Algebraic Expressions

A special class of expressions are polynomials.

Polynomials are sums and differences of terms which are a product of a constant or some parameter with a power of a variable with integer exponents.

Example: $P(x) = 2x^3 + x - 7 = 2x \cdot x \cdot x + x - 7$

Fractions of polynomials are rational expressions.

Example: $R(x) = P(x) / Q(x)$ with

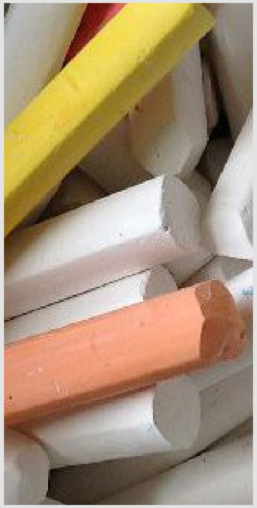
$$P(x) = a x^4 + 2x^2 - 1 \quad , \quad Q(x) = 3x^3 + 5x^2 - 6$$

Algebraic expressions contain roots of variables in addition.

addition subtraction multiplication	→	Polynomial + division	→	Rational Expression + root	→	

→ *Algebraic Expression*

$$2\,x^2+3\,,\quad x^3-7 \quad\rightarrow\quad \frac{2\,x^2+3}{x^3-7} \quad\rightarrow\quad \sqrt{2\,x^2+3}$$



Definition:

The following expression with variable x

$$a_0 + a_1 \cdot x + a_2 \cdot x^2 + \ldots + a_n x^n , \qquad a_i \in \mathbb{R}$$

is a polynomial of degree n. The a_i are called coefficients of the polynomial; the addends of the sum may be called terms of the polynomials.

The name "Polynomial" means, that the expression consists of several terms (poly: *greek* many). An expression with only two terms may be called binomial.

We recall the binomial formulas:

$$(a + b)^2 = a^2 + 2\,a\,b + b^2$$

$$(a + b)^3 = a^3 + 3\,a^2\,b + 3\,a\,b^2 + b^3$$

$$(a + b)^4 = a^4 + 4\,a^3\,b + 6\,a^2\,b^2 + 4\,a\,b^3 + b^4$$

The degree of a term is determined by the exponent of the variable, the largest exponent determines the degree of the polynomial.

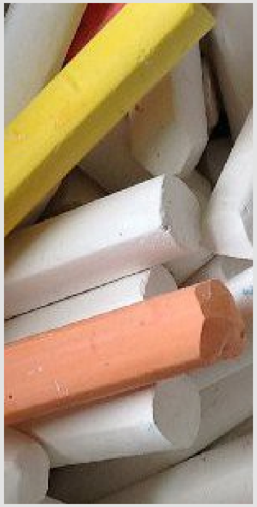
Example:

The polynomial $2 + x - 3x^2 + 6x^3$ consists of

- a term of degree *0*: 2
- a term of degree *1*: x
- a term of degree *2*: $-3x^2$
- a term of degree *3*: $6x^3$

It is a polynomial of degree *3*.

The exponents of a polynomial are integer and positive (including zero).



Determine the degree of the polynomials:

1. $2x^2 + 7x - 3x^6$
2. $\sqrt{10x} - x^3$
3. $\sqrt{10} + 2x - x^3$
4. $(x+4)^2$
5. $\sqrt{x} + 3x + x^4$
6. $\frac{1}{x}$
7. $x^{\frac{2}{3}} + x^2 - 12$
8. $(x+4)^5$
9. $x^{-3} + x^2\sqrt{2} - 8x$

Polynomials of degree n: solution 1

1.	$2x^2 + 7x - 3x^6$	polynomial, degree *6*
2.	$\sqrt{10x} - x^3$	not a polynomial, because of the root
3.	$\sqrt{10} + 2x - x^3$	polynomial, degree *3*
4.	$(x+4)^2$	polynomial, degree *2*
5.	$\sqrt{x} + 3x + x^4$	not a polynomial, because of the root
6.	$\frac{1}{x}$	not a polynomial
7.	$x^{\frac{2}{3}} + x^2 - 12$	not a polynomial, due to non integer exponent
8.	$(x+4)^5$	polynomial, degree *5*
9.	$x^{-3} + x^2\sqrt{2} - 8x$	not a polynomial, due to negative exponent

$$(x-1)\cdot(x^2+x-3) = x^3-4x+3$$

The product of two polynomials is obtained by multiplying them term by term. The result is again a polynomial. Its degree is given by the sum of the degrees of the two polynomial factors.

Consequently, multiplication of an *n-th* degree polynomial with an *m-th* degree polynomial, results in an *(n + m)-th* degree polynomial.

Multiplication term by term means, that each term of one of the polynomials is multiplied by all terms of the other polynomial. The resulting polynomial is then obtained by the sum of all these products.

$$2x^4 - 5x^2 - 12 = 0, \qquad u = x^2 \quad \rightarrow \quad 2u^2 - 5u - 12 = 0$$

$$u_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$u_{1,2} = \frac{5 \pm \sqrt{121}}{4} = \frac{5 \pm 11}{4}, \qquad u_1 = 4, \qquad u_2 = -\frac{3}{2}$$

$$2u^2 - 5u - 12 = 2(u - u_2)(u - u_1) = (2u + 3)(u - 4)$$

$$2x^4 - 5x^2 - 12 = (2x^2 + 3)(x^2 - 4) = (2x^2 + 3)(x - 2)(x + 2)$$

If a product is zero, then at least one the factors is zero !

1. factor: $(2x^2 + 3) = 0 \rightarrow x^2 = -3/2 \rightarrow$ there is no real solution
2. factor: $(x - 2) = 0 \rightarrow x = 2$
3. factor: $(x + 2) = 0 \rightarrow x = -2$

$$2x^4 - 5x^2 - 12 = (2x^2 + 3)(x - 2)(x + 2)$$

At first one might consider this a useless operation. However, we met a similar situation when reducing fractions. Nominator and denominator are decomposed into prime factors with the aim to find common factors which can be cancelled.

$$\frac{x^2 + 2x}{2x^4 - 5x^2 - 12} = \frac{x(x + 2)}{(2x^2 + 3)(x - 2)(x + 2)} = \frac{x}{(2x^2 + 3)(x - 2)}$$

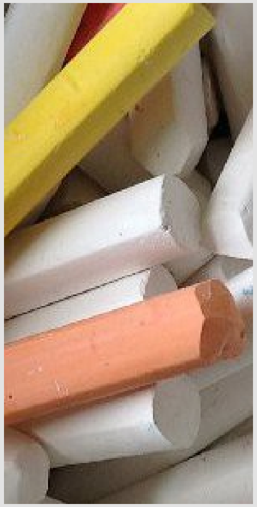
$$x^2 + 2xy + y^2 = (x + y)^2$$

$$x^2 - 2xy + y^2 = (x - y)^2$$

$$x^2 - y^2 = (x + y)(x - y)$$

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$



Factorise the polynomials to polynomial factors with integer exponents:

1. $12x^3 - 27x$
2. $y^3 + 1$
3. $8x^3 + y^3 \cdot z^3$
4. $a^2 + 10a + 25 - 49b^2$
5. $32x^3 - 18x$
6. $a^3 + 2a^2 b + a b^2 - a^2 - 2ab - b^2$

Factoring polynomials: solution 2

1. $12x^3 - 27x = 3x(4x^2 - 9) = 3x(2x - 3)(2x + 3)$

2. $y^3 + 1 = (y + 1)(y^2 - y + 1)$

3. $8x^3 + y^3 \cdot z^3 = (2x)^3 + (y \cdot z)^3 = (2x + y\ z)(4x^2 - 2xyz + y^2\ z^2)$

4. $a^2 + 10a + 25 - 49b^2 = (a + 5)^2 - (7b)^2 = (a + 5 - 7b)(a + 5 + 7b)$

5. $32x^3 - 18x = 2x(16x^2 - 9) = 2x((4x)^2 - 3^2) =$

   $= 2x(4x - 3)(2x + 3)$

6. $a^3 + 2a^2\ b + a\ b^2 - a^2 - 2ab - b^2 =$

   $= a(a^2 + 2a\ b + b^2) - (a^2 + 2ab + b^2) = a(a + b)^2 - (a + b)^2 =$

   $= (a - 1)(a + b)^2$