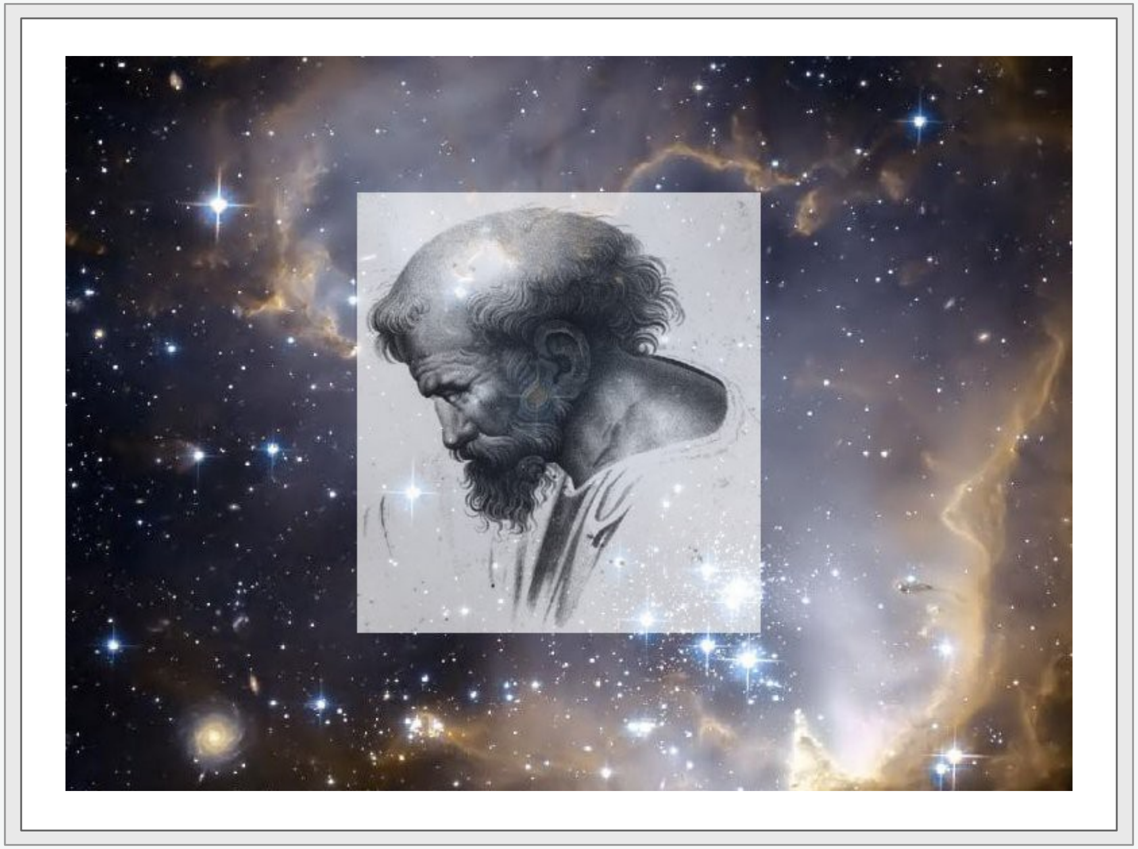


http://free-background-wallpaper.com/images/Wallpapers1280/star-backgrounds/

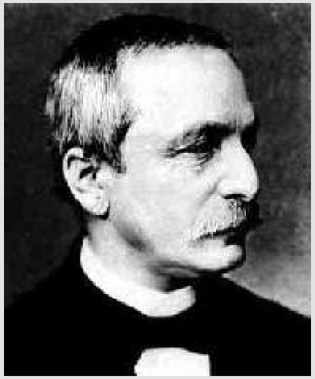
Numbers

The world is harmony and numbers.
Pythagoras

Richard Dedekind
(1831-1916)

Richard Dedekind a German mathematician.

"Numbers are created by human mind, they help to perceive more clearly the diversity of things ...."
R. Dedekind, 1887



Leopold Kronecker
(1823-1891)

"Die natürlichen Zahlen hat Gott gemacht, alles übrige ist Menschenwerk."

Leopold Kronecker

The history of the evolution of the number concept is not finished yet. Back in history and still today, numbers raise questions on their nature which are not yet finally answered.

http://www.bluevisions.ch/zb-bild064-080423triptychon3.jpg

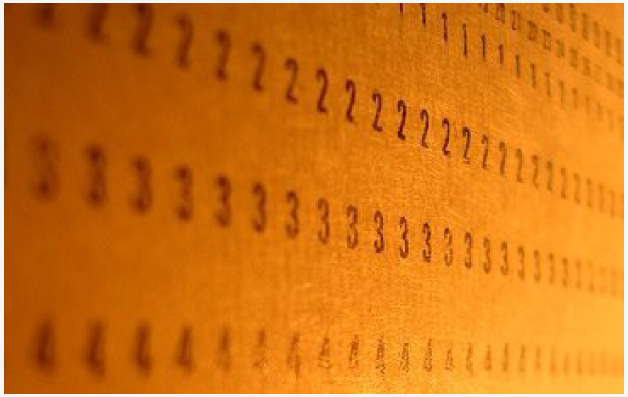
http://www.flickr.com/photos/artnoose/2263480871/

$\mathbb{N} = \{ 0, 1, 2, 3, 4, \ldots \}$ – natural numbers

$\mathbb{N}^* = \{ 1, 2, 3, 4, \ldots \}$ – positive natural numbers

2, 4, 6, 8 . . . – even numbers; *1, 3, 5, 7 . . .* – odd numbers

What kind of operations and relations exist on the set of natural numbers?



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Operations and relations on the set of natural numbers:

- addition, e.g. $27 + 3 = 30$
- multiplication, e.g. $7 \cdot 3 = 21$
- order relation, e.g. $27 < 30$ $(a < b$ oder $a = b$ oder $a > b)$

The quality $n = m$ is equivalent to $m = n$, and $n \leq m$ is equivalent to $m \geq n$. With symbols of propositional logic:

$$m = n \quad \Leftrightarrow \quad (m \leqslant n) \wedge (m \geqslant n)$$

Natural numbers are added in correspondence to counting, that is, two sets with a number of n and m objects are combined to a set which covers $m + n$ objects.

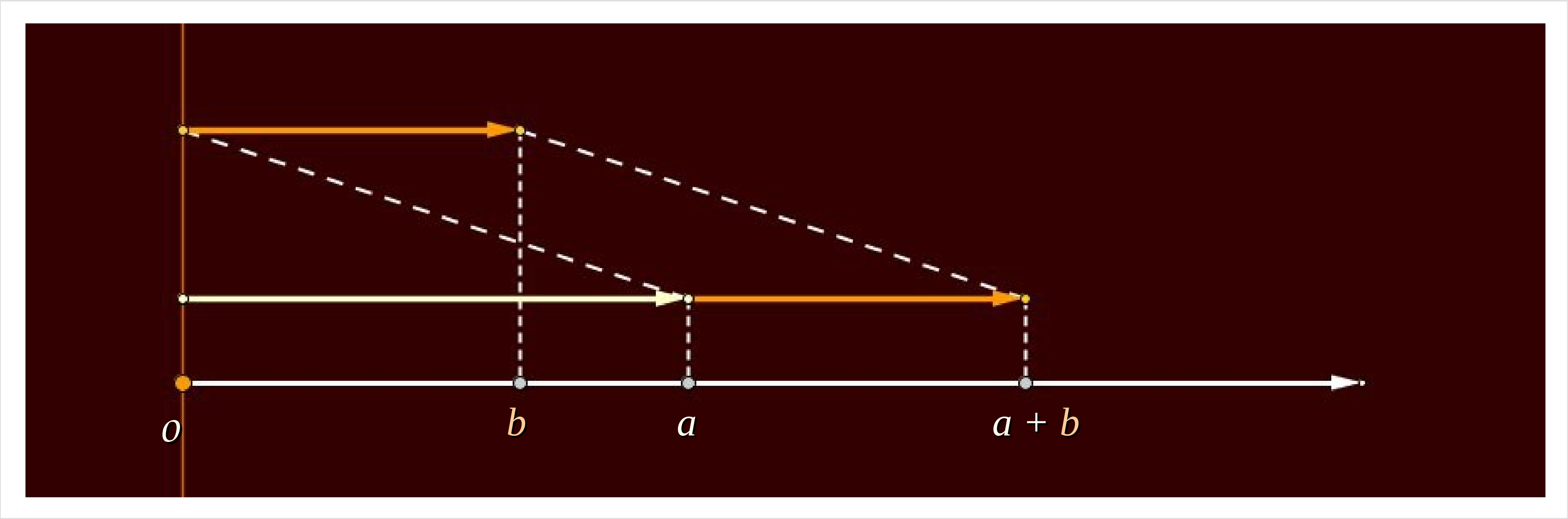


Fig 1: Representation of the addition of natural numbers

The addition of numbers a and b can be visualised by two arrows which are put together on a number line. Number a is shown as arrow, from 0 to a, and the arrow representing b is moved to the tip of the arrow of a. The combined arrow represents $a + b$.

$a + b = b + a$ – commutative property (also commutative law)

$(a + b) + c = a + (b + c)$ – associative property (also associative law)

Definition:

The multiplication of natural numbers a, b is defined as follows

$$a \cdot b = \underbrace{b + b + b + \ldots + b}_{a - times}$$

$$a \cdot b = \underbrace{a + a + a + \ldots + a}_{b - times}$$

a and b are called factors of the product $a \bullet b$.

$a \bullet b = b \bullet a$ – commutative property

$(a \bullet b) \bullet c = a \bullet (b \bullet c)$ – associative property

Properties of natural numbers

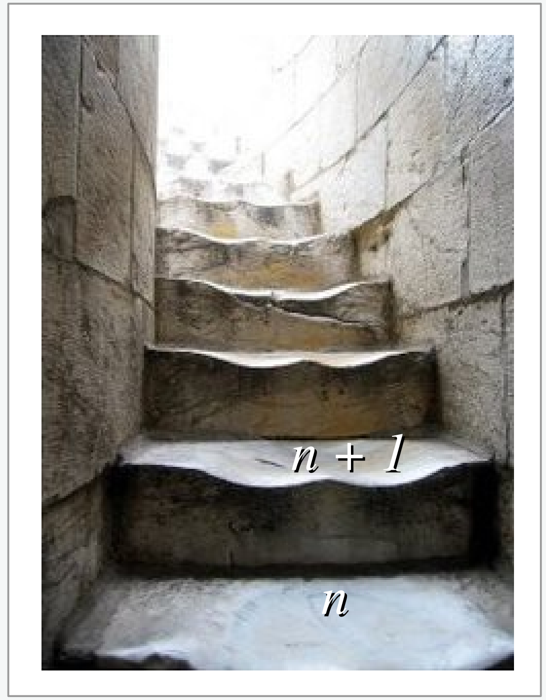
The element 0 (zero) plays some special role among the natural numbers, because it does not change the value of a number a, if added to it. However, there is no number b which, if added to a, yields zero.

Subtraction is not always possible inside the set of natural numbers $\mathbb{N}$:

$$27 - 6 = 21 \in \mathbb{N}, \qquad 6 - 27 = -21 \notin \mathbb{N}$$

To handle such situations, the number system was enlarged by negative numbers (appearing the first time in Chinese literature).

Largest natural number ?



http://thorngren.blogspot.com/2008/05/many-steps.html

The realisation, that counting can always be continued by adding 1, that is, understanding, that there is a number $n + 1$ for each natural number n, was an important step of mankind. To any natural number claimed to be the largest number, one can add 1 and claim to have found an even larger one. So obviously there is no largest natural number.

http://www.twistedtreecafe.com/ART Gallery 2007 August-Sept.htm

The notion, that there is no largest natural number, is tightly connected with the idea of infinity. The set of natural numbers is infinitely large. We can count and count and stop and add 1 again. The infinity means, that we never find an end. In other words, an infinite number of steps means, that we can always add a further step, no matter how many steps we did before.

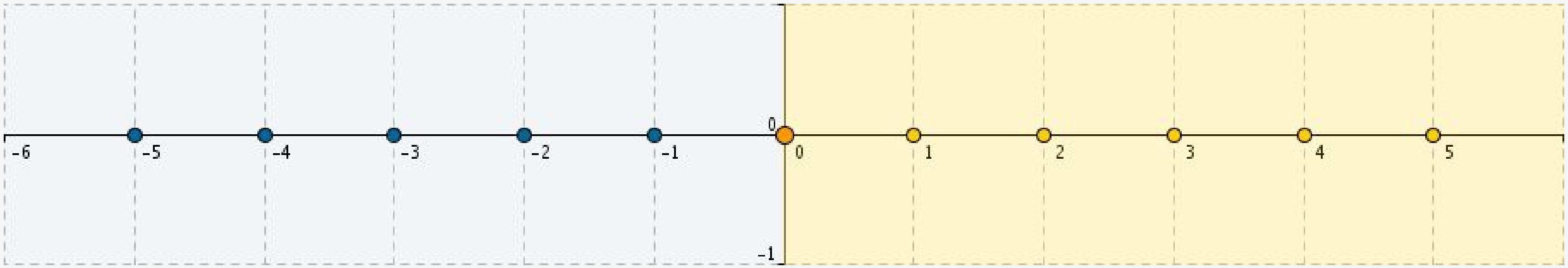


Fig 2: Representation of integer numbers on the number line

The negative numbers *(. . ., - 3, -2, -1)* form together with the natural numbers the enlarged set of integer numbers (in short: integers).

$$\mathbb{Z} = \{\, 0,\ \pm 1,\ \pm 2,\ \pm 3,\ \pm 4, \ldots \} \ - \ \text{integer numbers}$$

The integer numbers can be represented on a number line. The number line of the natural numbers is extended to the left. The point left from 0 in distance of one unit, is marked - 1.

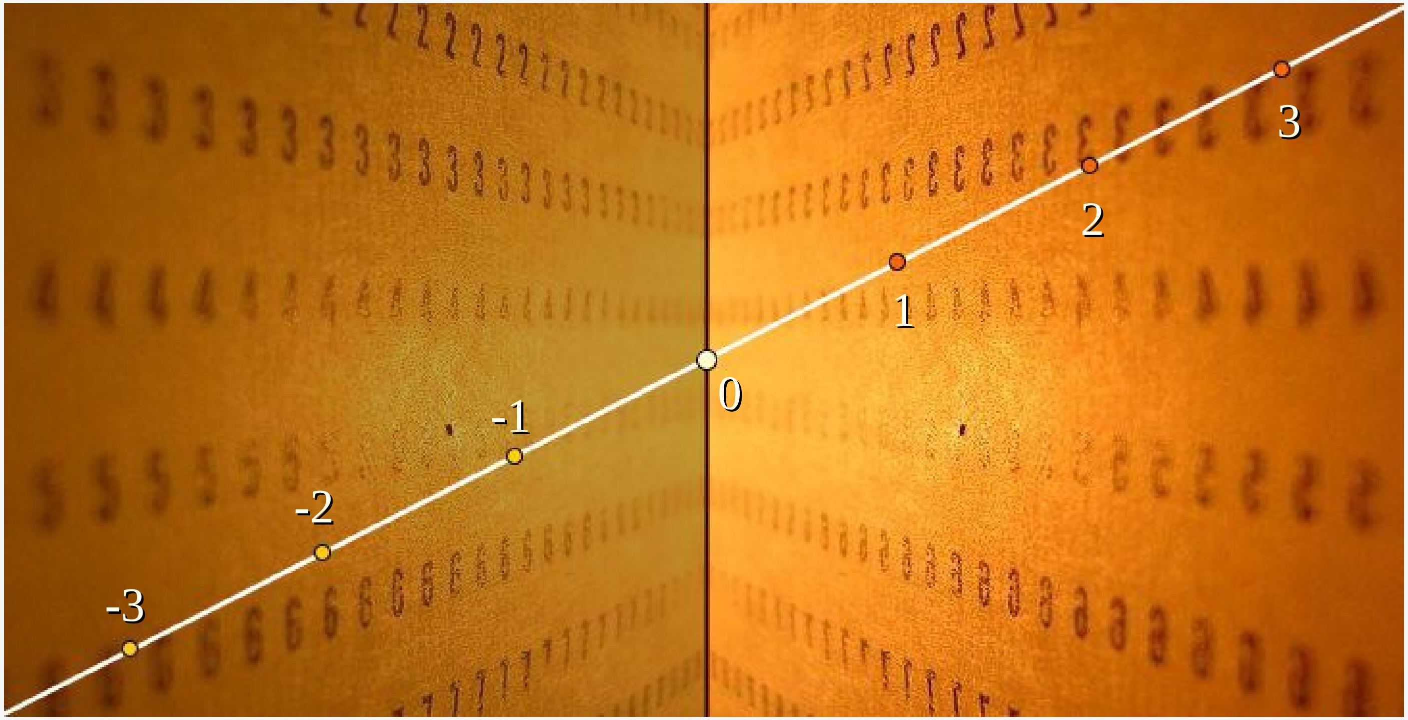


Fig 3: Representation of the integer numbers on the number line

We may also say, that we construct the negative numbers -1, -2, . . . from the natural numbers 1, 2, . . . by symmetry with respect to zero, such that to each natural number *n* there is a symmetric number *-n.* So we may also say, that we extend the number line of natural numbers to the number line of integers by point reflection with respect to zero.

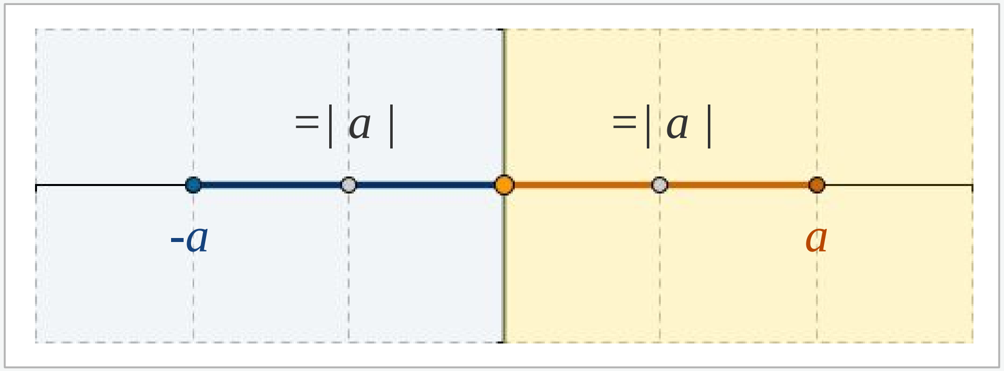


Fig 4: Representation of the absolute value of a number a

The absolute value of a number a is defined as the distance from zero on the number line (Fig. 4).

Definition:

The absolute value $|a|$ of a number a is defined as

$$|a| = \begin{cases} a\,, & a \geqslant 0 \\ -a\,, & a < 0 \end{cases}$$

Properties, structures, operations on the set of integer numbers:

- addition
- multiplication
- subtraction (existence of additive inverse)

Difference to the set of natural numbers:

To each integer number a, there is a uniquely determined integer number b, such that $a + b = 0$, $b = -a$.
b is called additive inverse of a.

- order relation

Division is not always possible on the set of integer numbers:

$$6 : 3 = 2 \in \mathbb{Z}, \qquad 3 : 6 = \frac{1}{2} \notin \mathbb{Z}$$