Gaussian Elimination

Case A (regular):

There is exactly one solution

Case B (singular):

There is no uniquely defined solution

- the system is inconsistent, there is no solution
- the equations are not independent, there is an infinite set of solutions

These characteristics of the resolvability of a system of 2 linear equations stay when considering systems of linear equations with an arbitrary number of equations and unknowns.

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases}$$

Determine analytically and graphically the solutions of the systems of linear equations given below

system 1: $\begin{cases} -0.5\,x + y = 1 \\ 2\,x - y = 2 \end{cases}$

system 2: $\begin{cases} -0.5\,x + y = 1 \\ -0.5\,x + y = 0 \end{cases}$

system 3: $\begin{cases} x + y = 2 \\ 2\,x + 2\,y = 4 \end{cases}$

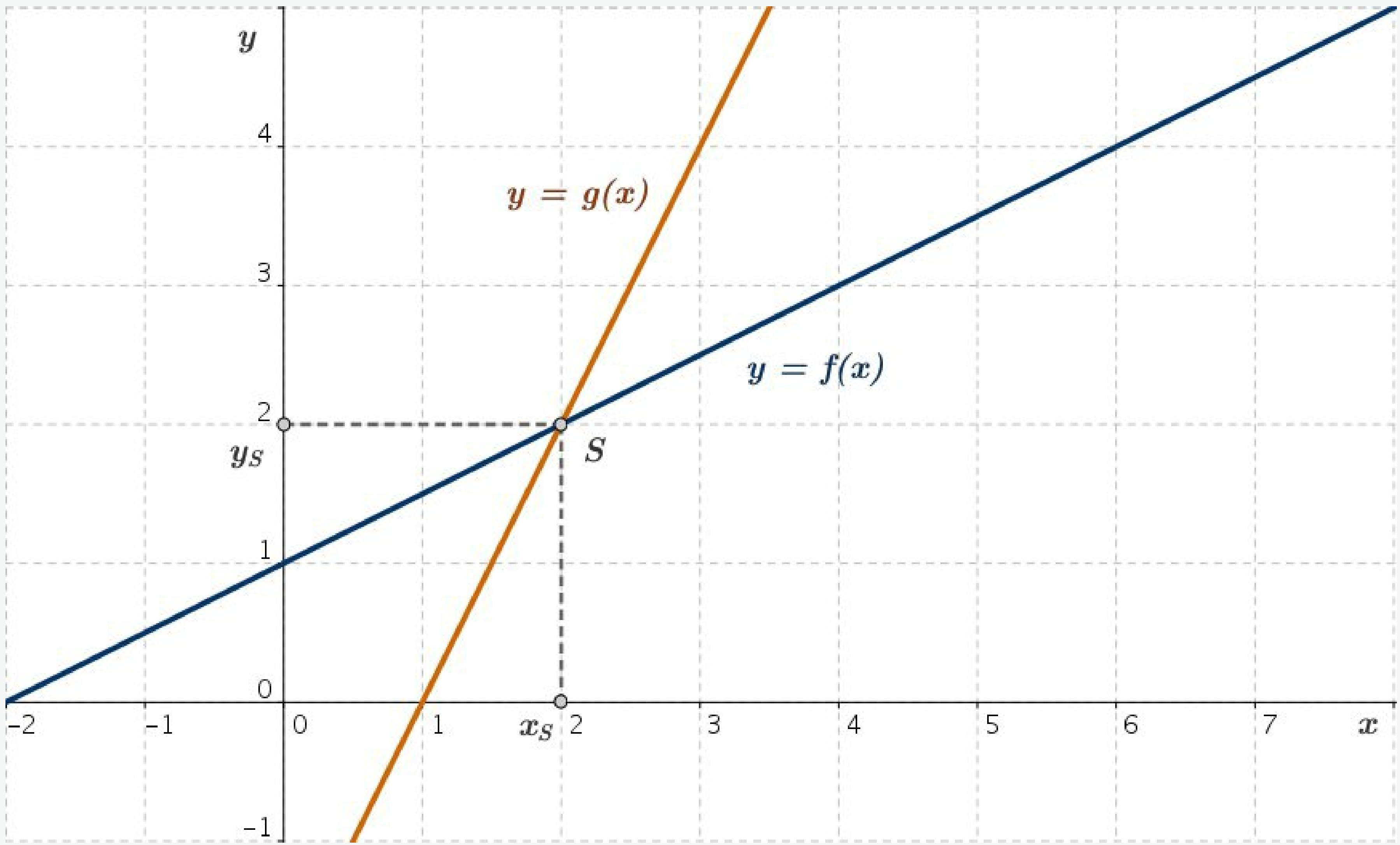


Fig. 3-1: Linear functions $f(x) = 1 + 0.5x$ and $g(x) = 2x - 2$

The system 1 has one solution $x = 2,\ y = 2$: $\begin{cases} -0.5\,x + y = 1 \\ 2\,x - y = 2 \end{cases}$

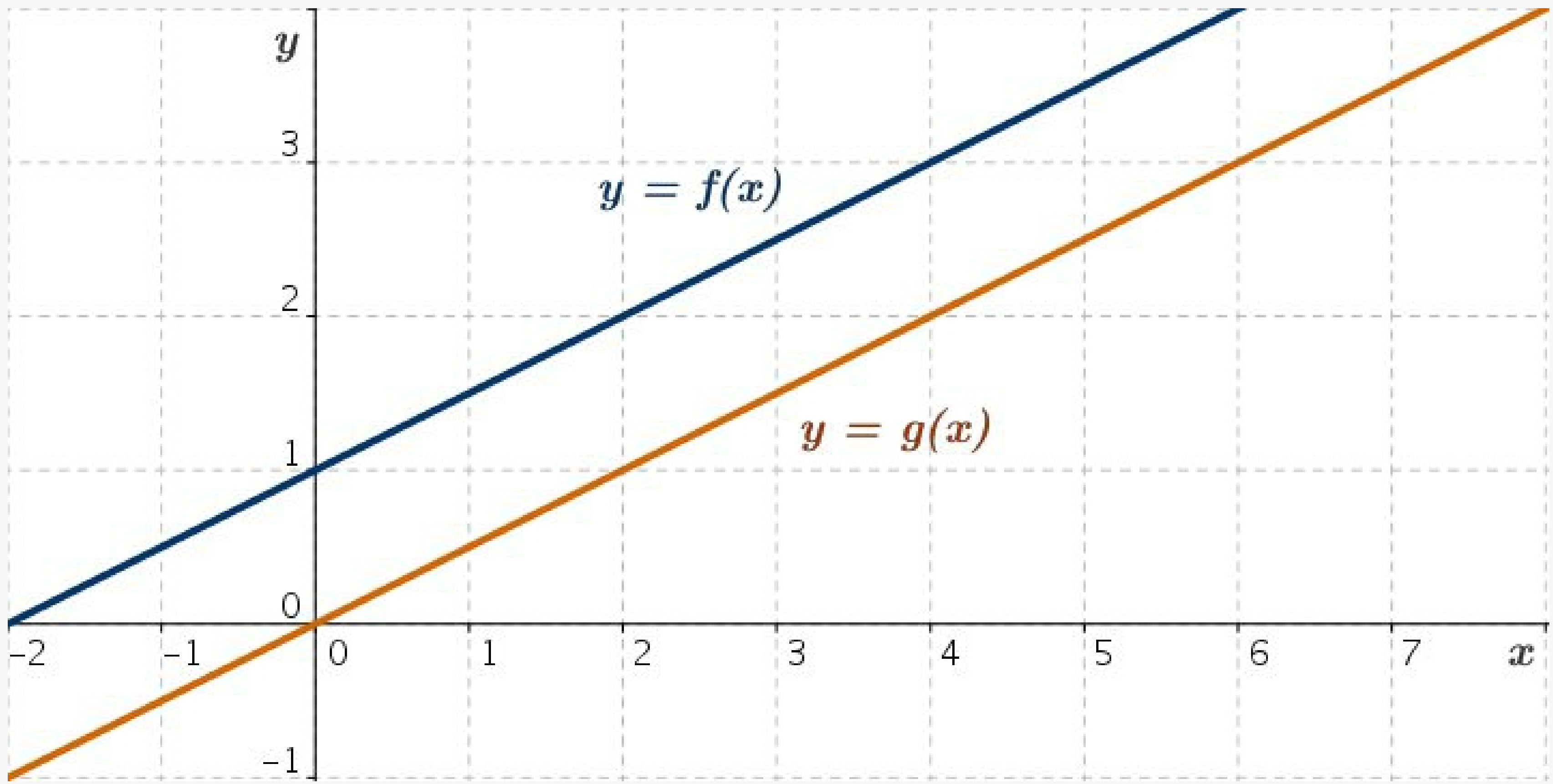


Fig. 3-2: Linear functions $f(x) = 1 + 0.5\,x$ *and* $g(x) = 0.5\,x$

$$\text{system 2:} \quad \begin{cases} -0.5\,x + y = 1 \\ -0.5\,x + y = 0 \end{cases}$$

The straight lines $y = 1 + 0.5\,x$ and $y = 0.5\,x$ are parallel. Therefore they have no common point. This system of equations has no solution.

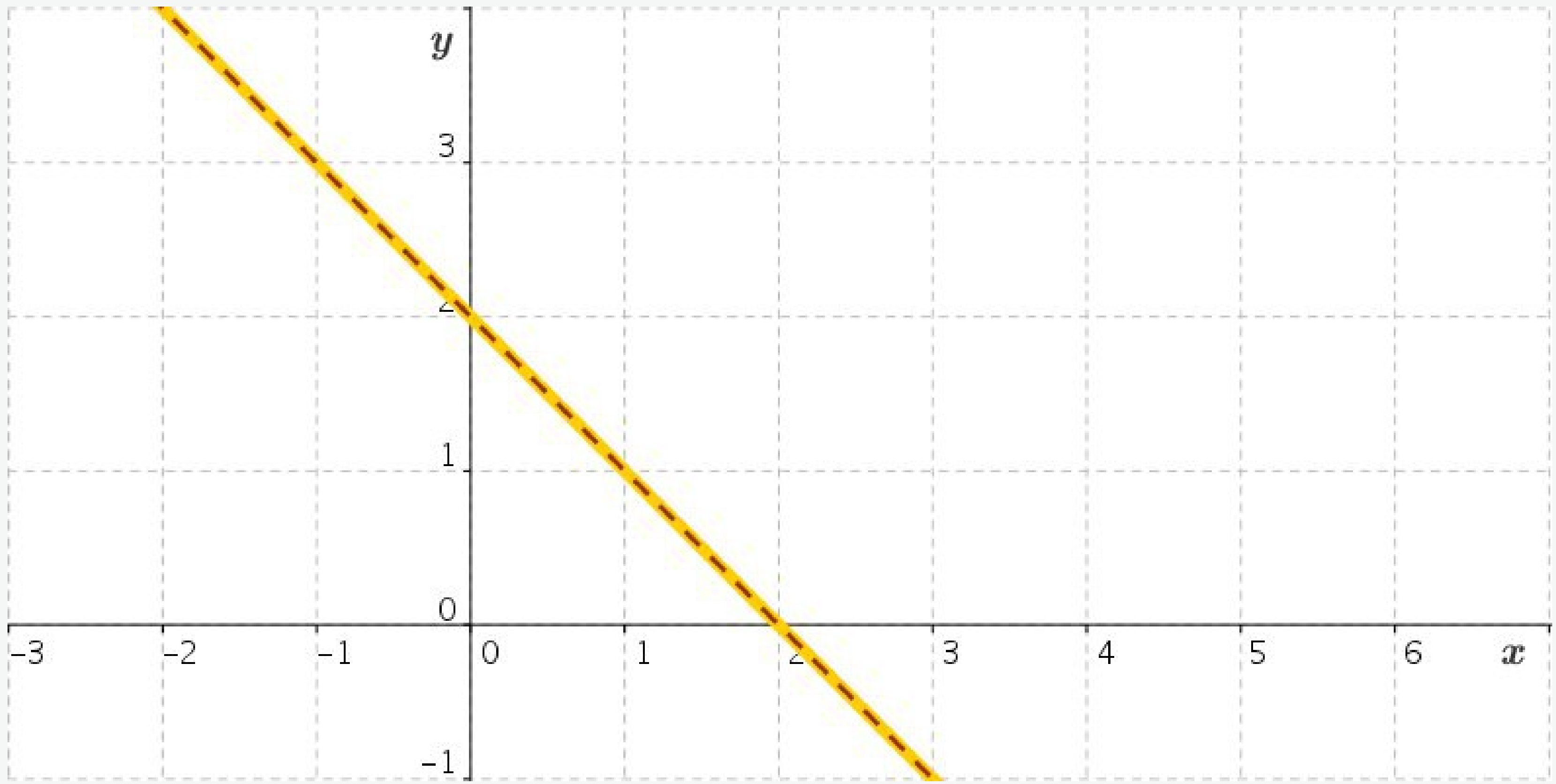


Fig. 3-3: linear functions $y = 2 - x$ and $2y = 4 - 2x$

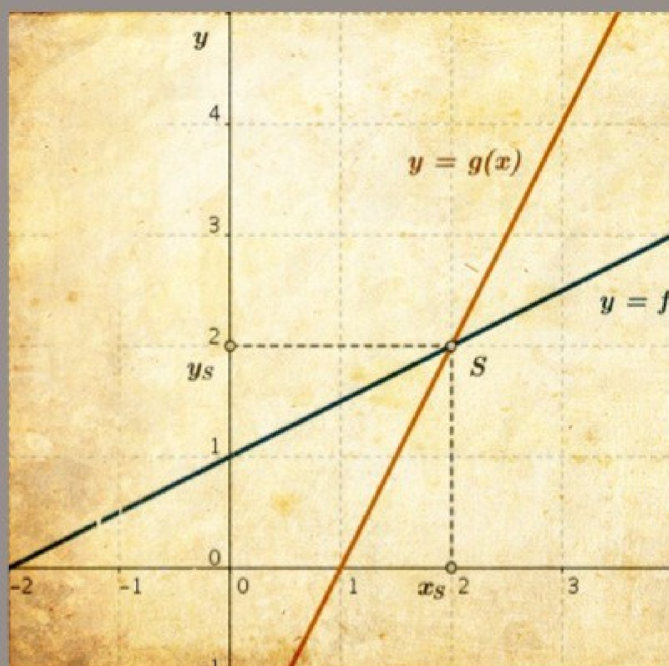
$$\text{system 3:} \quad \begin{cases} x + y = 2 \\ 2x + 2y = 4 \end{cases}$$

The two equations are equivalent, describing the same function. The number of solutions is infinite.

Carl Friedrich Gauß (1777-1855), brilliant German mathematician

Gaussian Elimination

Below we demonstrate the algorithm Gaussian Elimination by solving a system of three linear equations with three unknowns.


y
4
3
2
1
0
-1
y = g(x)
y = f
S
y_S
x_S
-2
-1
0
1
2
3

system:

$$\begin{cases} E_1: & -x + y + z = 0 \\ E_2: & x - 3y - 2z = 5 \\ E_3: & 5x + y + 4z = 3 \end{cases}$$

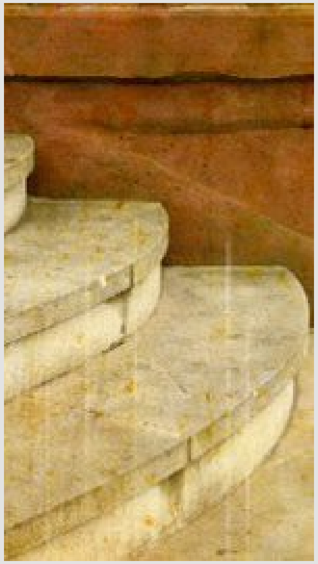
We call the first equation elimination row. It stays unchanged in the subsequent transformations. Below it is multiplied by a factor.

Step 1: elimination of x

$$\begin{cases} E_1 + E_2 = \tilde{E}_1: & -2y - z = 5 \\ 5E_1 + E_3 = \tilde{E}_2: & 6y + 9z = 3 \end{cases} \Leftrightarrow \begin{cases} -2y - z = 5 \\ 2y + 3z = 1 \end{cases}$$

Step 2: elimination of y

$$\tilde{E}_1 + \frac{1}{3}\tilde{E}_2 = E^*: \quad z = 3$$



The equation without x and the equation without x and y form together with the first equation a triangular system of equations from which the three unknowns can be calculated one by one from bottom to top.

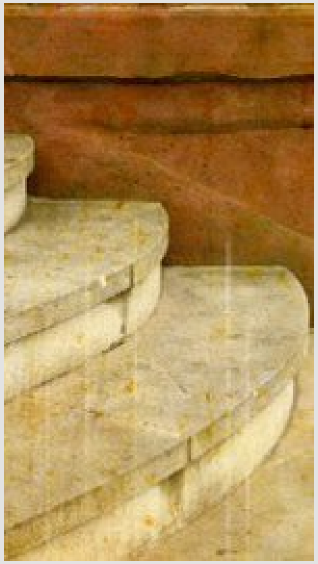
Triangular system of equations:

$$-x + y + z = 0$$
$$-2y - z = 5$$
$$z = 3$$

Unique solution: $x = -1, \quad y = -4, \quad z = 3$

or as triple of numbers: $(-1, -4, 3)$

or as column vector: $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ 3 \end{pmatrix}$



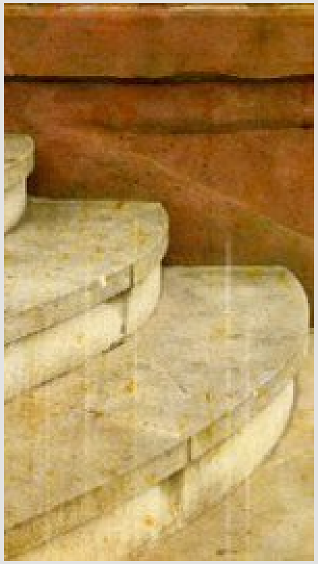
Solve the systems of equations given below:

Exercise 2:
$$\begin{cases} 2x - y + z = 8 \\ x + 2y + 2z = 6 \\ 4x - 2y - 3z = 1 \end{cases}$$

Exercise 3:
$$\begin{cases} x - z = 2 \\ 2x - y - 3z = -9 \\ -3x + y + 5z = 4 \end{cases}$$

Exercise 4:
$$\begin{cases} 2x - y - z = 4 \\ 3x + 4y - 2z = 11 \\ 3x - 2y + 4z = 11 \end{cases}$$

Exercise 5:
$$\begin{cases} x + y + 2z = -1 \\ 2x - y + 2z = -4 \\ 4x + y + 4z = -2 \end{cases}$$



Solve the systems of equations given below:

Exercise 6:

$$\begin{cases} x_1 + x_2 + 2x_3 + 3x_4 = 1 \\ 3x_1 - x_2 - x_3 - 2x_4 = -4 \\ 2x_1 + 3x_2 - x_3 - x_4 = -6 \\ x_1 + 2x_2 + 3x_3 - x_4 = -4 \end{cases}$$

Exercise 7:

$$\begin{cases} x_1 + 2x_2 + 3x_3 - 2x_4 = 6 \\ 2x_1 - x_2 - 2x_3 - 3x_4 = 8 \\ 3x_1 + 2x_2 - x_3 + 2x_4 = 4 \\ 2x_1 - 3x_2 + 2x_3 + x_4 = -8 \end{cases}$$

Gaussian Elimination: Solutions 2-7

Solution 2: $x = 2, \quad y = -1, \quad z = 3$

Solution 3: $x = -1, \quad y = 16, \quad z = -3$

Solution 4: $x = 3, \quad y = 1, \quad z = 1$

Solution 5: $x = 1, \quad y = 2, \quad z = -2$

Solution 6: $x_1 = x_2 = -1, \quad x_3 = 0, \quad x_4 = 1$

Solution 7: $x_1 = 1, \quad x_2 = 2, \quad x_3 = -1, \quad x_4 = -2$