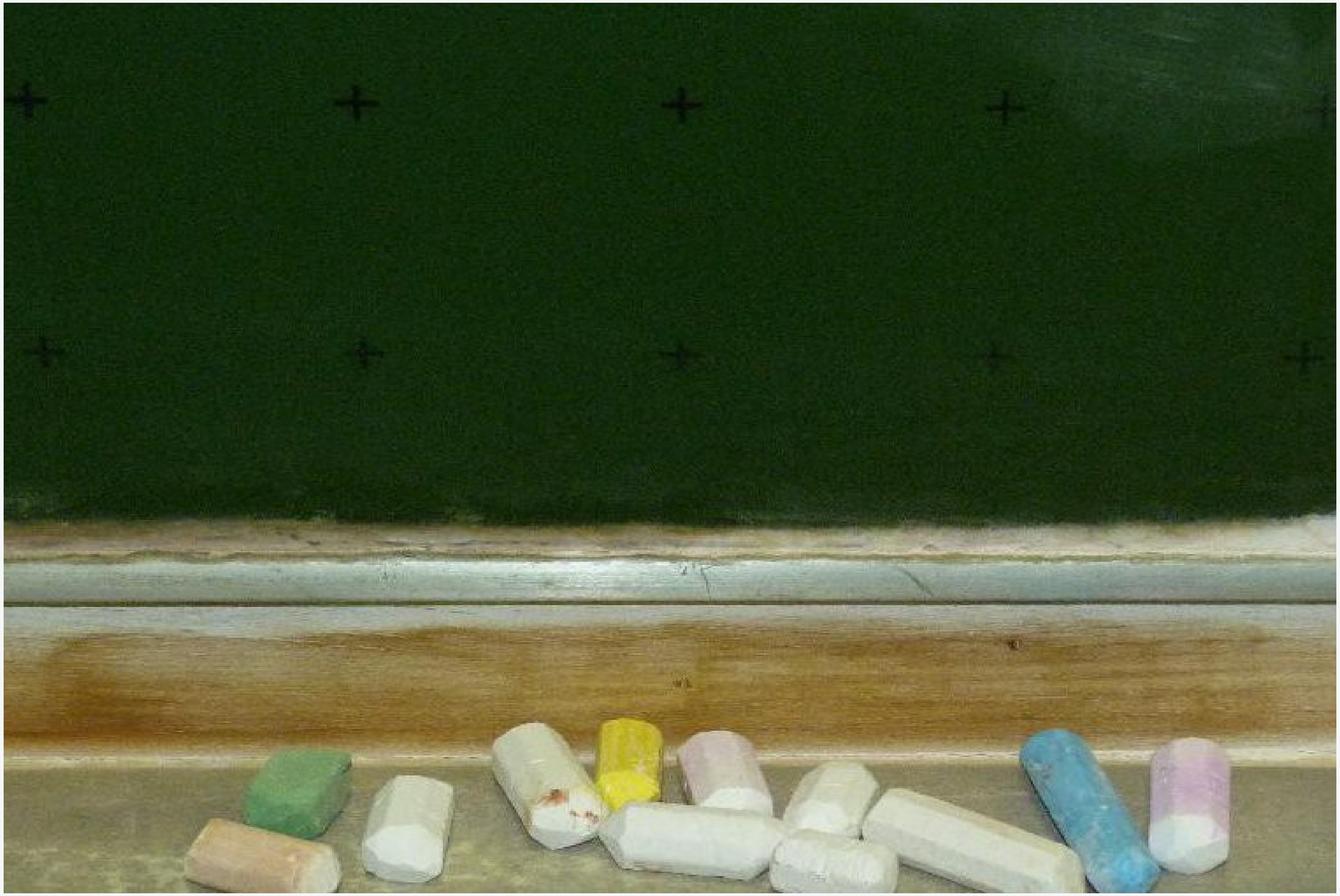


Logarithmic Calculus

First rule

$$\log_a(x \cdot y) = \log_a x + \log_a y$$

The logarithm of a <u>product</u> is the <u>sum</u> of the logarithms of the factors

$$a,\ x,\ y > 0$$

Second rule

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

The logarithm of a ratio is the difference of the logarithms of numerator and denominator

$$a\,,\ x\,,\ y > 0$$

Third rule

$$\log_a(x^n) = n \log_a x$$

The logarithm of the n-th power of a number is n times the logarithm of that number

$$a\,,\; x > 0$$

$$\log_a (x \cdot y) = \log_a x + \log_a y$$

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a (x^n) = n \log_a x$$

$$\log_a \sqrt[n]{x} = \frac{1}{n} \log_a x$$

Rules of logarithmic calculus

- The natural logarithm is very important in many applications. It is the logarithm to the base e, i.e. to Euler's number:

$$\log_e x \equiv \ln x$$

<u>*pronounced*</u>: *the natural logarithm of x*

- The logarithm to the base $a = 10$ is called the common logarithm, also known as the decadic logarithm or the Briggsian logarithm:

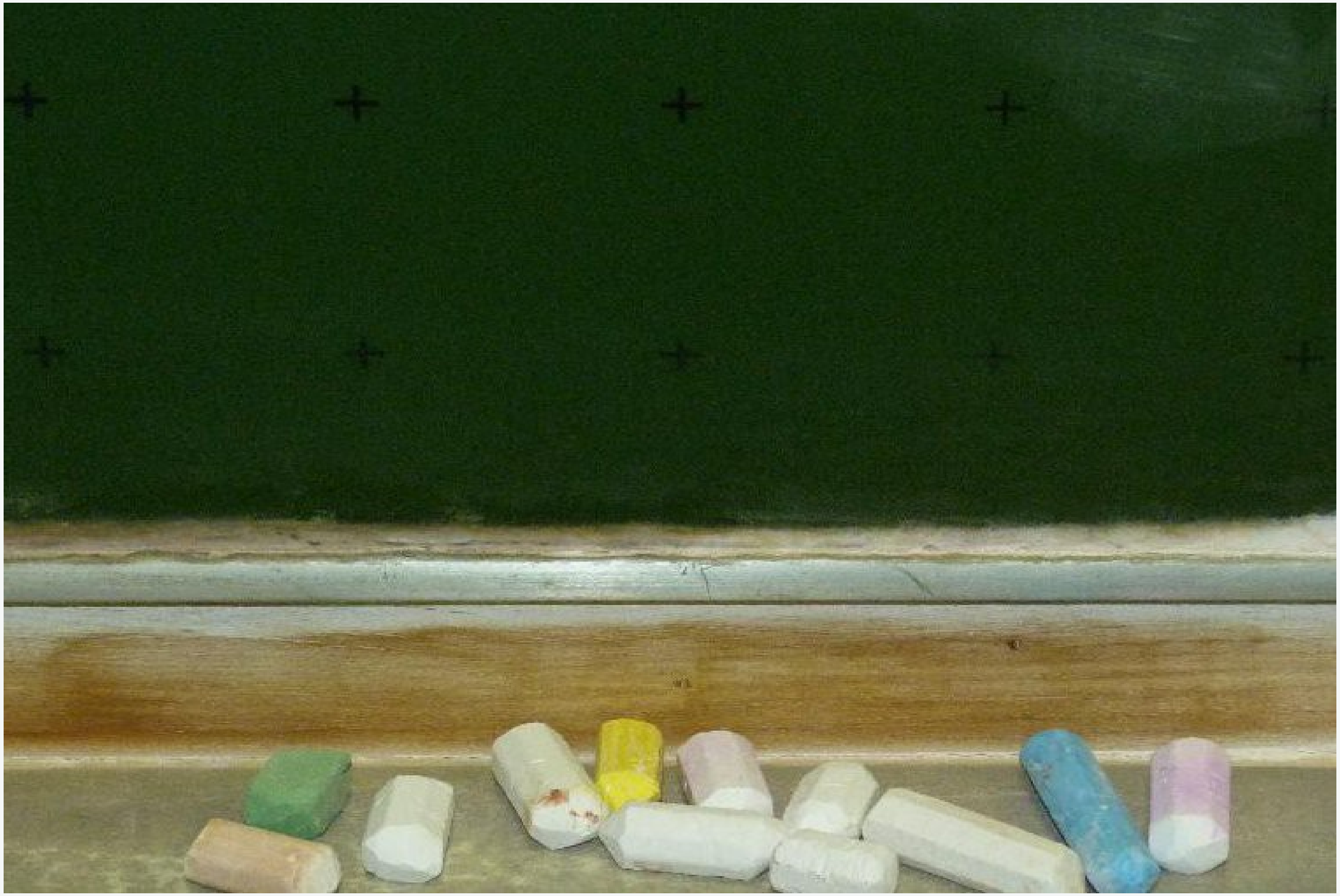
$$\log_{10} x \equiv lg\, x$$

<u>*pronounced*</u>: *the common logarithm of x*

- The logarithm to the base $a = 2$ is called the binary logarithm:

$$\log_2 x \equiv lb\, x$$

<u>*pronounced*</u>: *the binary logarithm of x*



Exercise 1:

Transform the following power equations into logarithmic equations:

a) $2^5 = 32, \quad 2^7 = 128, \quad 2^{-3} = \frac{1}{8}$

b) $3^3 = 27, \quad 3^4 = 81, \quad 3^{-2} = \frac{1}{9}$

c) $4^0 = 1, \quad 4^3 = 64, \quad 4^{-2} = \frac{1}{16}$

Exercise 2:

Transform the following logarithmic equations into power equations:

$\log_3 9 = 2, \quad \log_7 49 = 2, \quad \log_6 6 = 1, \quad \log_8 1 = 0, \quad \log_4 2 = \frac{1}{2}$

Logarithms: *Solution 1*

a) $2^5 = 32$, $\log_2 32 = 5$, $2^7 = 128$, $\log_2 128 = 7$,

$2^{-3} = \frac{1}{8}$, $\log_2 \frac{1}{8} = -3$,

b) $3^3 = 27$, $\log_3 27 = 3$, $3^4 = 81$, $\log_3 81 = 4$,

$3^{-2} = \frac{1}{9}$, $\log_3 \frac{1}{9} = -2$,

c) $4^0 = 1$, $\log_4 1 = 0$, $4^3 = 64$, $\log_4 64 = 3$,

$4^{-2} = \frac{1}{16}$, $\log_4 \frac{1}{16} = -2$,

Logarithms: Solution 2

$$\log_3 9 = 2, \qquad 3^2 = 9$$

$$\log_7 49 = 2, \qquad 7^2 = 49$$

$$\log_6 6 = 1, \qquad 6^1 = 6$$

$$\log_8 1 = 0, \qquad 8^0 = 1$$

$$\log_4 2 = \frac{1}{2}, \qquad 4^{\frac{1}{2}} = \sqrt{4} = 2$$

Exercise 3:

Evaluate the indicated expressions without using a calculator:

$a)\ \log_2 32, \quad \log_2 64, \quad \log_2 \frac{1}{16}, \quad \log_2 \frac{1}{128}, \quad \log_2 2^{-4}, \quad \log_2 1$

$b)\ \log_4 4, \quad \log_4 16, \quad \log_4 \frac{1}{64}, \quad \log_8 64, \quad \log_8 \frac{1}{8}, \quad \log_8 8^{-3}$

$c)\ \log_6 36, \quad \log_5 125, \quad \log_{16} \frac{1}{16}, \quad \log_7 1, \quad \log_7 \left(\frac{1}{7}\right)^{3}, \quad \log_7 \left(\frac{1}{49}\right)^{2}$

$d)\ lg\, 100, \quad lg\, 100000, \quad lg\, \frac{1}{10}, \quad lg\, \frac{1}{1000}, \quad lg\, 0.01, \quad lg\, 0.0001$

Exercise 4:

Find such number x that:

$a)\ \log_x 25 = 2, \quad \log_x 27 = 3, \quad \log_x \frac{1}{9} = -2, \quad \log_x \frac{1}{9} = -1,$

$b)\ \log_x 0.25 = -2, \quad \log_x 0.64 = -2, \quad \log_x \sqrt{2} = -4, \quad \log_x \sqrt{2} = \frac{1}{4}.$

$$a)\ \log_2 32 = 5, \qquad \log_2 64 = 6, \qquad \log_2 \frac{1}{16} = -4,$$
$$\log_2 \frac{1}{128} = -7, \qquad \log_2 2^{-4} = -4, \qquad \log_2 1 = 0,$$

$$b)\ \log_4 4 = 1, \qquad \log_4 16 = 2, \qquad \log_4 \frac{1}{64} = -3,$$
$$\log_8 64 = 2, \qquad \log_8 \frac{1}{8} = -1, \qquad \log_8 8^{-3} = -3,$$

$$c)\ \log_6 36 = 2, \qquad \log_5 125 = 3, \qquad \log_{16} \frac{1}{16} = -1,$$
$$\log_7 1 = 0, \qquad \log_7 \left(\frac{1}{7}\right)^3 = -3, \qquad \log_7 \left(\frac{1}{49}\right)^2 = -4,$$

$$d)\ lg\, 100 = 2, \qquad lg\, 100000 = 5, \qquad lg\, \frac{1}{10} = -1,$$
$$lg\, \frac{1}{1000} = -3, \qquad lg\, 0.01 = -2, \qquad lg\, 0.0001 = -4$$

Logarithms: Solution 4

$a)\ \log_x 25 = 2, \qquad x^2 = 25, \qquad x = 5,$

$\log_x 27 = 7, \qquad x^3 = 27, \qquad x = 3,$

$\log_x \frac{1}{9} = -2, \qquad x^{-2} = \frac{1}{9} = \frac{1}{3^2} = 3^{-2}, \qquad x = 3$

$\log_x \frac{1}{9} = -1, \qquad x^{-1} = \frac{1}{9} = 9^{-1}, \qquad x = 9$

$b)\ \log_x 0.25 = -2, \qquad x^{-2} = 0.25 = (0.5)^2 = \left(\frac{1}{2}\right)^2 = 2^{-2}, \qquad x = 2$

$\log_x 0.64 = -2, \qquad x^{-2} = 0.64 = 0.8^2 = \left(\frac{4}{5}\right)^2 = \left(\frac{5}{4}\right)^{-2}, \qquad x = \frac{5}{4}$

$\log_x \sqrt{2} = -4, \qquad x^{-4} = \sqrt{2} = 2^{1/2}, \qquad x = 2^{-1/8}$

$\log_x \sqrt{2} = \frac{1}{4}, \qquad x^{1/4} = \sqrt{2} = 2^{1/2}, \qquad x = (2^{1/2})^4 = 2^2 = 4$

Exercise 5:

Evaluate the expressions without using a calculator:

$a)\ \log_2 32 + \log_2 \frac{1}{8},\qquad b)\ \log_4 64 + \log_4 \frac{1}{16},\qquad c)\ lg\,100000 - lg\,\frac{1}{100} + lg\,\frac{1}{10}$

$d)\ \log_2 16 + lg\,\frac{1}{1000},\qquad e)\ \log_5 25 + \log_3 \frac{1}{9},\qquad f)\ \log_6 1 - \log_7 1 - \log_8 1$

$g)\ \log_4 \frac{1}{4} - \log_5 \frac{1}{5},\qquad h)\ \left(\log_4 \frac{1}{4}\right)^2 + \left(\log_9 \frac{1}{9}\right)^2,\qquad i)\ \log_7 49 + 2\log_6 \frac{1}{6}$

$$a)\ \log_2 32 + \log_2 \frac{1}{8} = 5 - 3 = 2$$

$$b)\ \log_4 64 + \log_4 \frac{1}{16} = 3 - 2 = 1$$

$$c)\ lg\,100000 - lg\,\frac{1}{100} + lg\,\frac{1}{10} = 5 - (-2) + (-1) = 6$$

$$d)\ \log_2 16 + lg\,\frac{1}{1000} = 4 + (-3) = 1$$

$$e)\ \log_5 25 + \log_3 \frac{1}{9} = 2 + (-2) = 0$$

$$f)\ \log_6 1 - \log_7 1 - \log_8 1 = 0 - 0 - 0 = 0$$

$$g)\ \log_4 \frac{1}{4} - \log_5 \frac{1}{5} = -1 - (-1) = 0$$

$$h)\ \left(\log_4 \frac{1}{4}\right)^2 + \left(\log_9 \frac{1}{9}\right)^2 = (-1)^2 + (-1)^2 = 2$$

$$i)\ \log_7 49 + 2\log_6 \frac{1}{6} = 2 + 2(-1) = 0$$

Decompose the given expressions as much as possible using the rules of logarithmic calculus:

a) $\log_3(5\,x)$, $\log_3(3\,x)$, $\log_2(4\,x)$

b) $\log_3\frac{1}{3}$, $\log_3\frac{1}{9}$, $2\,\log_3\frac{1}{27}$

c) $\log(a\,b)$, $\log(a\,b\,c)$, $\log(a\,b\,c\,d)$

d) $\log\frac{a}{b}$, $\log\frac{a\,c}{b}$, $\log\frac{a\,b}{c\,d}$

e) $\log a^2\,b$, $\log a^3\,b^2$, $\log a^5\,b^3\,c$

f) $\log\frac{a^2\,c}{b}$, $\log\frac{a\,c}{b\,d^3}$, $\log\frac{\sqrt{a}\,b}{c^4}$

a) $\log_3(5\,x) = \log_3 5 + \log_3 x\,, \qquad \log_3(3\,x) = \log_3 3 + \log_3 x = 1 + \log_3 x$

$\log_2(4\,x) = \log_2 4 + \log_2 x = \log_2 2^2 + \log_2 x = 2 + \log_2 x$

b) $\log_3 \dfrac{1}{3} = \log_3 1 - \log_3 3 = -1, \qquad \log_3 \dfrac{1}{3} = \log_3 3^{-1} = -\log_3 3 = -1,$

$\log_3 \dfrac{1}{9} = \log_3 1 - \log_3 9 = 0 - \log_3 3^2 = -2\log_3 3 = -2$

$2\,\log_3 \dfrac{1}{27} = 2(\log_3 1 - \log_3 27) = 2(\log_3 1 - \log_3 3^3) = -2\cdot 3\,\log_3 3 = -6$

c) $\log(a\,b) = \log a + \log b\,, \qquad \log(a\,b\,c) = \log a + \log b + \log c$

$\log(a\,b\,c\,d) = \log a + \log b + \log c + \log d$

d) $\log \dfrac{a}{b} = \log a - \log b\,, \qquad \log \dfrac{a\,c}{b} = \log a + \log c - \log b$

$\log \dfrac{a\,b}{c\,d} = \log a + \log b - \log c - \log d$

e) $\log a^2 b = \log a^2 + \log b = 2\log a + \log b$

$\log a^3 b^2 = \log a^3 + \log b^2 = 3\log a + 2\log b$

$\log a^5 b^3 c = \log a^5 + \log b^3 + \log c = 5\log a + 3\log b + \log c$

f) $\log \dfrac{a^2 c}{b} = \log a^2 + \log c - \log b = 2\log a + \log c - \log b$

$\log \dfrac{a\,c}{b\,d^3} = \log(a\,c) - \log(b\,d^3) = \log a + \log c - \log b - \log d^3 =$

$= \log a + \log c - \log b - 3\log d$

$\log \dfrac{\sqrt{a}\,b}{c^4} = \log(\sqrt{a}\,b) - \log(c^4) = \log a^{1/2} + \log b - 4\log c =$

$= \dfrac{1}{2}\log a + \log b - 4\log c$