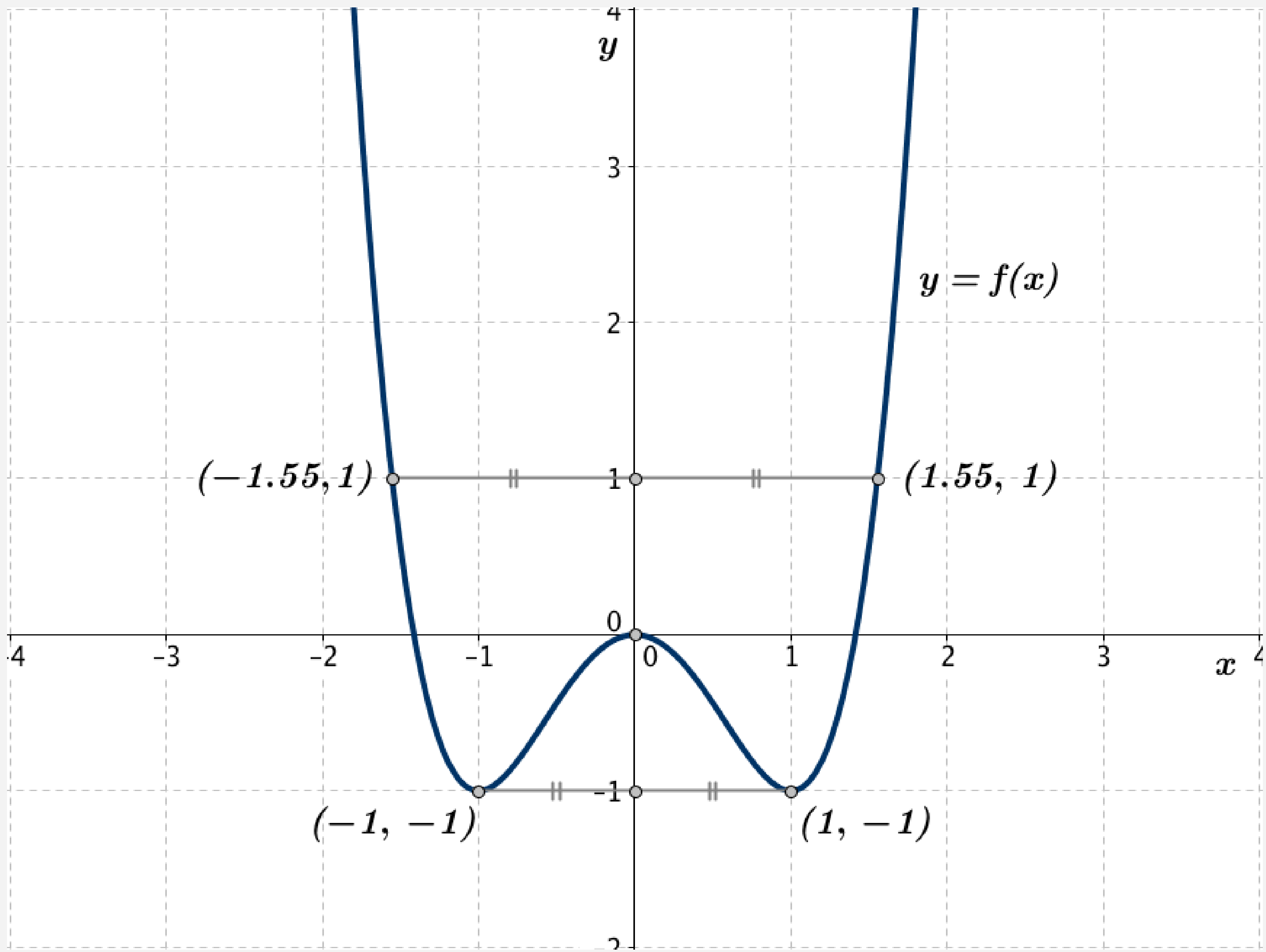

Fig. 3-1: The graph of y = f (x) (a) is symmetric with respect to the y-axis. The function is even

Algebraic solution:

$$f(x) = x^3 - 4x$$

$$f(-x) = (-x)^3 - 4(-x) = -x^3 + 4x = -(x^3 - 4x) = -f(x)$$

Raising (-*x*) to an odd power *n* yields the negative of raising +*x* to the power *n*. The function contains two odd powers, 1 and 3, of -*x*.

$$(-x)^3 = (-x)\cdot(-x)\cdot(-x) = -x^3$$

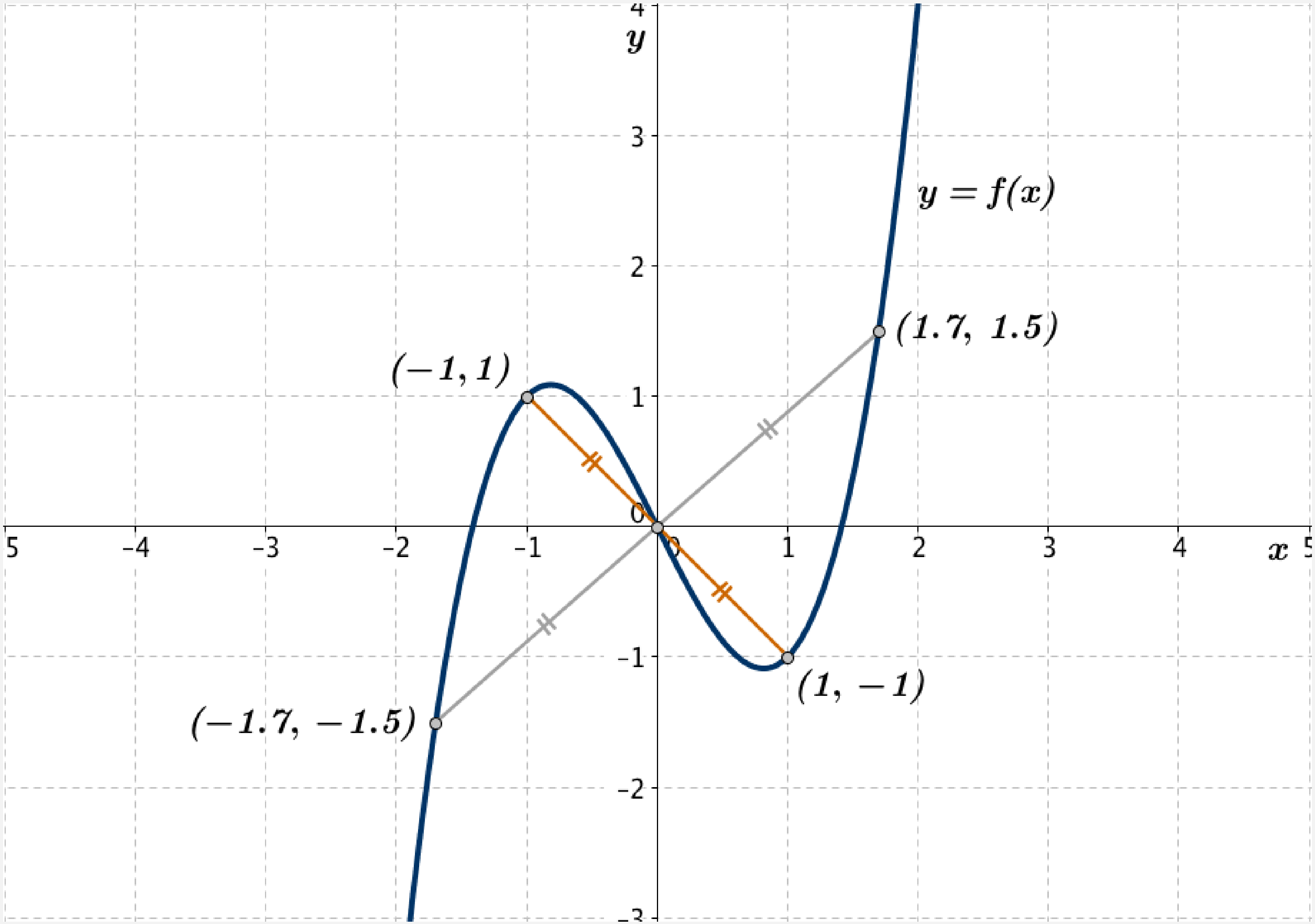

Fig. 3-2: The graph of $y = f(x)$ (b) is symmetric with respect to the origin. The function is odd

Determine, whether a function $y = f(x)$ is even, odd or neither in the given domains:

$$f(x) = \frac{x^2}{2} - 2, \qquad a)\ D_f = \mathbb{R}, \qquad b)\ D_f = [-2, 3]$$

Explain how symmetry properties can be influenced by changing the function domain.

Algebraic proof:

$$f(x) = \frac{x^2}{2} - 2$$

$$f(-x) = \frac{(-x)^2}{2} - 2 = \frac{x^2}{2} - 2 = f(x)$$

The algebraic check of the function term indicates, that the function is even. However this algebraic check is not enough. The following figures 4-1 and 4-2 show, that also the function domain can influence the symmetry.

$a)\ D_f = \mathbb{R}$

Here we have a symmetric domain, the function graph (Fig. 4-1) is symmetric with respect to the *y*-axis.

$b)\ D_f = [-2,\ 3]$

In this case the function graph (Fig. 4-2) is not symmetric with respect to the *y*-axis, as the domain is not symmetric. To point (3, 2.5) there is no symmetric point (-3, 2.5) on the function graph.

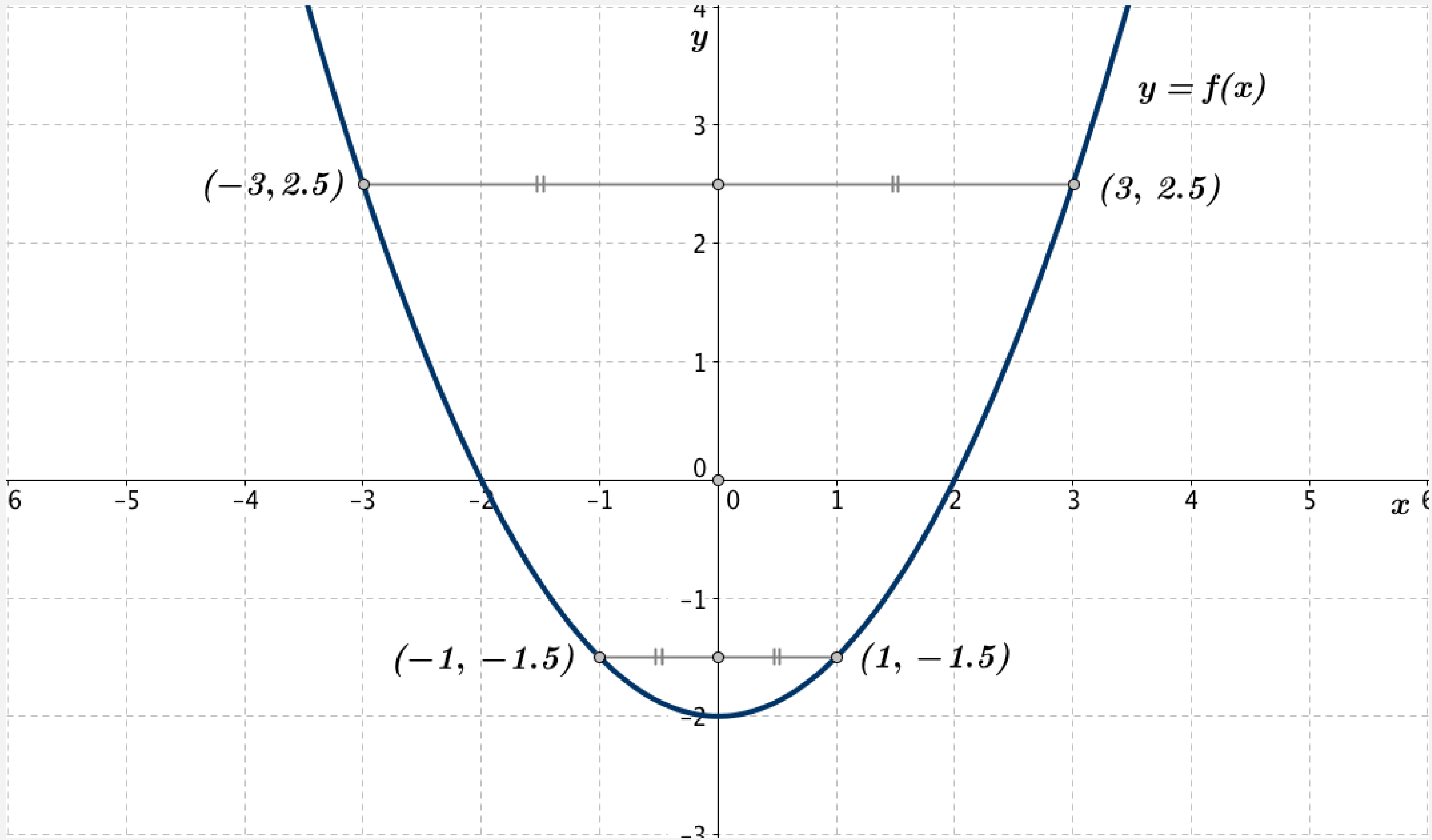

Fig. 4-1: The graph of the function $y = f(x)$ is symmetric with respect to the y-axis

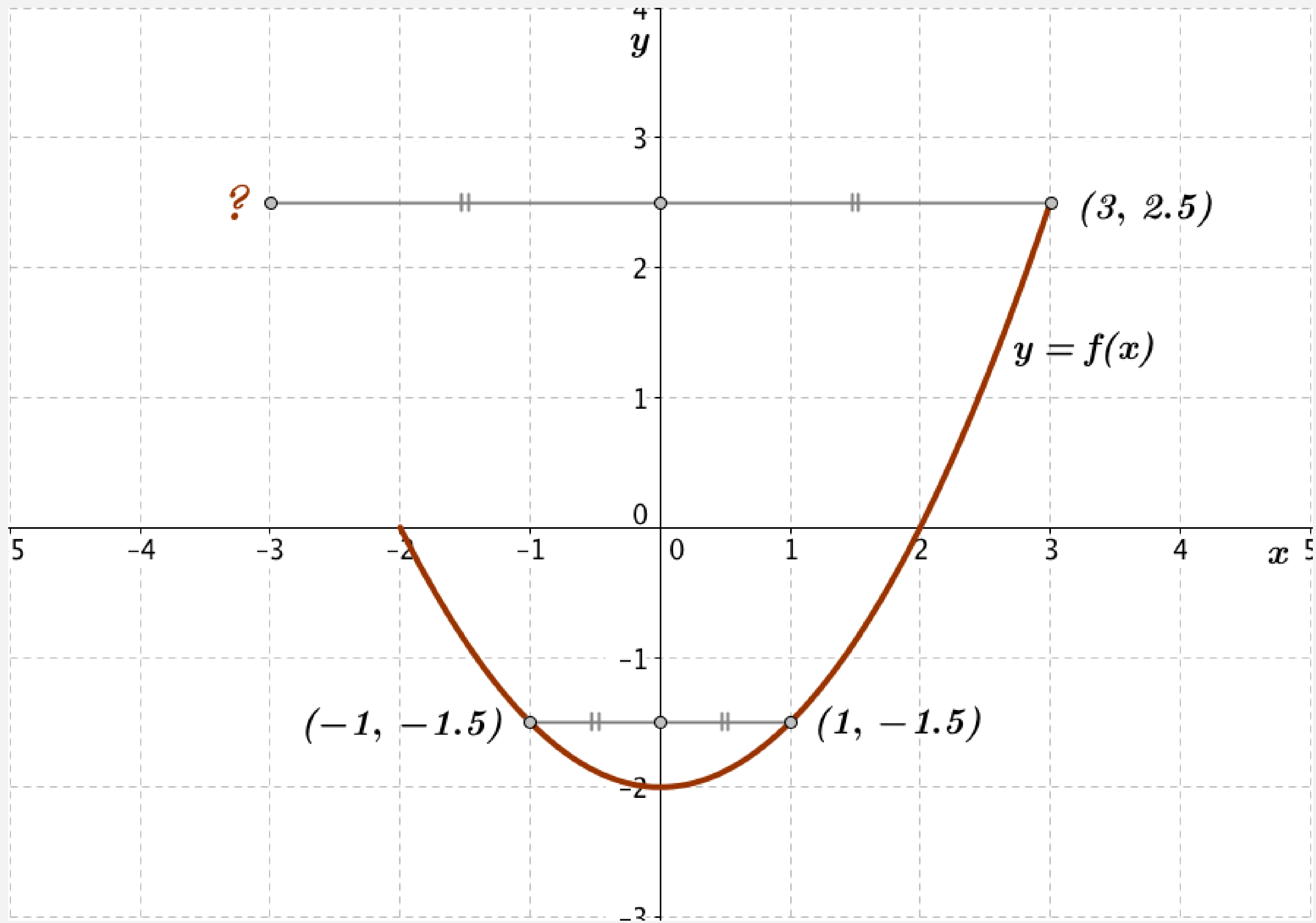

Fig. 4-2: The graph of the function y = f (x) is not symmetric with respect the y-axis

Exercise 5: Determine

1) which of the given functions are polynomial functions

$a)\ f(x) = x^2 - 2$

$b)\ f(x) = x^3 - 1$

$c)\ f(x) = 5x^2 - 3x + \sqrt{x}$

$d)\ f(x) = x^2 - 2x - 1$

$e)\ f(x) = -x^{5/2} + 7x^2 - 11x$

$g)\ f(x) = x^5 - 4x^3 + x$

2) which of the polynomial functions are even, odd or neither.

Exercise 6:

Formulate the condition for a given polynomial function, to be even odd or neither of them.

1) A polynomial $y = f(x)$ is an expression constructed from variables and constants, using only the operations of addition, subtraction and multiplication. A polynomial can have non-negative integer exponents only. The polynomial functions of this exercise are:

$$a)\ \ f(x) = x^2 - 2$$

$$b)\ \ f(x) = x^3 - 1$$

$$d)\ \ f(x) = x^2 - 2x - 1$$

$$g)\ \ f(x) = x^5 - 4x^3 + x$$

They have integer exponents only. The functions *c*) and *e*) are no polynomials. The last term of *c*) is *x* raised to power 1/2 and the first term of *e*) is *x* raised to 5/2.

$$c)\ \ f(x) = 5x^2 - 3x + \sqrt{x}$$

$$e)\ \ f(x) = -x^{5/2} + 7x^2 - 11x$$

2) To determine which of the polynomial functions is even, odd or neither, we find for each function *f (-x)*.

$a)\ \ f(x) = x^2 - 2, \qquad f(-x) = (-x)^2 - 2 = x^2 - 2 = f(x)$

$b)\ \ f(x) = x^3 - 1, \qquad f(-x) = (-x)^3 - 1 = -x^3 - 1 \neq f(x)$

$d)\ \ f(x) = x^2 - 2x - 1,$

$$f(-x) = (-x)^2 - 2(-x) - 1 = x^2 + 2x - 1 \neq f(x)$$

$g)\ \ f(x) = x^5 - 4x^3 + x,$

$$f(-x) = (-x)^5 - 4(-x)^3 + (-x) = -x^5 + 4x^3 - x =$$

$$= -(x^5 - 4x^3 + x) = -f(x)$$

The function *a*) is even, the function *g*) is odd, and the functions *b*) and *d*) are neither even nor odd.

There is also another possibility to test, whether a function is even, odd or neither. As we have shown algebraically the function *a*) is even:

$$a)\ \ f(x) = x^2 - 2, \qquad f(-x) = f(x)$$

Let us take two values of x, $x = 1$ and $x = -1$, which are symmetric about the origin and evaluate the function for both x-values:

$$f(-1) = (-1)^2 - 2 = 1 - 2 = -1, \qquad f(1) = 1^2 - 2 = 1 - 2 = -1$$

$$f(-1) = f(1)$$

$$b)\ \ f(x) = x^3 - 1, \qquad f(-x) \neq f(x)$$

$$f(-1) = (-1)^3 - 1 = -1 - 1 = -2, \qquad f(1) = 1^3 - 1 = 1 - 1 = 0$$

$$f(-1) \neq \pm f(1)$$

$d)\ \ f(x) = x^2 - 2x - 1, \qquad f(-x) \neq f(x)$

$$f(-1) = (-1)^2 - 2\cdot(-1) - 1 = 1 + 2 - 1 = 2$$

$$f(1) = 1^2 - 2\cdot 1 - 1 = 1 - 2 - 1 = -2$$

$$f(-1) = -f(1)$$

We have shown already, that this function is neither even nor odd. However, for these particular *x*-values the function shows the property of being odd.

To remember!

If we just evaluate the function at two *x*-values which are symmetric about the origin, we *have to be sure*, that the same result will be obtained for all other pair of symmetric *x*-values of the function domain.

The figure on the next page shows the graph of this function with two points:

$$P_1 = (1,\ f(1)) = (1, -2), \qquad P_2 = (-1,\ f(-1)) = (-1, 2)$$

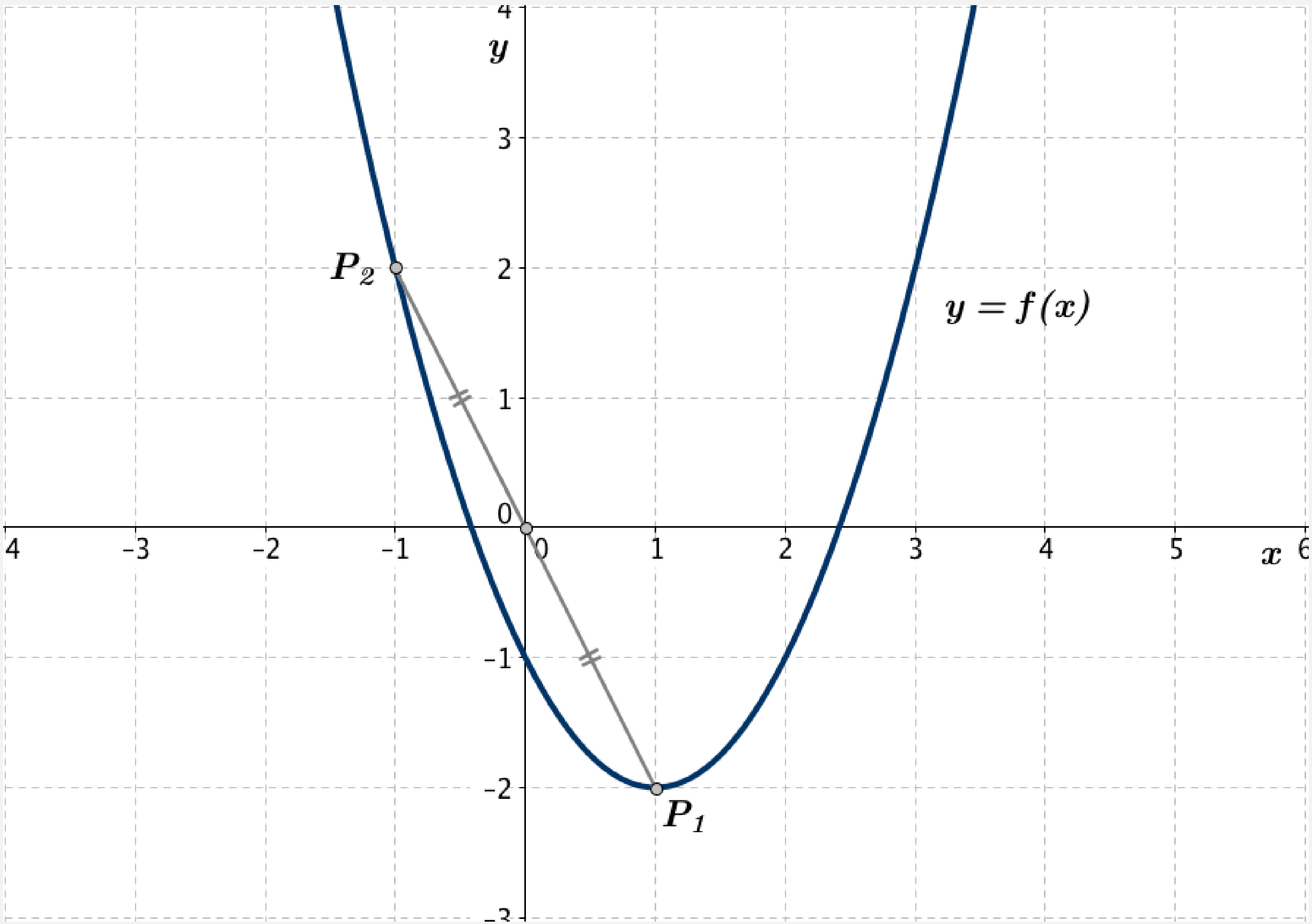

Fig. 4: Graph of a neither even nor odd function $y = f(x)$ with two points

$$(-x)^n = \begin{cases} x^n, & \text{if } n \text{ is even} \\ -x^n, & \text{if } n \text{ is odd} \end{cases}$$

Terms with *even* powers of x will remain the same, when x is replaced by $-x$.

Terms with *odd* powers of x will change sign, when x is replaced by $-x$.

Function symmetry: Solution 6

Polynomial functions with terms containing only even powers of the variable *x* and multiple or additive constants are even functions. For example, the following functions are even:

$$f(x) = \frac{x^2}{2} - 2$$

$$g(x) = -3x^4 + 6x^2 - 1$$

$$h(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$$

These functions are presented in Fig. 5-1.

A multiple constant is a numerical or constant factor in an algebraic term. For example, 1/2 is the multiple constant in the function *f (x)*. - 3 and 6 are multiple constants in the function *g (x)*.

The constant - 2 in the function *f (x)* and -1 in the function *g (x)* are additive constants.

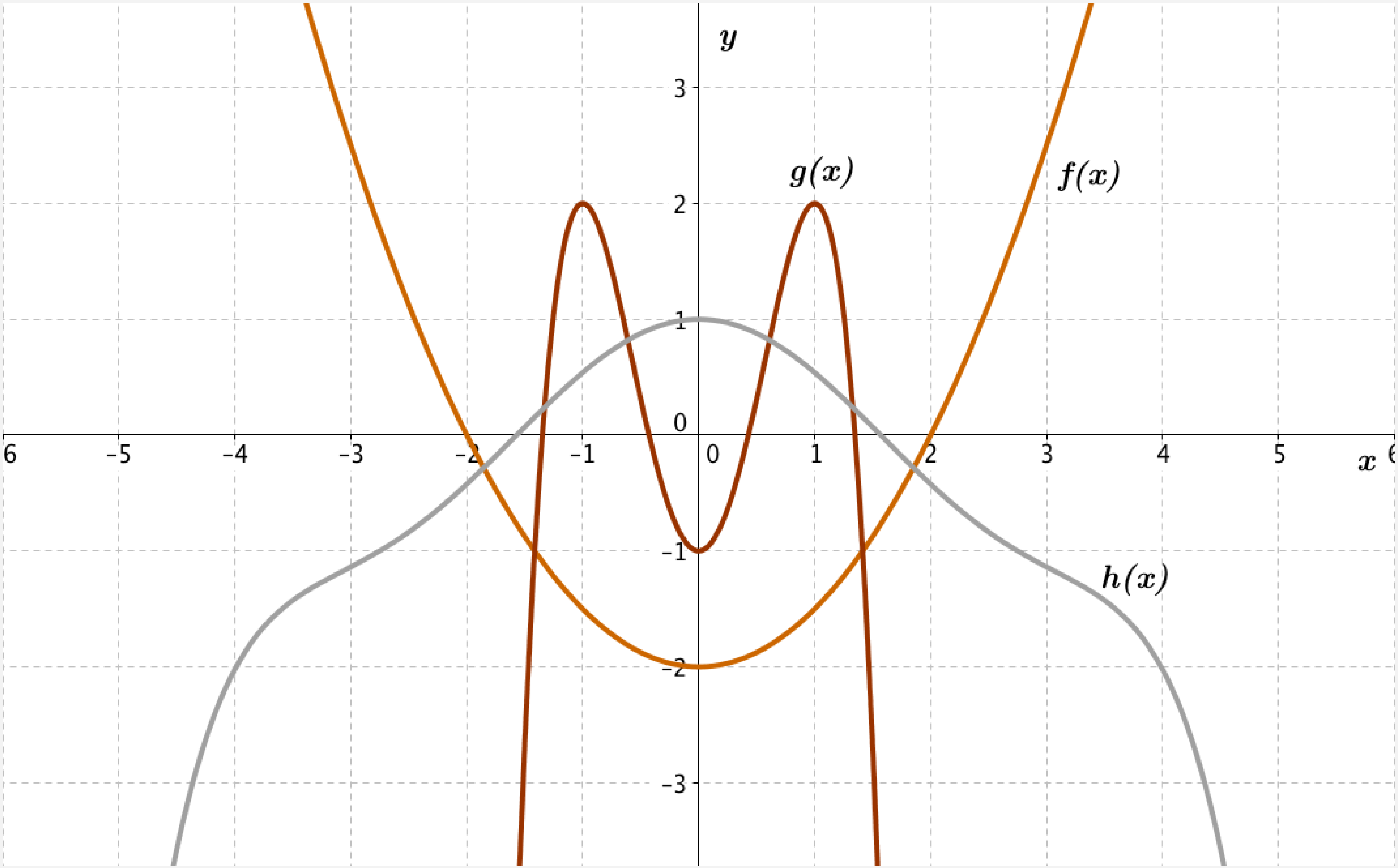

Fig. 5-1: Graphs of even functions

Polynomial functions with terms containing only odd powers of the variable x are odd functions. For example, odd functions are:

$$f(x) = \frac{x^3}{6}$$

$$g(x) = -6x^5 + 9x^3 - x$$

$$h(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$$

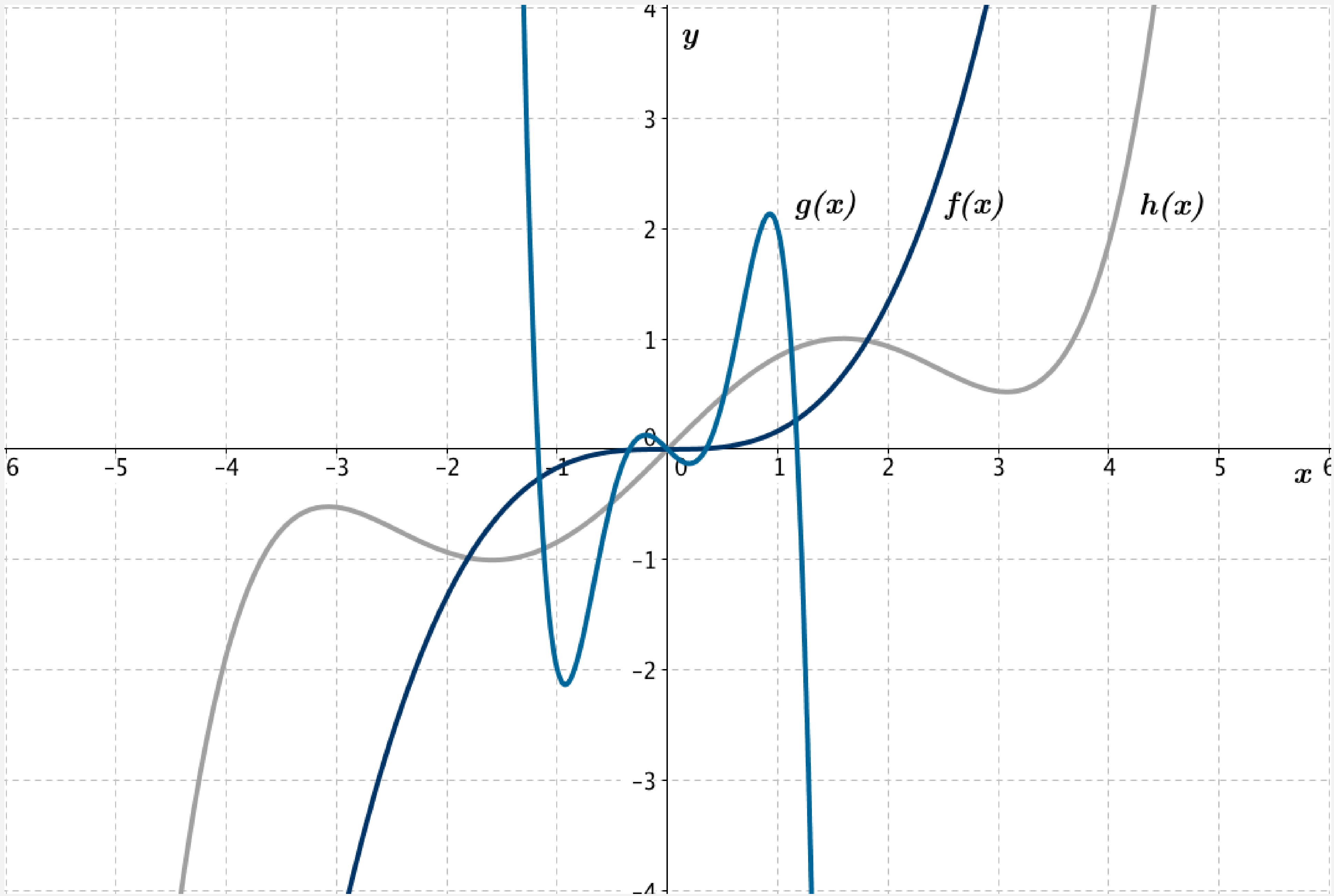

Fig. 5-2: Graphs of odd functions

Determine which of the functions given below are even, odd or neither:

a) $f(x) = \dfrac{1}{x}$, $\quad g(x) = \dfrac{1}{x-3}$, $\quad h(x) = \dfrac{2x}{x+7}$

b) $f(x) = \dfrac{1}{x^2}$, $\quad g(x) = \dfrac{x}{x^2+1}$, $\quad h(x) = \dfrac{5x^3}{x^2-16}$

c) $f(x) = \dfrac{1}{x^3}$, $\quad g(x) = \dfrac{3}{x^3-4x}$, $\quad h(x) = \dfrac{2x^3-x^2}{x^2+5}$

d) $f(x) = \dfrac{x^2+7}{x^2-3x^4}$, $\quad g(x) = \dfrac{x^3-11x}{x^4+12}$, $\quad h(x) = \dfrac{5x^3}{x^7-9x^3}$

e) $f(x) = \dfrac{1}{|x|}$, $\quad g(x) = \dfrac{|x|}{x}$, $\quad h(x) = \dfrac{1}{|x|+1/2}$

e) $f(x) = \dfrac{1}{x^2+2|x|+1}$, $\quad g(x) = \dfrac{1}{x^2-0.8|x|+1/2}$

Formulate the conditions for a rational function to be even or odd.

Definition:

A rational function is a function which can be defined by a rational fraction, i.e. an algebraic fraction such that both the numerator and the denominator are polynomials.

$$f(x) = \frac{P(x)}{Q(x)}, \qquad Q(x) \neq 0$$

Example of functions *b*):

$$f(x) = \frac{1}{x^2}, \qquad P_f(x) = 1, \qquad Q_f(x) = x^2$$

$$g(x) = \frac{x}{x^2 + 1}, \qquad P_g(x) = x, \qquad Q_g(x) = x^2 + 1$$

$$h(x) = \frac{5x^3}{x^2 - 16}, \qquad P_h(x) = 5x^3, \qquad Q_h(x) = x^2 - 16$$

a) $f(x) = \dfrac{1}{x}, \qquad g(x) = \dfrac{1}{x-3}, \qquad h(x) = \dfrac{2x}{x+7}$

$$f(-x) = -\frac{1}{x} = -f(x), \qquad g(-x) = \frac{1}{-x-3} = -\frac{1}{x+3}$$

$$h(-x) = -\frac{2x}{-x+7} = \frac{2x}{x-7}$$

The function *f (x)* is odd.

b) $f(x) = \dfrac{1}{x^2}, \qquad g(x) = \dfrac{x}{x^2+1}, \qquad h(x) = \dfrac{5x^3}{x^2-16}$

$$f(-x) = \frac{1}{x^2} = f(x), \qquad g(-x) = \frac{(-x)}{(-x)^2+1} = -\frac{x}{x^2+1} = -g(x)$$

$$h(-x) = \frac{5(-x)^3}{(-x)^2-16} = -\frac{5x^3}{x^2-16} = -h(x)$$

The function *f (x)* is even, the functions *g (x)* and *h (x)* are odd.

Function symmetry: Solution 7c

c) $f(x) = \dfrac{1}{x^3}$, $\quad g(x) = \dfrac{3}{x^3 - 4x}$, $\quad h(x) = \dfrac{2x^3 - x^2}{x^2 + 5}$

$$f(-x) = \frac{1}{(-x)^3} = -\frac{1}{x^3} = -f(x)$$

$$g(x) = \frac{3}{x^3 - 4x} = \frac{3}{(-x)^3 - 4(-x)} = \frac{3}{-x^3 + 4x} = -\frac{3}{x^3 - 4x} = -g(x)$$

$$h(-x) = \frac{2(-x)^3 - (-x)^2}{(-x)^2 + 5} = \frac{-2x^3 - x^2}{x^2 + 5} = \frac{-(2x^3 + x^2)}{x^2 + 5}$$

The function *f (x)* and *g (x)* are odd.

Function symmetry: Solution 7d

$$d)\ \ f(x)=\frac{x^2+7}{x^2-3x^4},\qquad g(x)=\frac{x^3-11x}{x^4+12},\qquad h(x)=\frac{5x^3}{x^7-9x^3}$$

$$f(-x)=\frac{(-x)^2+7}{(-x)^2-3(-x)^4}=\frac{x^2+7}{x^2-3x^4}=f(x)$$

$$g(-x)=\frac{(-x)^3-11\cdot(-x)}{(-x)^4+12}=\frac{-x^3+11x}{x^4+12}=-\frac{x^3-11x}{x^4+12}=-g(x)$$

$$h(-x)=\frac{5\cdot(-x)^3}{(-x)^7-9(-x)^3}=\frac{5x^3}{x^7-9x^3}=h(x)$$

The functions *f (x)* and *h (x)* are even, the function *g (x)* is odd.

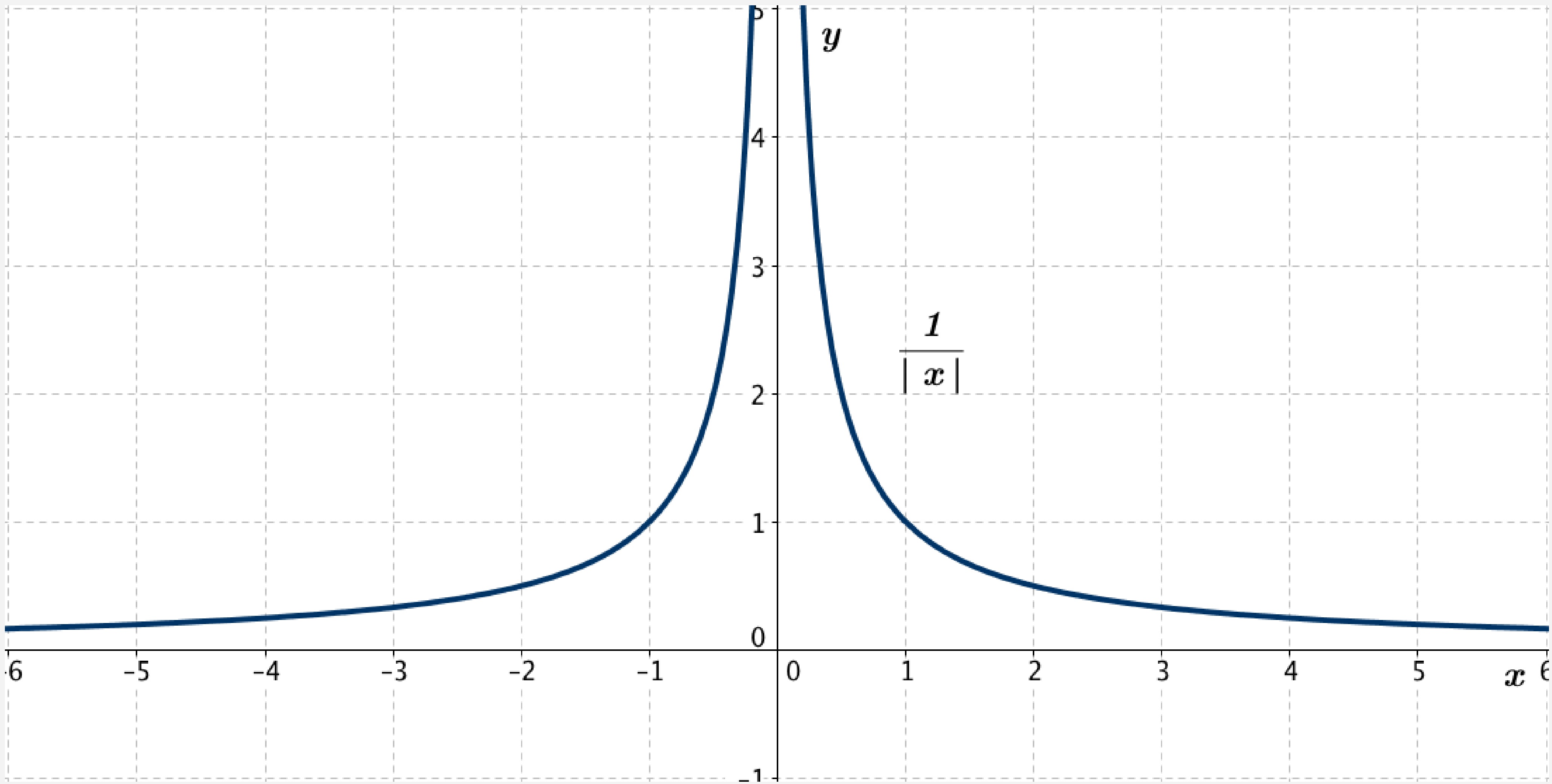

Fig. 7-1: Graph of the even function y = f (x)

$$f(x) = \frac{1}{|x|}, \qquad f(-x) = \frac{1}{|-x|} = \frac{1}{|x|} = f(x)$$

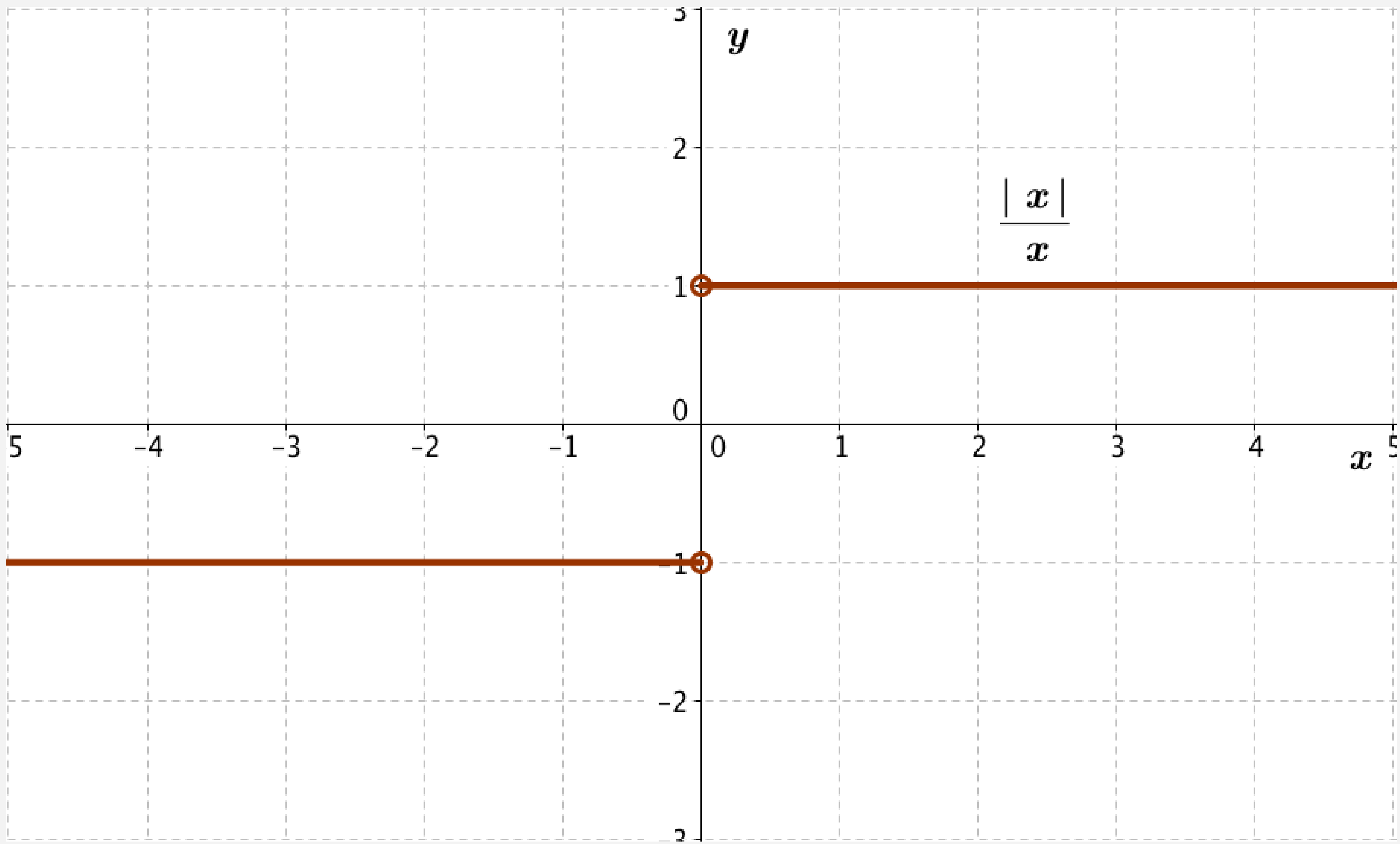

Fig. 7-2: Graph of the odd function y = g (x)

$$g(x) = \frac{|x|}{x}, \qquad g(-x) = \frac{|-x|}{-x} = -\frac{|x|}{x} = -g(x)$$

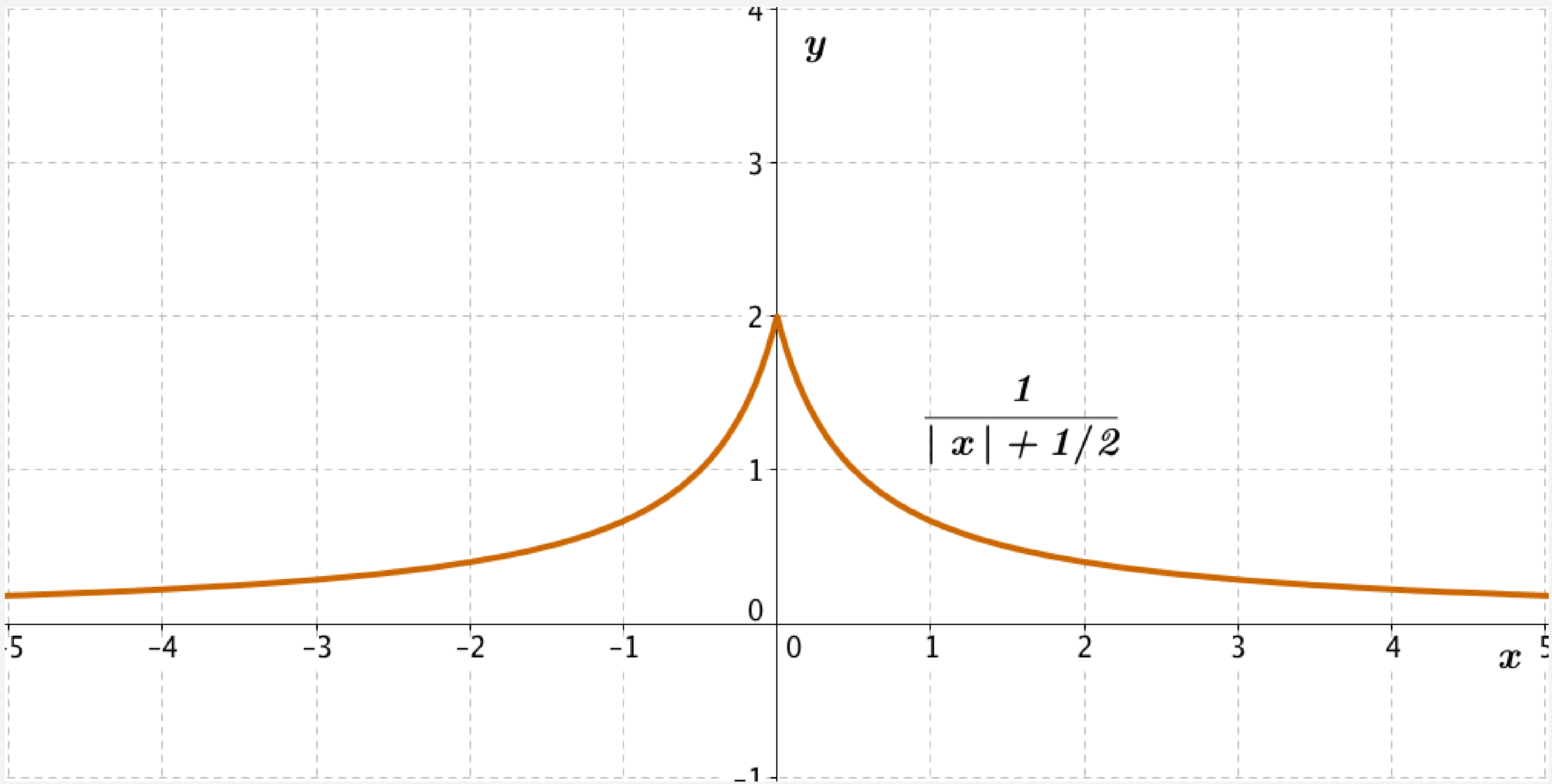

Fig. 7-3: Graph of the even function y = h (x)

$$h(x) = \frac{1}{|x| + 1/2}, \qquad h(-x) = \frac{1}{|-x| + 1/2} = \frac{1}{|x| + 1/2} = h(x)$$

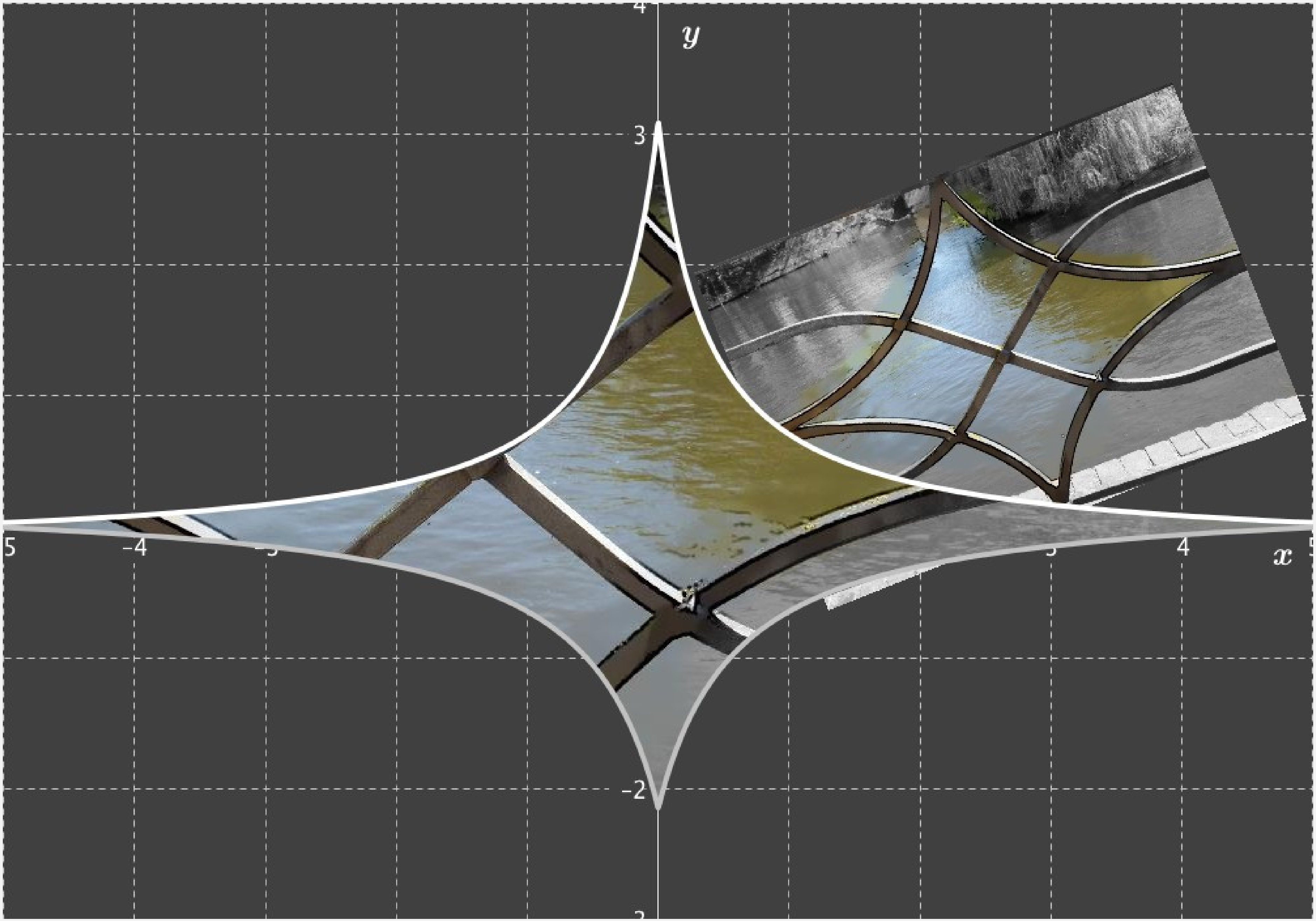

Fig. 7-4: Even functions, which we may observe (Lüneburg)

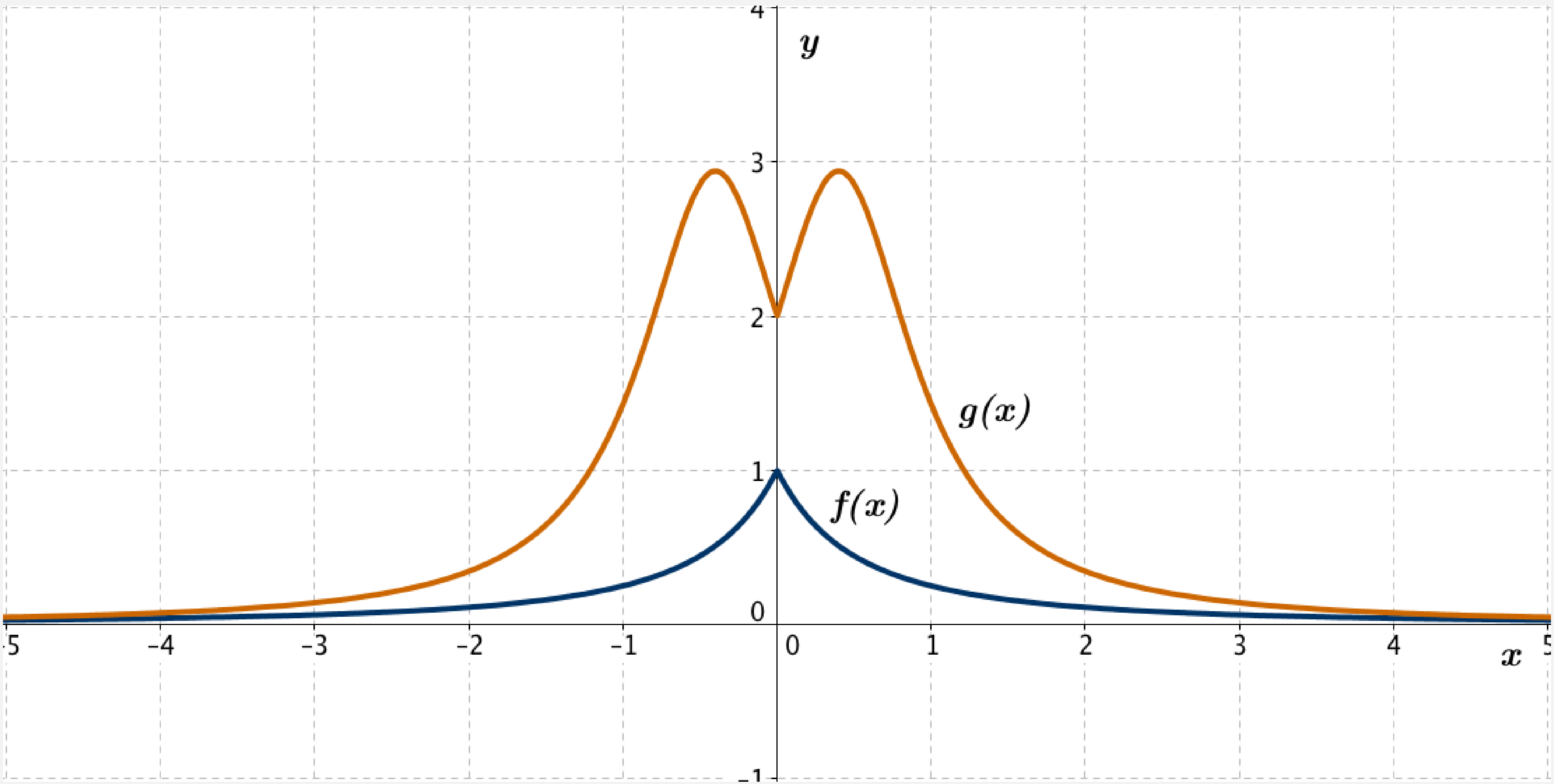

Fig. 7-4: Graphs of even functions

$$f(x) = \frac{1}{x^2 + 2|x| + 1}, \qquad g(x) = \frac{1}{x^2 - 0.8|x| + 1/2}$$

$$f(x) = \frac{P(x)}{Q(x)}, \qquad Q(x) \neq 0$$

A rational function $y = f(x)$ is even, when

- the numerator $P(x)$ and the denominator $Q(x)$ are even functions, for example, $y = f(x)$ *a)* and $y = f(x)$ *d)*:

$$a)\ f(x) = \frac{1}{x^2}, \qquad d)\ f(x) = \frac{x^2 + 7}{x^2 - 3x^4}, \qquad e)\ f(x) = \frac{1}{|x|}$$

- the numerator $P(x)$ and the denominator $Q(x)$ are odd functions, for example, $y = h(x)$ *d)*:

$$h(x) = \frac{5x^3}{x^7 - 9x^3}$$

$$f(x) = \frac{P(x)}{Q(x)}, \qquad Q(x) \neq 0$$

A rational function $y = f(x)$ is odd, when

- the numerator $P(x)$ is odd and the denominator $Q(x)$ is even, for example:

$$b)\ \ g(x) = \frac{x}{x^2 + 1}, \qquad h(x) = \frac{5x^3}{x^2 - 16}$$

- the numerator $P(x)$ is even and the denominator $Q(x)$ is odd, for example:

$$a)\ \ f(x) = \frac{1}{x}, \qquad c)\ \ g(x) = \frac{3}{x^3 - 4x}, \qquad e)\ \ g(x) = \frac{|x|}{x}$$