Rational numbers were introduced, because they allow to solve equations of the type

$$q\ x = p\,, \qquad x = \frac{p}{q}\,, \qquad q \neq 0\,, \qquad p\,,\, q \in \mathbb{Z}$$

We can imagine x to be an ordered pair $x = (p, q)$. Such numbers are also called fractions or quotients.

The representation of a rational number is not unique!

$$\frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \frac{1}{2}$$

Fractions can be reduced, or the numerator and the denominator can be multiplied by the same factor, without changing the value of the fraction. The representation is unique only, if numerator and denominator are relatively prime.

$\frac{p}{1} = p$ – therefore the rational numbers include the integers.

Ending decimal fractions : $\frac{3}{5} = 0.6\,,\quad \frac{7}{4} = 1.75$

Repeating decimal fractions :

$$\frac{1}{3} = 0.333\ldots = 0.\overline{3}\,,\quad \frac{19}{9} = 2.111\ldots = 2.\overline{1}$$

The period is marked by placing a bar over the repeating decimal digits (or possibly underlining them) :

$$3.\overline{7} = 3 + \frac{7}{10} + \frac{7}{10^2} + \frac{7}{10^3} + \frac{7}{10^4} + \ldots$$

$$0.05\overline{31} = \frac{5}{10^2} + \frac{31}{10^4} + \frac{31}{10^6} + \frac{31}{10^8} + \ldots$$

Statement:

Each rational number can be represented by an ending or repeating decimal fraction.

Conversely, each ending or repeating decimal fraction can be represented as fraction p/q.

Operations and relations on the set $\mathbb{Q}$ of rational numbers :

- adddition
- multiplication
- subtraction (existence of additive inverse)
- division (existence of multiplicative inverse)

Difference to the set of integer numbers:

For every rational number *a,* there is a uniquely defined rational number b, such that $a \cdot b = 1$. The number b is then the multiplicative inverse of a

$$a \cdot b = 1 , \qquad b = a^{-1}$$

- order relation

Any result we get by addition, substraction, multiplication and division of rational numbers, is again a rational number. In mathematical terms: the set $\mathbb{Q}$ of rational numbers is "closed" under these arithmetic operations, which means that these operations do not lead to results outside $\mathbb{Q}$

$$x^2 = 2\,, \qquad x = \pm\sqrt{2} \qquad \text{What is } \sqrt{2}\,?$$

$$\sqrt{2} \simeq 1.41, \qquad 1.41^2 = 1.9881\,; \qquad \sqrt{2} \simeq 1.414\,, \qquad 1.414^2 = 1.999386$$

We may use MAPLE, to calculate the decimal expansion of $\sqrt{2}$ to 415 digits:

$$x = 1{,}41421356237309504880168872420969807856967187 53\ldots 699$$

$$x^2 = 1{,}99999999999999999999999999999999999999999999999\ldots 010$$

This number satisfies the equation $x^2 = 2$ to high precision, but not exactly. By finite decimal expansion, we will never get a number, the square of which is exactly *2*.

The length of the diagonal of a square with sides of length 1 is $\sqrt{2}$, as indicated in the figure. It can not be written as fraction of two integers. That is, $\sqrt{2}$ is not a rational number (geometrically shown already around 500 BC, with numbers about 200 years later by Euclid).

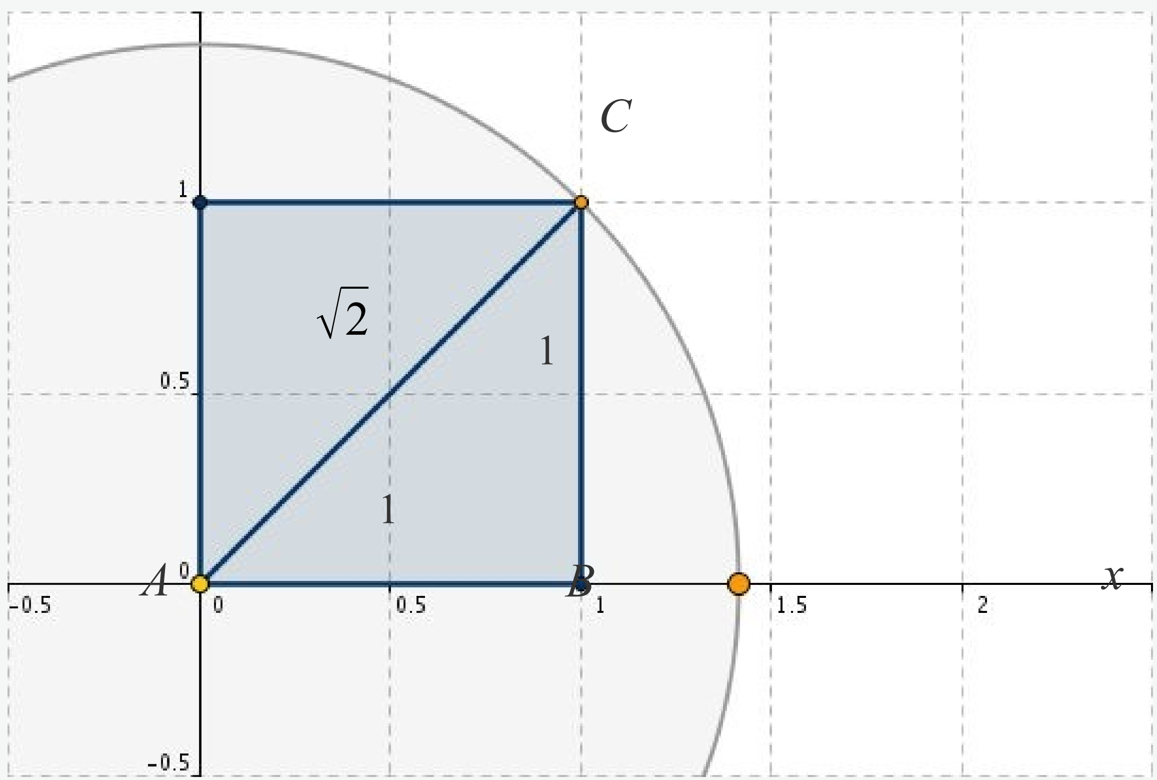


Fig 5: The number $\sqrt{2}$ as diagonal and on the number line

$$|AC|^2 = |AB|^2 + |BC|^2 = 1^2 + 1^2 = 2 \quad \Rightarrow \quad |AC| = \sqrt{2}$$

Square root of two

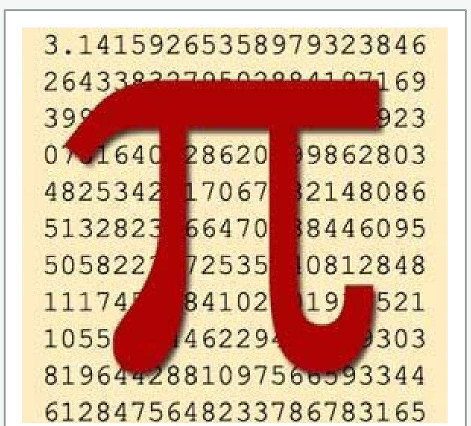
http://www.flickr.com/photos/animal168/3515233003/

Fig. 6: $\sqrt{2}$ is not element of the set of rational numbers

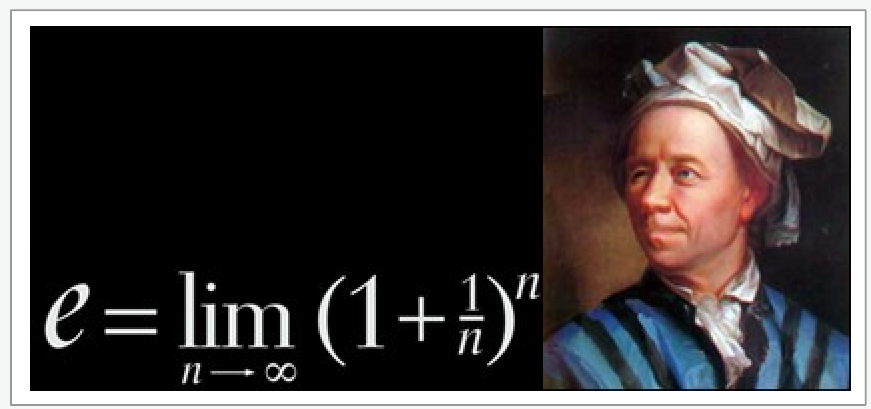
Irrational numbers are infinite, not repeating decimal fractions.

Most of the outputs of the root function, of the logarithmic functions or trigonometric functions, as well as the numbers π and e are irrational numbers.

Archimedes' constant	$\pi = 3.141592654\ldots$
Euler's number	$e = 2{,}718281828459\ldots$



1)



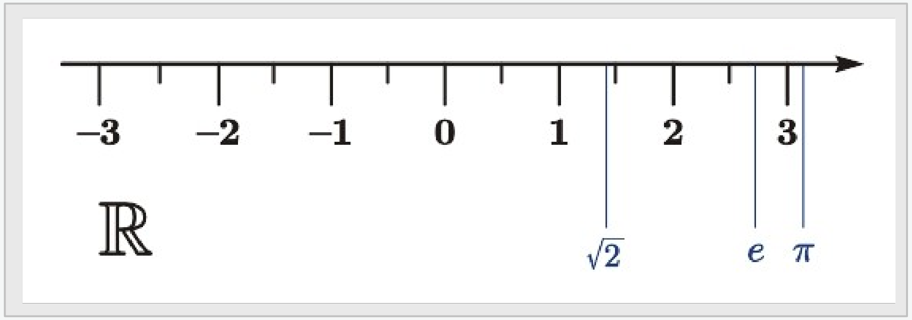
2)

1) http://i032.radikal.ru/0804/82/af2caf626175.jpg

2) http://images.zeit.de/bilder/2007/24/wissen/wissenschaft/euler/euler-artikel.jpg

The sets of rational and irrational numbers form together the set of real numbers $\mathbb{R}$.

The set of real numbers can be seen as the set of all points on the number line. The points corresponding to real numbers cover the line completely.



Operations and relations on the set $\mathbb{R}$ of real numbers :

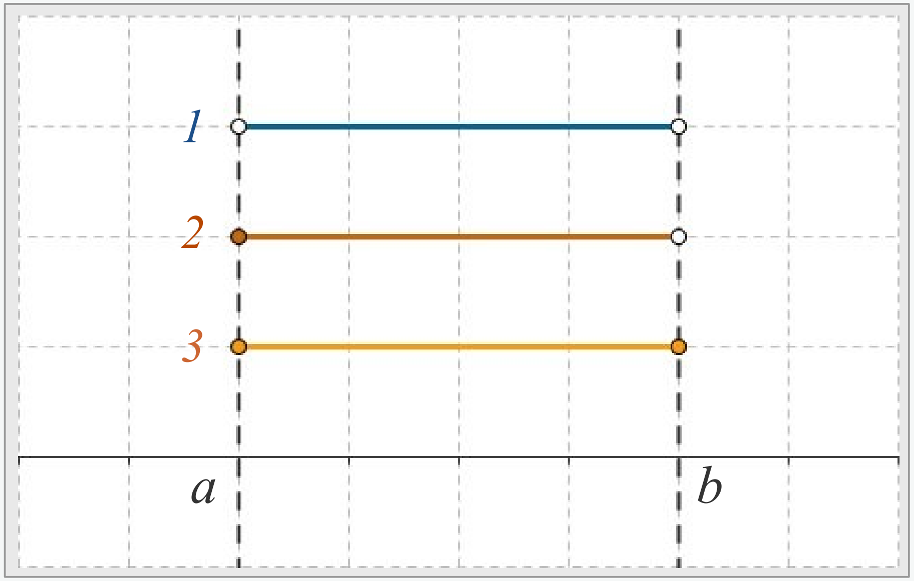
- addition
- multiplication
- subtraction (existence of additive inverse)
- division (existence of multiplicative inverse)
- order relation

Commutative property: $a + b = b + a\,,\qquad a \cdot b = b \cdot a$

Associative property: $(a + b) + c = a + (b + c)\,,\qquad (a \cdot b) \cdot c = a \cdot (b \cdot c)$

Distributive property: $a \cdot (b + c) = a \cdot b + a \cdot c$

Finite intervals $(a < b)$:



1) open interval

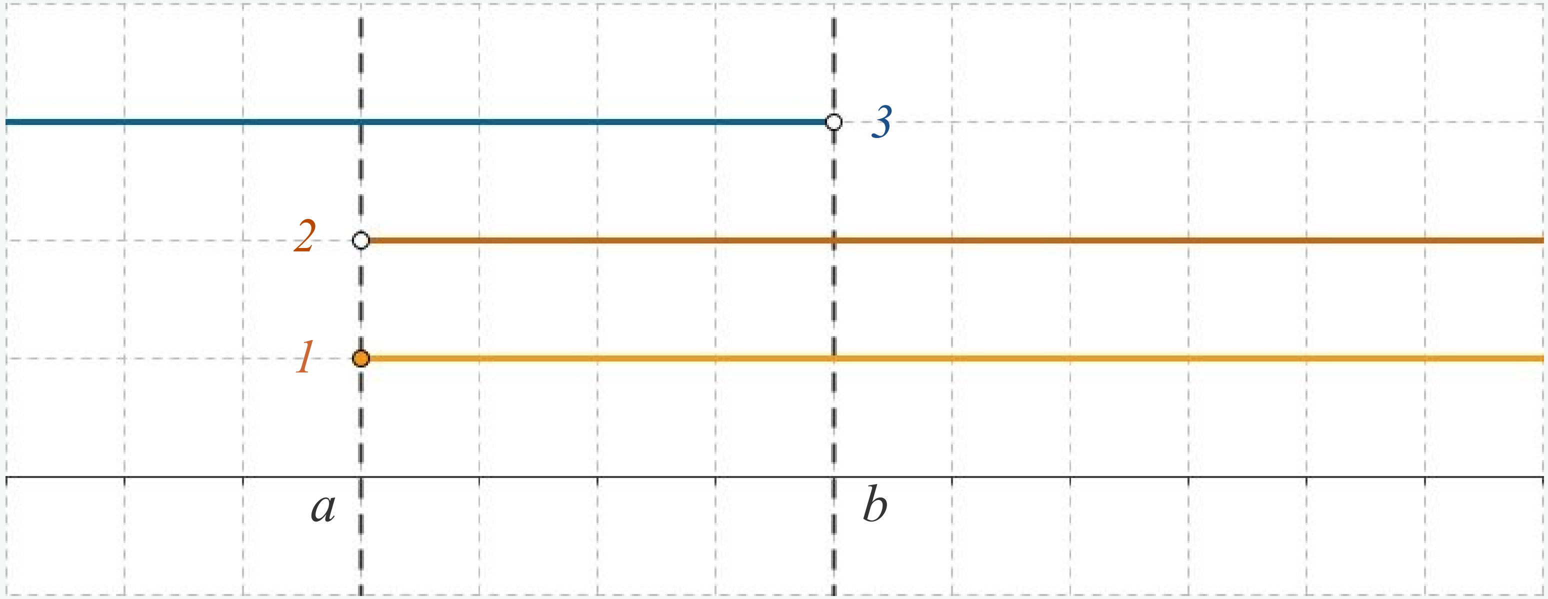
$(a\,,b) = \{x \mid a < x < b\}$

2) left closed, right open interval

$[a\,,b) = \{x \mid a \leqslant x < b\}$

3) closed interval

$[a\,,b] = \{x \mid a \leqslant x \leqslant b\}$



1. $[a, \infty) = \{x \mid a \leqslant x < \infty\}$

2. $(a, \infty) = \{x \mid a < x < \infty\}$

3. $(-\infty, b) = \{x \mid -\infty < x < b\}$

$$\mathbb{R} = (-\infty, \infty)$$