Binomial Theorem

The notion expression plays an important role in algebra

Definition:

Any meaningful connected series of mathematical symbols is called “expression”. Simple expressions like a single number or a single variable of a series of a product of numbers or variables are also called “term”.

Expressions without variables are e.g.:

1	$19 + 27$	$12 : 5$

Expressions with variables are e.g:

x	$1 + x$	$5x - 3$	$7x + b$

The following is not an expression:

$2 = 7$

Many famous mathematical theorems go with the name of the supposed discoverer, e.g. the Pythagorean theorem. This is different with the Binomial Theorem which was not introduced by Mr. Binom. The name is due to the two terms in the expression. A binomial is an expression consisting of two elements which are not the same.

Some examples of binomials

$$a + b, \quad 2a - 7b, \quad x - 4y$$

Binomials can be distinguished by their degree. The degree of a binomial is given by the the exponent of its outer bracket. $(a - b)$ is a binomial of first degree, $(x + y)^2$ is a binomial of second degree etc. The higher the degree of a binomial, the more involved its evaluation.

Binomials of second degree

Famous are the binomials of second degree. They are known as first, second and third binomial formulas.

first binomial formula

$$(a + b)^2 = a^2 + 2\,a\,b + b^2$$

second binomial formula

$$(a - b)^2 = a^2 - 2\,a\,b + b^2$$

third binomial formula

$$(a + b)\,(a - b) = a^2 - b^2$$

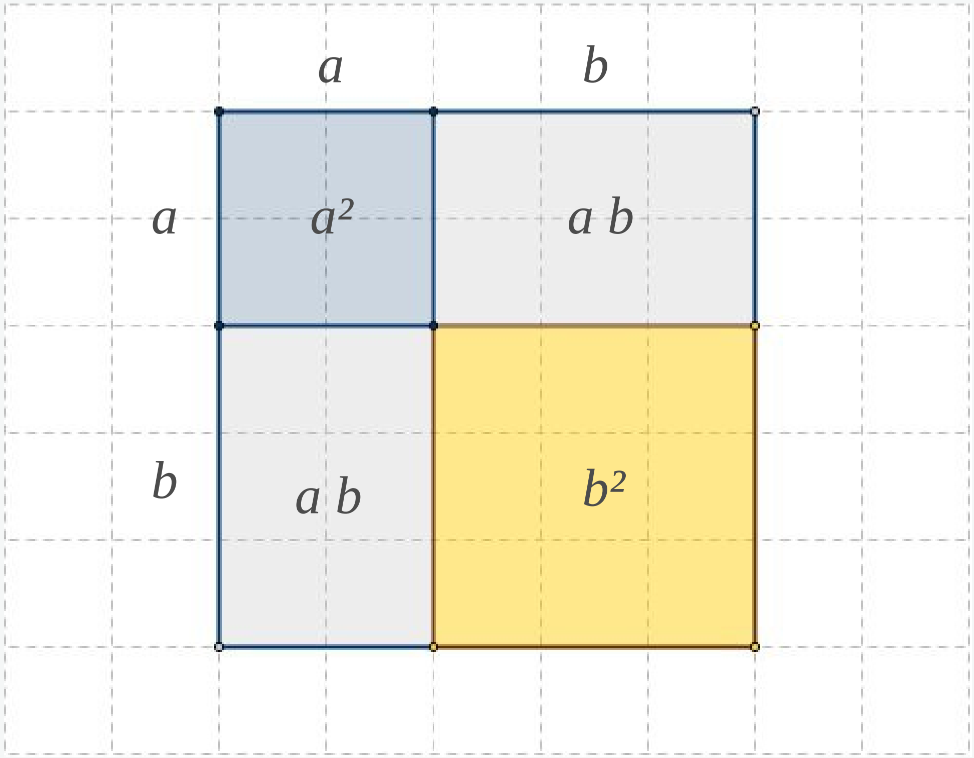


Fig.1 : Graphical demonstration of the first binomial formula

That the first binomial formula is correct, can also be verified graphically.

Binomials of higher degree

We want to develop an evaluation of $(a+b)^n$ for any n. To get the results for $n = 0, 1, 2, 3, 4, \ldots$ we begin with $n = 0$ and multiply the results in each case by $a + b$

$$(a+b)^0 = 1$$

$$(a+b)^1 = a + b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

The factors in front of the terms with powers of a and b are known as binomial coefficients.

As an example, we write down more explicitly the binomial of degree 4:

$$(a+b)^4 = a^4 b^0 + 4a^3 b + 6a^2 b^2 + 4ab^3 + a^0 b^4$$

We notice:

1) The highest exponents correspond to the degree of the binomial.

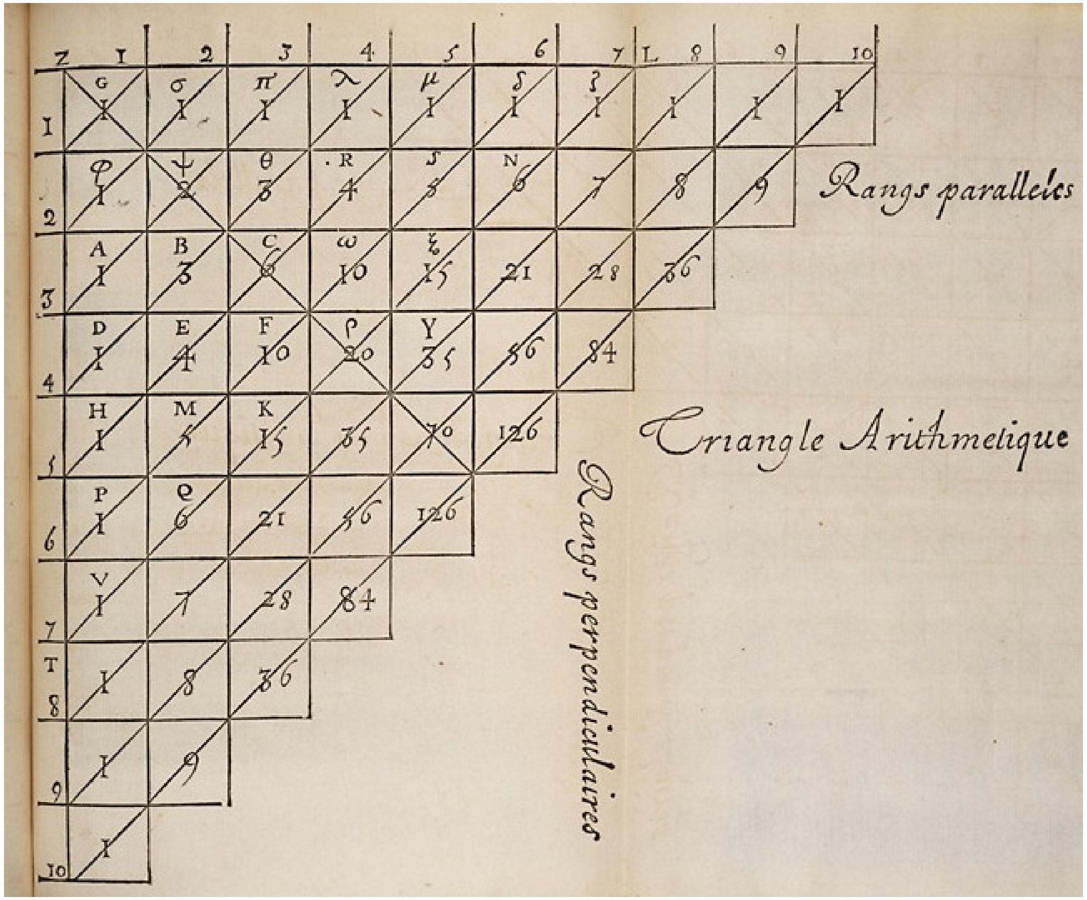
2) a begins with the highest exponent. The exponent is reduced by 1 at each term until it arrives at the value zero.

3) For b it is just the other way round.

These observations are valid for all degrees.

We now investigate the binomial coefficients, that is the numbers in front of the powers of a and b. If we write them line by line, we build a triangle. We notice that there are regularities. There is a one at the tip, and the first and last number of each row are one as well. The other numbers are the sum of the pair just above. This is the simple recipe to reproduce Pascal's triangle.

Fig.2 : Blaise Pascal (1623–1662), french mathematician, physicist, man of letters and philosopher



http://upload.wikimedia.org/wikipedia/commons/6/66/TrianguloPascal.jpg

Fig.3 : Blaise Pascal's version of the triangle

$$(a+b)^n =$$

$$= a^n + \binom{n}{1} a^{n-1} \cdot b^1 + \binom{n}{2} a^{n-2} \cdot b^2 + \ldots + \binom{n}{n-1} a^1 \cdot b^{n-1} + b^n =$$

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} \cdot b^k$$

Binomial coefficient (*"n choose k"*):

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad (k, n \in \mathbb{N}^*, \quad k \leqslant n)$$

$$\binom{n}{0} = 1, \qquad \binom{n}{1} = n, \qquad \binom{n}{n} = 1$$

Factorial function: $n! = 1 \cdot 2 \cdot 3 \ldots n \qquad 0! = 1$

The formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} \cdot b^k$$

may also be used to calculate $(a-b)^n$

$$(a-b)^n =$$

$$= a^n - \binom{n}{1} a^{n-1} \cdot b^1 + \binom{n}{2} a^{n-2} \cdot b^2 - \ldots + \binom{n}{n-1} a^1 \cdot b^{n-1} + (-1)^n b^n =$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} a^{n-k} \cdot b^k$$

There is a minus sign in front of all terms where b has an uneven exponent.

Blaise Pascal

Evaluate the following binomials:

Exercise 1: $(x + y)^5$

Exercise 2: $(x + y)^6$

Exercise 3: $(x - y)^3$

Exercise 4: $(x - y)^5$

Exercise 5: $(a + 2b)^5$

Binomial theorem: solution 1

$$(x+y)^5 = x^5 + \binom{5}{1} x^4 \cdot y + \binom{5}{2} x^3 \cdot y^2 + \binom{5}{3} x^2 \cdot y^3 + \binom{5}{4} x \cdot y^4 + y^5$$

$$\binom{5}{1} = \frac{5!}{1!\,(5-1)!} = \frac{5!}{4!} = 5$$

$$\binom{5}{2} = \frac{5!}{2!\,(5-2)!} = \frac{5!}{2!\,3!} = 10$$

$$\binom{5}{3} = \frac{5!}{3!\,(5-3)!} = \frac{5!}{3!\,2!} = 10$$

$$\binom{5}{4} = \frac{5!}{4!\,(5-4)!} = \frac{5!}{4!\,1!} = 5$$

$$(x+y)^5 = x^5 + 5\,x^4 \cdot y + 10 \cdot x^3 \cdot y^2 + 10 \cdot x^2 \cdot y^3 + 5 \cdot x \cdot y^4 + y^5$$

Binomial theorem: solution 2

$$(x+y)^6 =$$

$$= x^6 + \binom{6}{1} x^5 \cdot y + \binom{6}{2} x^4 \cdot y^2 + \binom{6}{3} x^3 \cdot y^3 + \binom{6}{4} x^2 \cdot y^4 + \binom{6}{5} x \cdot y^5 + y^6$$

$$\binom{6}{1} = \frac{6!}{1!\,(6-1)!} = \frac{6!}{5!} = 6$$

$$\binom{6}{2} = \frac{6!}{2!\,(6-2)!} = \frac{6!}{2!\;4!} = \frac{5\cdot 6}{2} = 15$$

$$\binom{6}{3} = \frac{6!}{3!\,(6-3)!} = \frac{6!}{3!\;3!} = \frac{4\cdot 5\cdot 6}{2\cdot 3} = 20$$

$$\binom{6}{4} = \frac{6!}{4!\,(6-4)!} = \frac{6!}{4!\;2!} = \frac{5\cdot 6}{2} = 15$$

$$\binom{6}{5} = \frac{6!}{5!\,(6-5)!} = \frac{6!}{5!} = 6$$

$$(x+y)^6 = x^6 + 6\,x^5 \cdot y + 15 \cdot x^4 \cdot y^2 + 20 \cdot x^3 \cdot y^3 + 15 \cdot x^2 \cdot y^4 + 6\,x \cdot y^5 + y^6$$

Solution 3: $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

Solution 4: $(x-y)^5 =$

$$= x^5 - 5x^4 \cdot y + 10 \cdot x^3 \cdot y^2 - 10 \cdot x^2 \cdot y^3 + 5 \cdot x \cdot y^4 - y^5$$

Solution 5: $(a+2b)^5 =$

$$= a^5 + \binom{5}{1} a^4 \cdot (2b) + \binom{5}{2} a^3 \cdot (2b)^2 + \binom{5}{3} a^2 \cdot (2b)^3 +$$

$$+ \binom{5}{4} a \cdot (2b)^4 + (2b)^5 = a^5 + 5a^4 \cdot (2b) + 10a^3 \cdot (2b)^2 +$$

$$+ 10a^2 \cdot (2b)^3 + 5a \cdot (2b)^4 + (2b)^5 = a^5 + 10a^4 \cdot b + 40a^3 \cdot b^2 +$$

$$+ 40a^3 \cdot b^2 + 80a^2 \cdot b^3 + 80a \cdot b^4 + 32b^5$$

$$(a+2b)^5 = a^5 + 10a^4 \cdot b + 40a^3 \cdot b^2 + 40a^3 \cdot b^2 + 80a^2 \cdot b^3 +$$

$$+ 80a \cdot b^4 + 32b^5$$