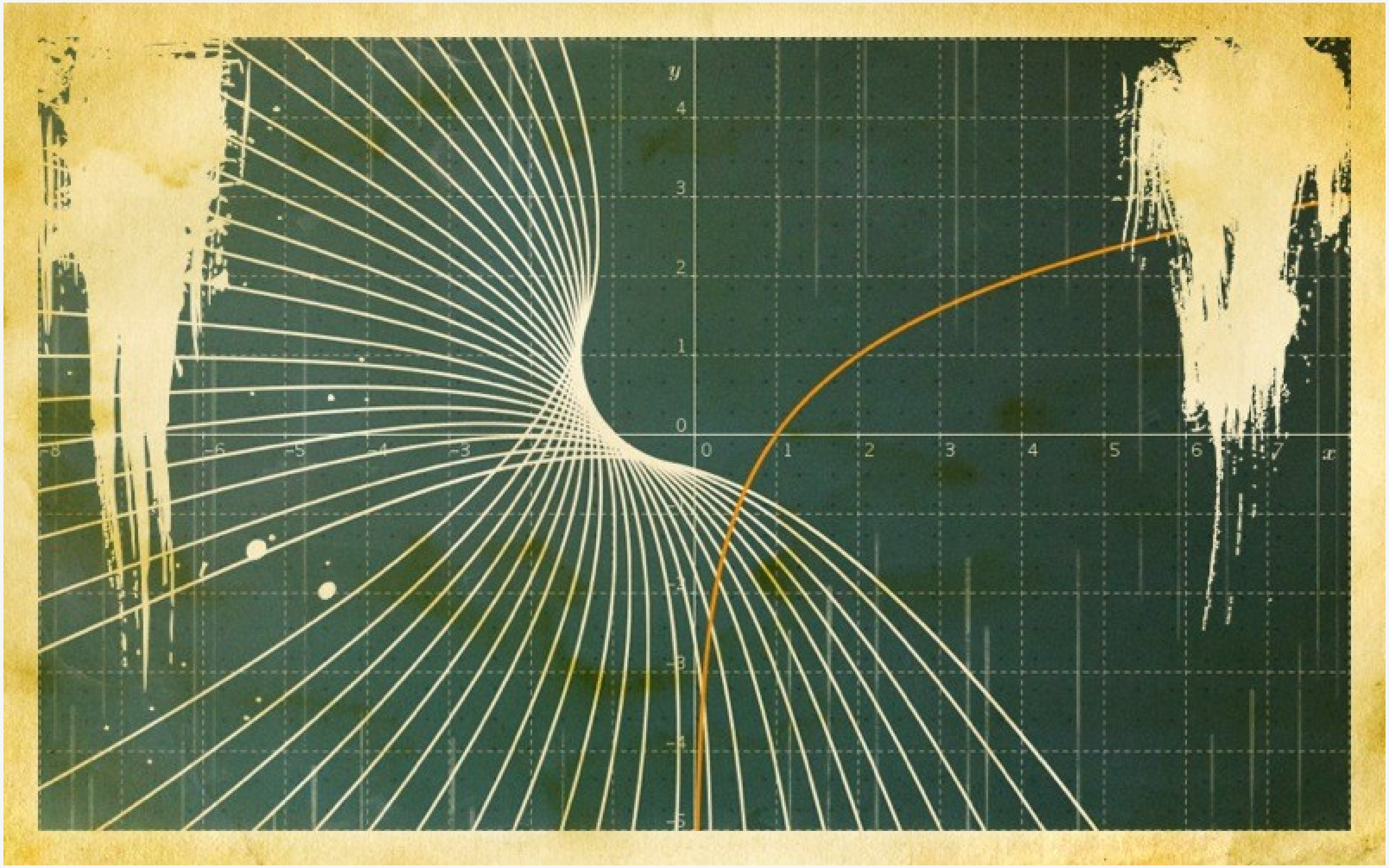
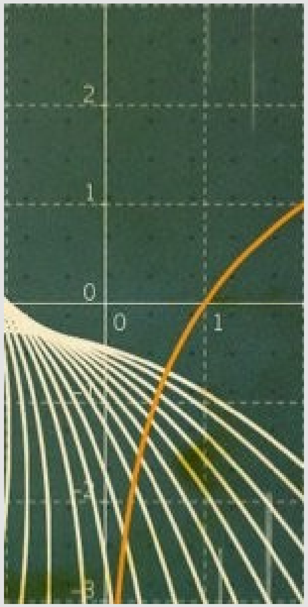


Logarithm as Inverse of Exponentiation


y = log2(x)
P1
P2
P3
P4
P5
P6
P7



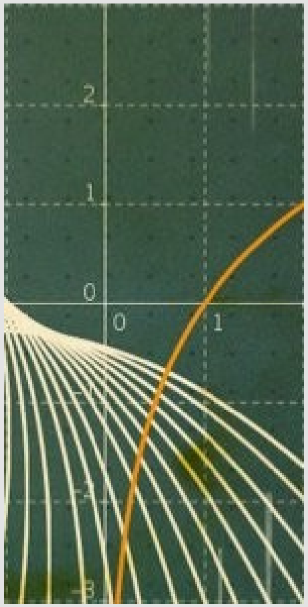
Inverse function:

- Which criteria must be satisfied by a function having an inverse function ?
- Why do we need to know an inverse function ?

Exponential function:

- Definition of an exponential function and its properties.

At the end of this section, we should know



- the logarithm as a mathematical tool,
- the logarithm as the inverse function of the exponential function,
- the basic properties of the logarithm,
- the evaluation of some simple logarithms without a calculator.

The following equations should be solved algebraically:

1) $x^2 = 4,$ 2) $\sqrt{x} = 2.$

1) To solve this quadratic equation we have to apply the root operation:

$$x^2 = 4, \qquad x = \pm\sqrt{4}, \qquad x_1 = -2, \qquad x_2 = 2.$$

2) This root equation is solved by squaring both parts of the equation:

$$\sqrt{x} = 2, \qquad \left(\sqrt{x}\right)^2 = 2^2, \qquad x = 4.$$

To solve the quadratic equation we have applied the root operation, to solve the root equation we have squared both sides of the equation. These algebraical solutions are possible, because the functions

$$y = x^2, \qquad y = \sqrt{x}$$

are invertible for $x \geq 0$.

Exponential function $y = 2^x$

$$2^x = 8$$

This equation can not be solved algebraically, as a function of type

$$y = a^x$$

is not an algebraic function. Such a function is a transcendent function.

The base of the exponential function $y = 2^x$ is the constant *2,* and the exponent is the variable here. To evaluate the function, you may simply proceed as follows: take some values of x and estimate y by plugging x into the function equation. Here some *(x, y)*-pairs:

$$(x, y) = (x, 2^x): \quad \left(-6, \frac{1}{64}\right), \left(-5, \frac{1}{32}\right), \left(-4, \frac{1}{16}\right), \left(-3, \frac{1}{8}\right),$$
$$\left(-2, \frac{1}{4}\right), \left(-1, \frac{1}{2}\right), (0, 1), (1, 2), (2, 4).$$

The graph of the function is following.

Exponential function $y = 2^x$

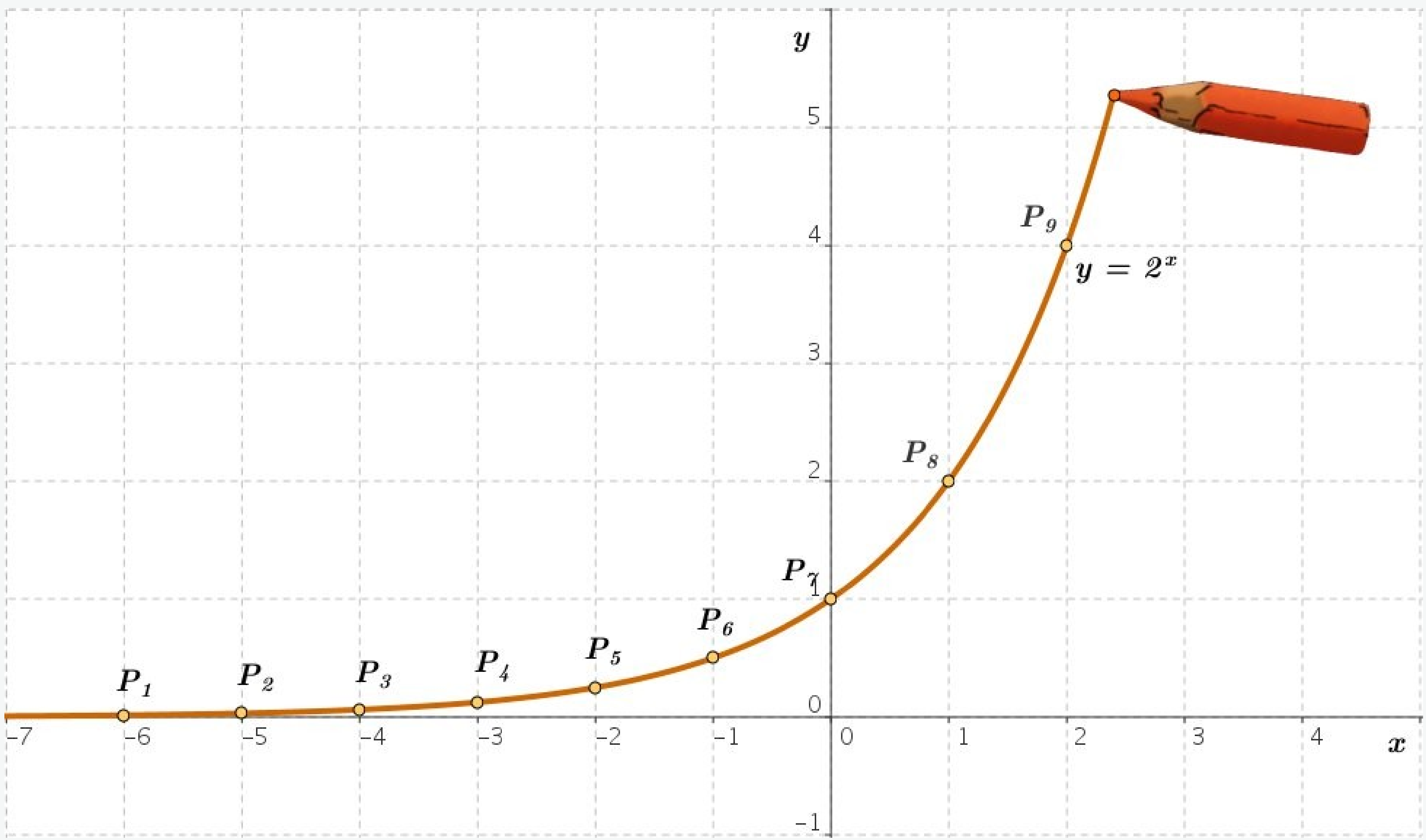


Fig. 1-1: The graph of the exponential function with base 2

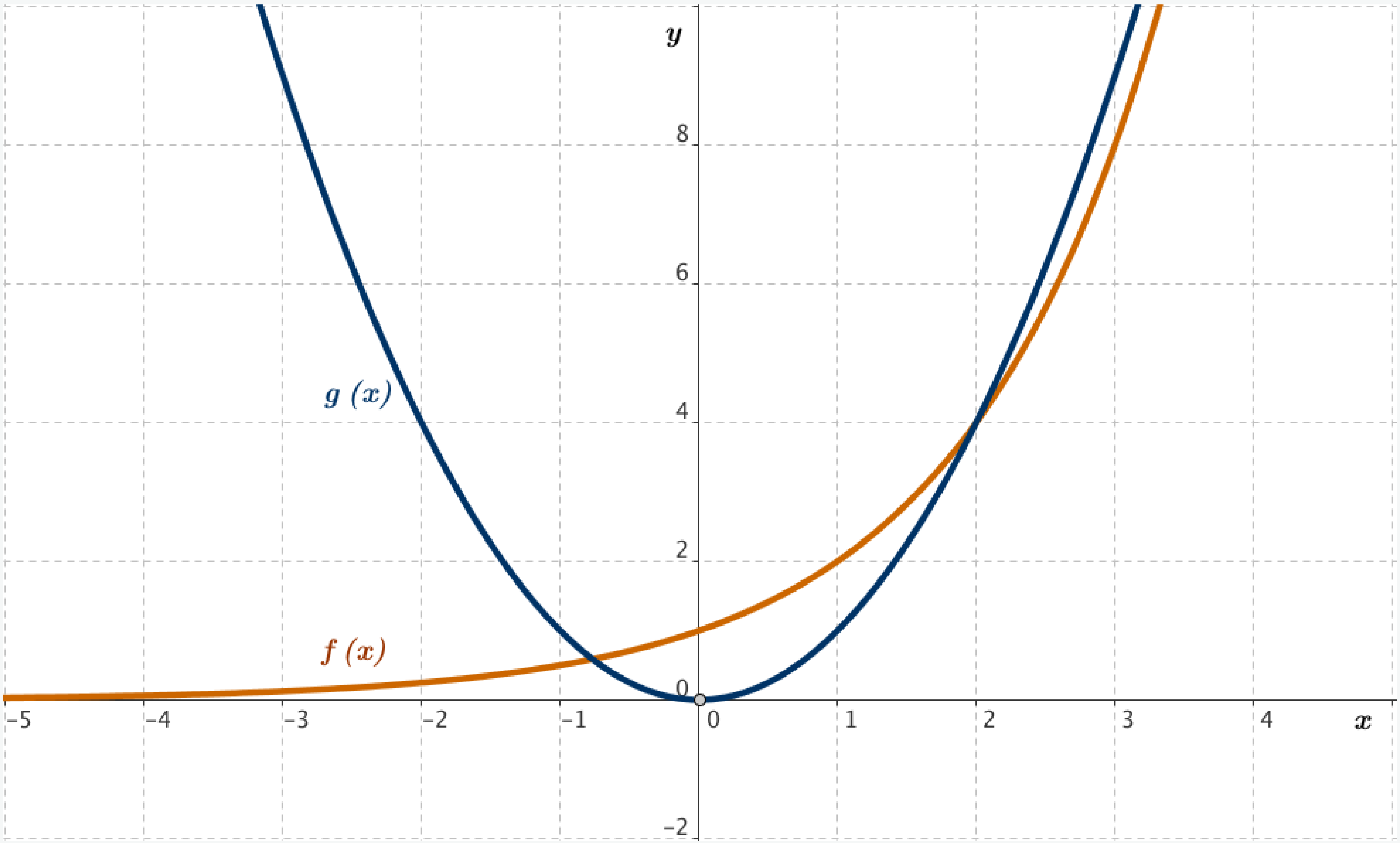


Fig. 1-2: The graphs of the exponential function with base 2 and of the quadratic function $y = x^2$ *are different.*

This exponential function is a monotonic one in the whole domain. This means, that there is an inverse function. If we know this inverse function, we can solve the exponential equation

$$y = 2^x$$

with respect to x.

Right now we don't know the analytic expression of the inverse function, but we can draw its graph by reflecting the curve of $y = 2^x$ over the line $y = x$. If we want to draw the graph of the inverse function using points, we have to interchange x and y in the pairs of $y = 2^x$. Some of them are:

$$(x,\ y):\quad \left(\frac{1}{64},\ -6\right),\ \left(\frac{1}{32},\ -5\right),\ \left(\frac{1}{16},\ -4\right),\ \left(\frac{1}{8},\ -3\right),\ \left(\frac{1}{4},\ -2\right),$$

$$\left(\frac{1}{2},\ -1\right),\ (1,\ 0),\ (2,\ 1),\ (4,\ 2).$$

Inverse of the exponential function $y = 2^x$

Definition:

The logarithmic function $y = \log_2 x$ is the inverse function of the exponential function $y = 2^x$

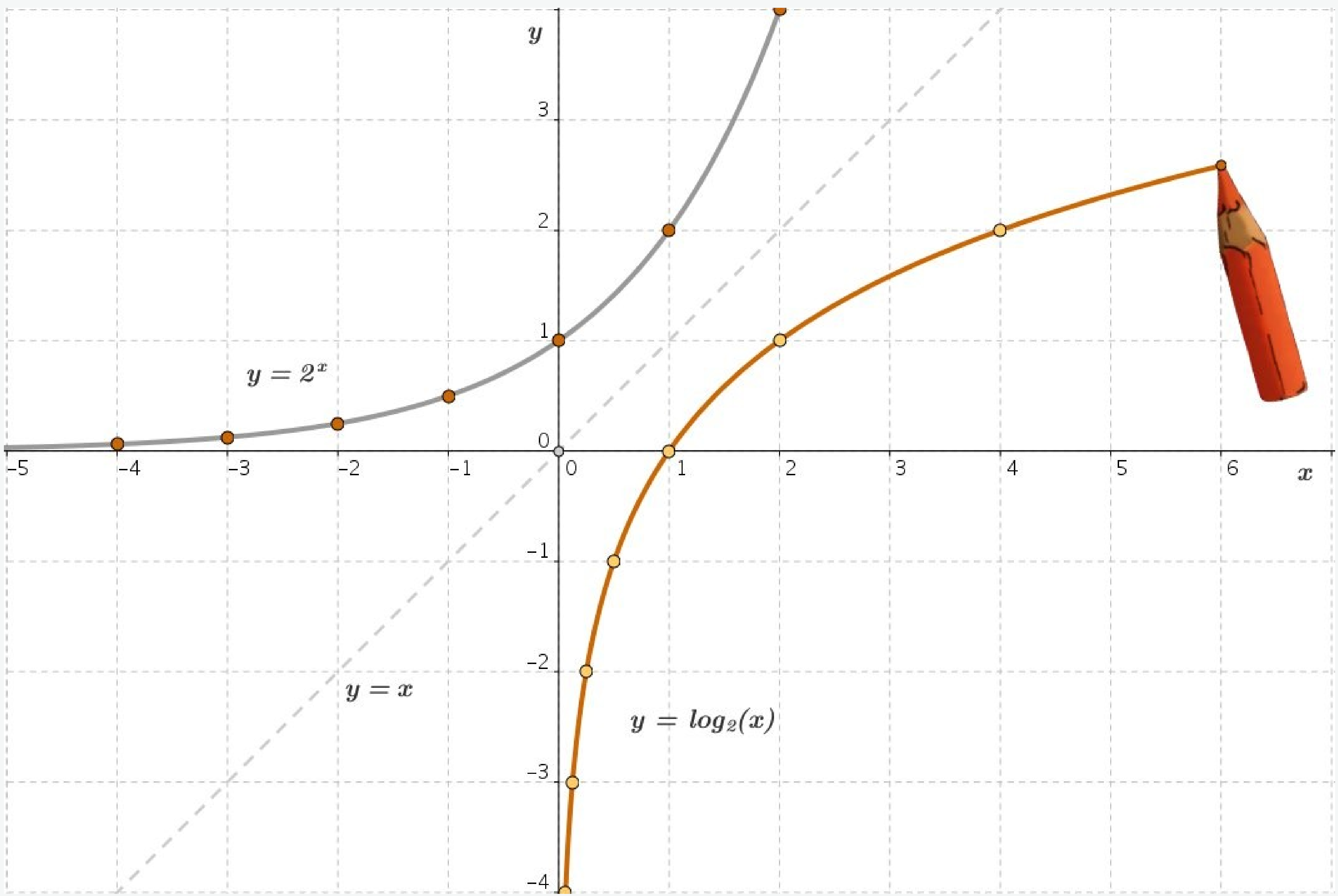


Fig. 2-1: The graph of the exponential function with base 2 (grey) and its inverse (orange) the logarithmic function to base 2

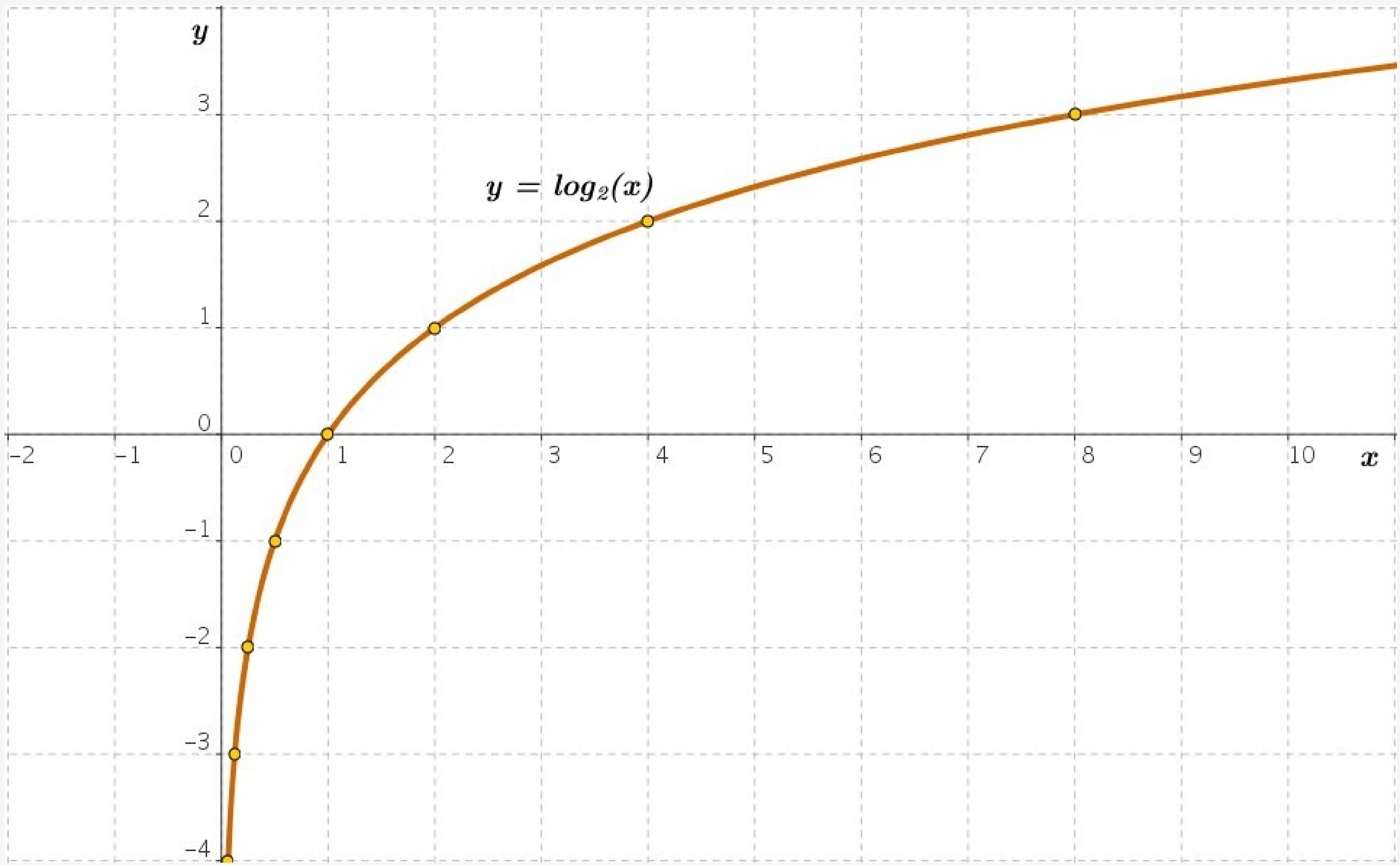
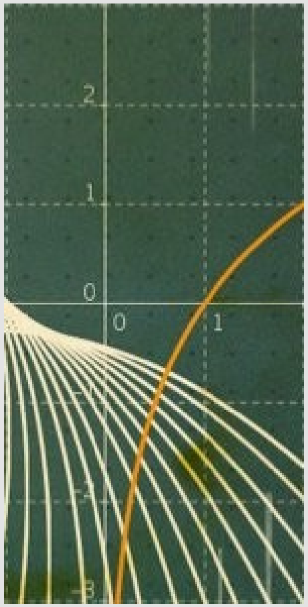


Fig. 2-2: The logarithmic function to base 2



Definition:

For each positive number y, the logarithm to base 2 of y, of y, denoted as

$$\log_2 y\,,$$

is defined to be the number x such that

$$y = 2^x$$

Examples:

$\log_2 8 = 3$ because $2^3 = 8$

$\log_2 4 = 2$ because $2^2 = 4$

$\log_2 2 = 1$ because $2^1 = 2$

$\log_2 \frac{1}{2} = -1$ because $2^{-1} = \frac{1}{2}$

Logarithm: The Definition

$$10^3 = 1\,000 \quad \Leftrightarrow \quad \log_{10} 1\,000 = 3$$

$$10^{-2} = 0.01 \quad \Leftrightarrow \quad \log_{10} 0.01 = -2$$

$$10^6 = 1\,000\,000 \quad \Leftrightarrow \quad \log_{10} 1\,000\,000 = 6$$

$$10^0 = 1 \quad \Leftrightarrow \quad \log_{10} 1 = 0$$

$$2^5 = 32 \quad \Leftrightarrow \quad \log_2 32 = 5$$

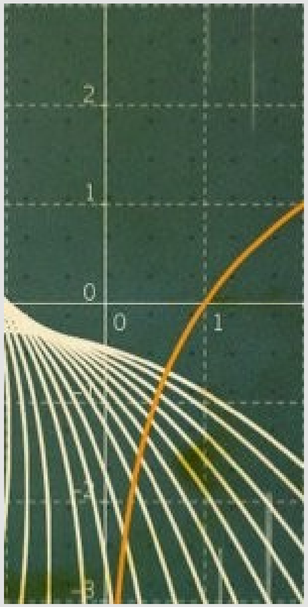
$$2^{-5} = \frac{1}{32} \quad \Leftrightarrow \quad \log_2 \frac{1}{32} = -5$$

$$32^{\frac{1}{5}} = \sqrt[5]{32} = 2 \quad \Leftrightarrow \quad \log_{32} 2 = \frac{1}{5}$$

$$5^3 = 125 \quad \Leftrightarrow \quad \log_5 125 = 3$$

$$4^{-3} = \frac{1}{64} \quad \Leftrightarrow \quad \log_4 \frac{1}{64} = -3$$

$$256^{\frac{1}{2}} = \sqrt{256} = 16 \quad \Leftrightarrow \quad \log_{256} 16 = \frac{1}{2}$$



Definition:

For each positive number y, the logarithm to base a of y, denoted as

$$\log_a y,$$

is defined to be the number x such that

$$y = a^x$$

$x = \log_a y, \quad a^x = y$

$x, y \in \mathbb{R}, \quad a > 0, \quad y > 0$

$x = \log_a y$ pronounced as: logarithm of y to base a

a – base, x – logarithm of the number y

Logarithmus: Geschichte

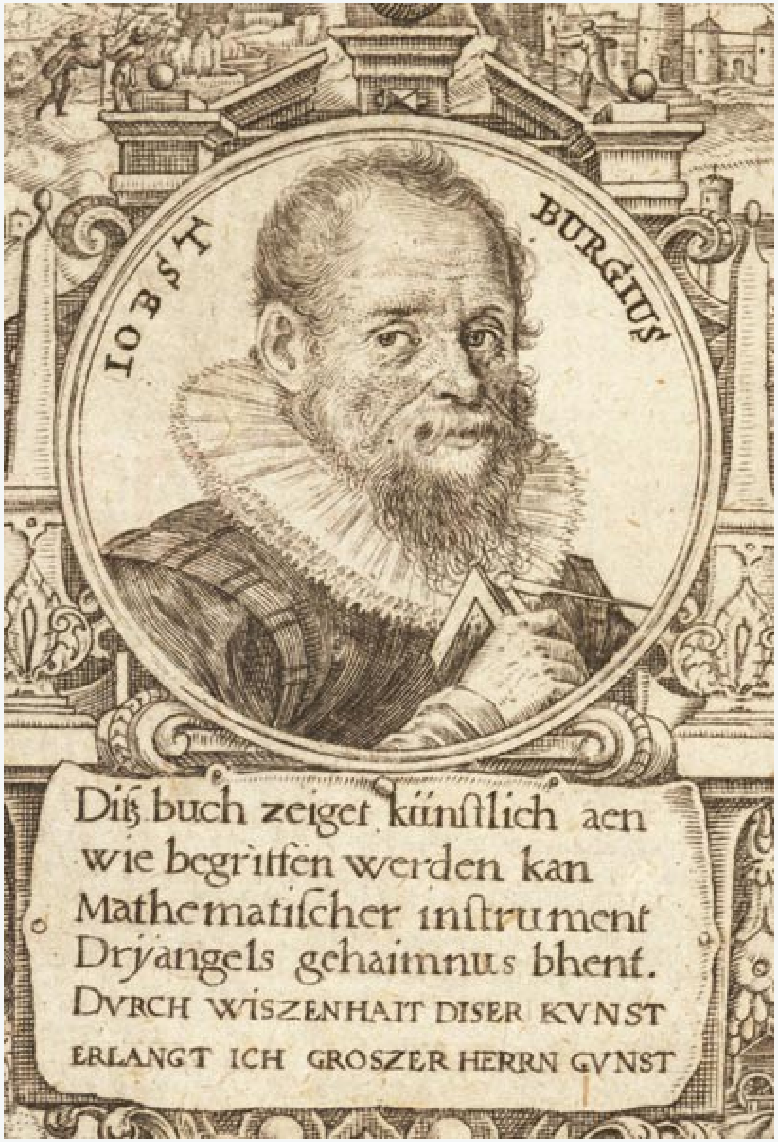
The method of logarithms was publicly propounded by John Napier in 1614 in a book entitled *Mirifici Logarithmorum Canonis Descriptio* (Description of the Wonderful Rule of Logarithms). Jost Buergli independently invented logarithms, but published six years after Napier.

Wikipedia

John Napier (1550 - 1617)

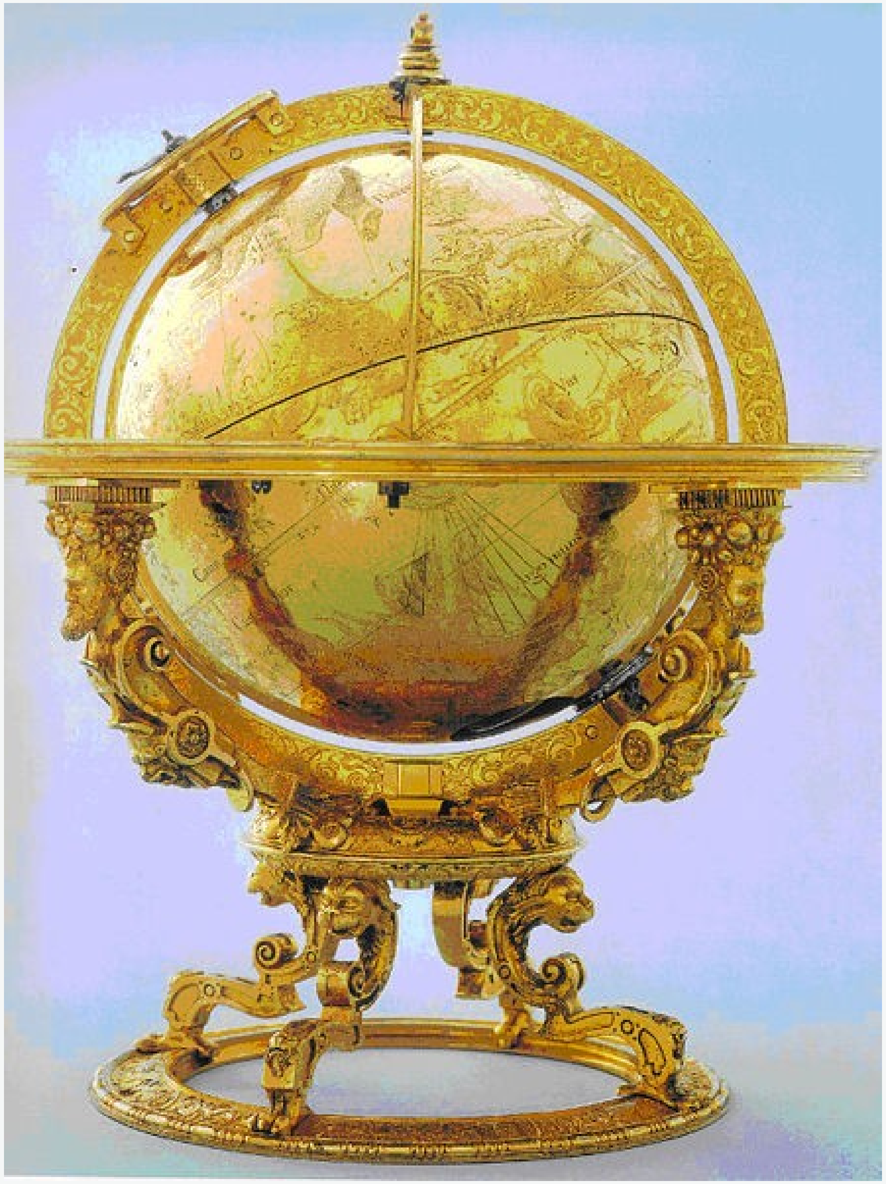
John Napier, philosopher and mathematician

Jost Bürgi (1552 - 1632)



Jost Bürgi was a Swiss watchmaker, constructor of instruments and an astronomer

Jost Bürgi (1552 - 1632)



Mechanical sky globe, constructed by Bürgi 1594 in Kassel, now at Swiss National Museum, Zurich.