Quadratic Equations

Definition:

A quadratic equations is a second order polynomial equation with a single unknown. That is, the equation contains the square of the unknown, but no higher power.

A quadratic equation can be written

$$a\,x^2 + b\,x + c = 0$$

with $a \neq 0.$

Dividing by a, one obtains the monic form of the quadratic equation:

$$x^2 + \frac{b}{a}\,x + \frac{c}{a} = 0\,, \qquad p \equiv \frac{b}{a}\,, \qquad q \equiv \frac{c}{a}$$

$$\Rightarrow \quad \boxed{x^2 + p\,x + q = 0}$$

Completing the Square is a method to rewrite terms containing the unknown squared, with the aim to get a squared binomial.

$$x^2 + p\,x = \underbrace{x^2 + 2\cdot\frac{p}{2}\,x + \left(\frac{p}{2}\right)^2}_{\left(x+\frac{p}{2}\right)^2} - \left(\frac{p}{2}\right)^2$$

In this step we add a term such, that we can build a square using the 1. binomial formula

$$\Rightarrow \quad x^2 + p\,x = \left(x + \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2$$

Completing the square: Example

Quadratic equation:	$2x^2 - 12x = 32$
Monic quadratic equation:	$x^2 - 6x = 16$
Completing the square:	$x^2 - 6x + 9 = 16 + 9$
	$(x - 3)^2 = 25$
Extracting the root:	$x - 3 = \pm\sqrt{25} = \pm 5$
Solving for x:	$x = 3 - 5, \quad x = 3 + 5$
Set of solutions:	$S = \{-2, 8\}$

Solution of quadratic equation $x^2 + px + q = 0$

Method of completing the square:

$$x^2 + px + q = \left(x + \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2 + q = 0 \quad \Rightarrow$$

$$\left(x + \frac{p}{2}\right)^2 = \left(\frac{p}{2}\right)^2 - q \quad \Rightarrow \quad x + \frac{p}{2} = \pm\sqrt{\left(\frac{p}{2}\right)^2 - q}$$

Solutions (roots) of the equation:

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

$$x^2 + px + q = 0$$

Solution: p-q-formula

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

Discriminant:

$$D = \left(\frac{p}{2}\right)^2 - q$$

$D > 0$: two real solutions

$D = 0$: one (double) solution

$D < 0$: no real solution

$$a\,x^2 + b\,x + c = 0 \quad \Leftrightarrow \quad x^2 + p\,x + q = 0\,, \qquad p \equiv \frac{b}{a}\,, \qquad q \equiv \frac{c}{a}$$

We know already the solutions of the monic equation (*p*-*q*-formula):

$$x_{1,\,2} = -\,\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}\quad ,$$

therefore, we can write down directly
the solutions of the *a*-*b*-*c* equation:

$$x_{1,\,2} = -\frac{b}{2\,a} \pm \sqrt{\left(\frac{b}{2\,a}\right)^2 - \frac{c}{a}}$$

or

$$x_{1,\,2} = -\frac{b}{2\,a} \pm \sqrt{\frac{b^2 - 4\,a\,c}{4\,a^2}}\,, \qquad x_{1,\,2} = \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}$$

$$a x^2 + b x + c = 0 \qquad (a \neq 0)$$

Solutions (roots):

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Discriminant:

$$D = b^2 - 4ac$$

$D > 0$: two real solutions

$D = 0$: one (double) solution

$D < 0$: no real solution

Exercises

Determine the roots of the following quadratic equations:

Exercise 1: $-2x^2 - 4x + 6 = 0$

Exercise 2: $x^2 - 2x - 15 = 0$

Determine the intercepts of the following parabolas with the x-axis (x-intercept):

Exercise 3: $f(x) = 4x^2 + x - \frac{3}{2}$

Exercise 4: $f(x) = 2x^2 + 6x + \frac{5}{2}$

Exercise 5: $f(x) = 2x^2 + 4x + 2$

Exercise 6: $f(x) = 2x^2 + 4x + \frac{5}{2}$

Quadratic equations: Solution 1

1. $-2x^2 - 4x + 6 = 0 \quad | \times \left(-\frac{1}{2}\right)$

2. $x^2 + 2x - 3 = 0$

Solution by p-q-formula

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

$p = 2, \quad q = -3$

$D > 0$: two real solutions

$x_{1,2} = -1 \pm \sqrt{4} = -1 \pm 2 \quad \Rightarrow$

$x_1 = -3, \quad x_2 = 1$

3. $S = \{-3, 1\}$

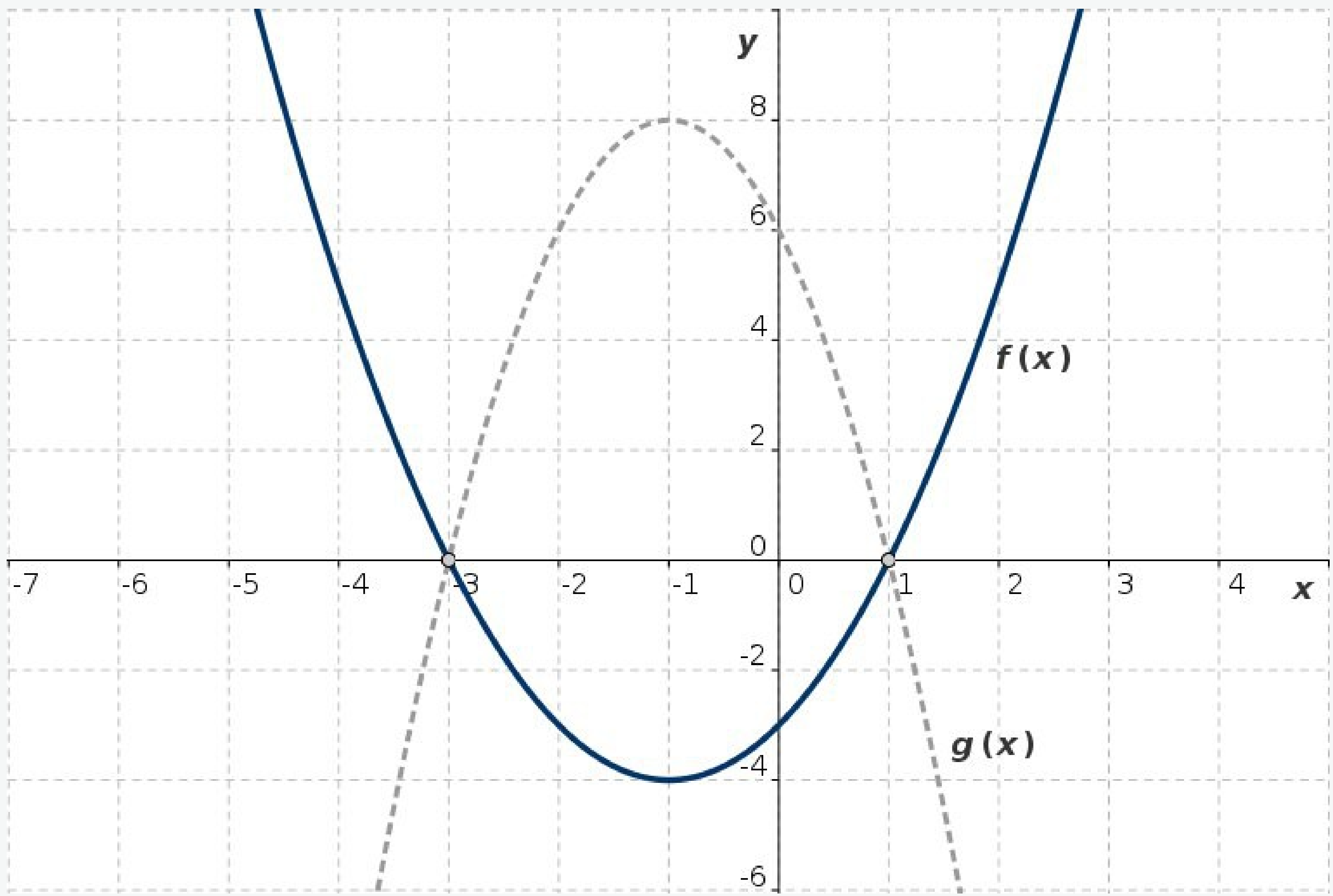

Fig. L1: Functions $y = f(x)$ and $y = g(x)$

$$f(x) = x^2 + 2x - 3, \qquad g(x) = -2x^2 - 4x + 6$$

1. $x^2 - 2x - 15 = 0$

2. Solution by p-q-formula

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

$$p = -2, \qquad q = -15$$

$D > 0$: two real solutions

$$x_{1,2} = 1 \pm \sqrt{1^2 - (-15)} = 1 \pm \sqrt{16} = 1 \pm 4 \quad \Rightarrow$$

$$x_1 = -3, \qquad x_2 = 5$$

3. $S = \{-3, 5\}$

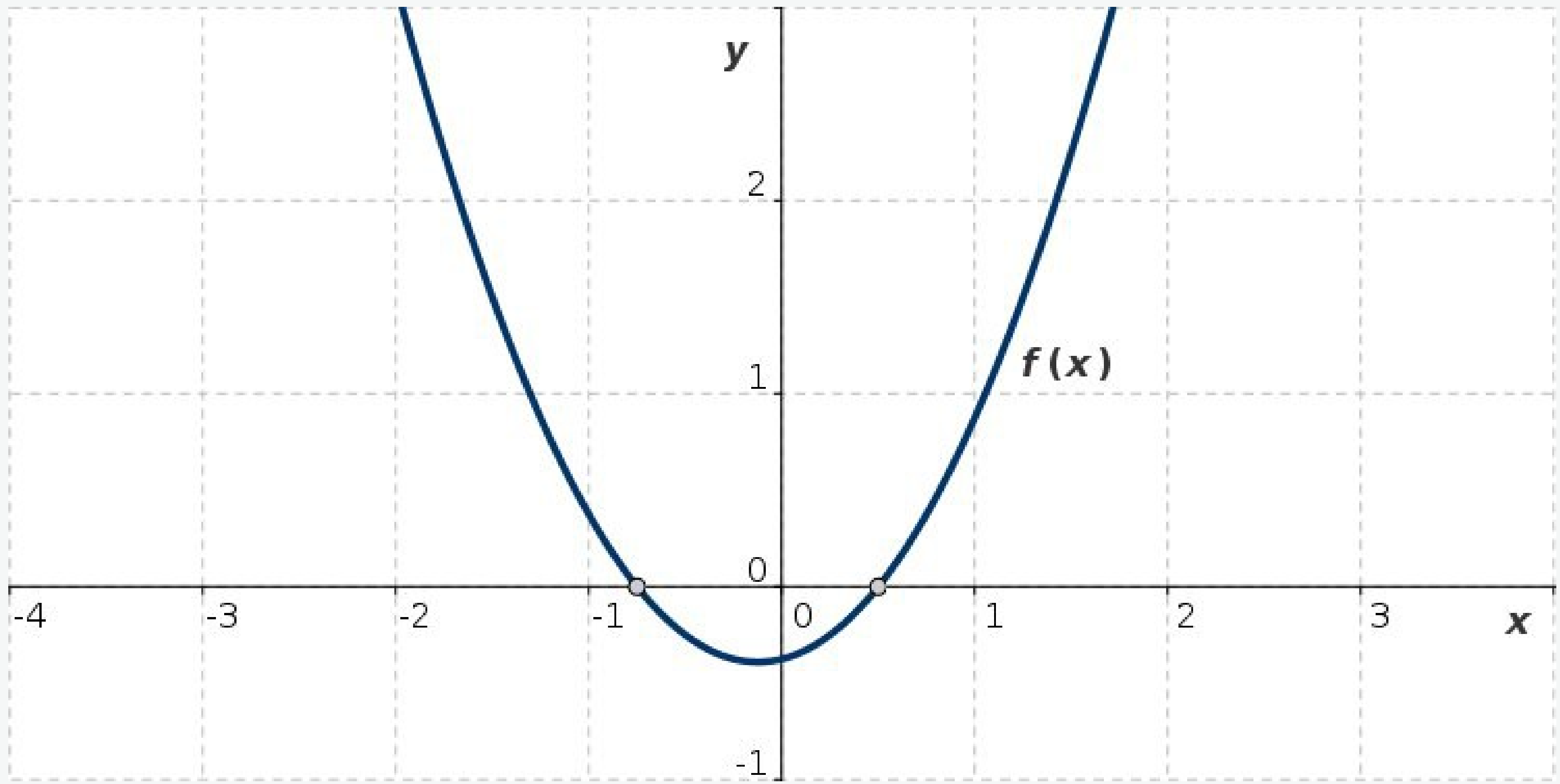

Fig. L3: Function $y = f(x)$

$$f(x) = 4x^2 + x - \frac{3}{2}, \qquad 4x^2 + x - \frac{3}{2} = 0 \quad \Leftrightarrow \quad x^2 + \frac{x}{4} - \frac{3}{8} = 0$$

$$x_1 = -\frac{3}{4}, \qquad x_2 = \frac{1}{2}$$

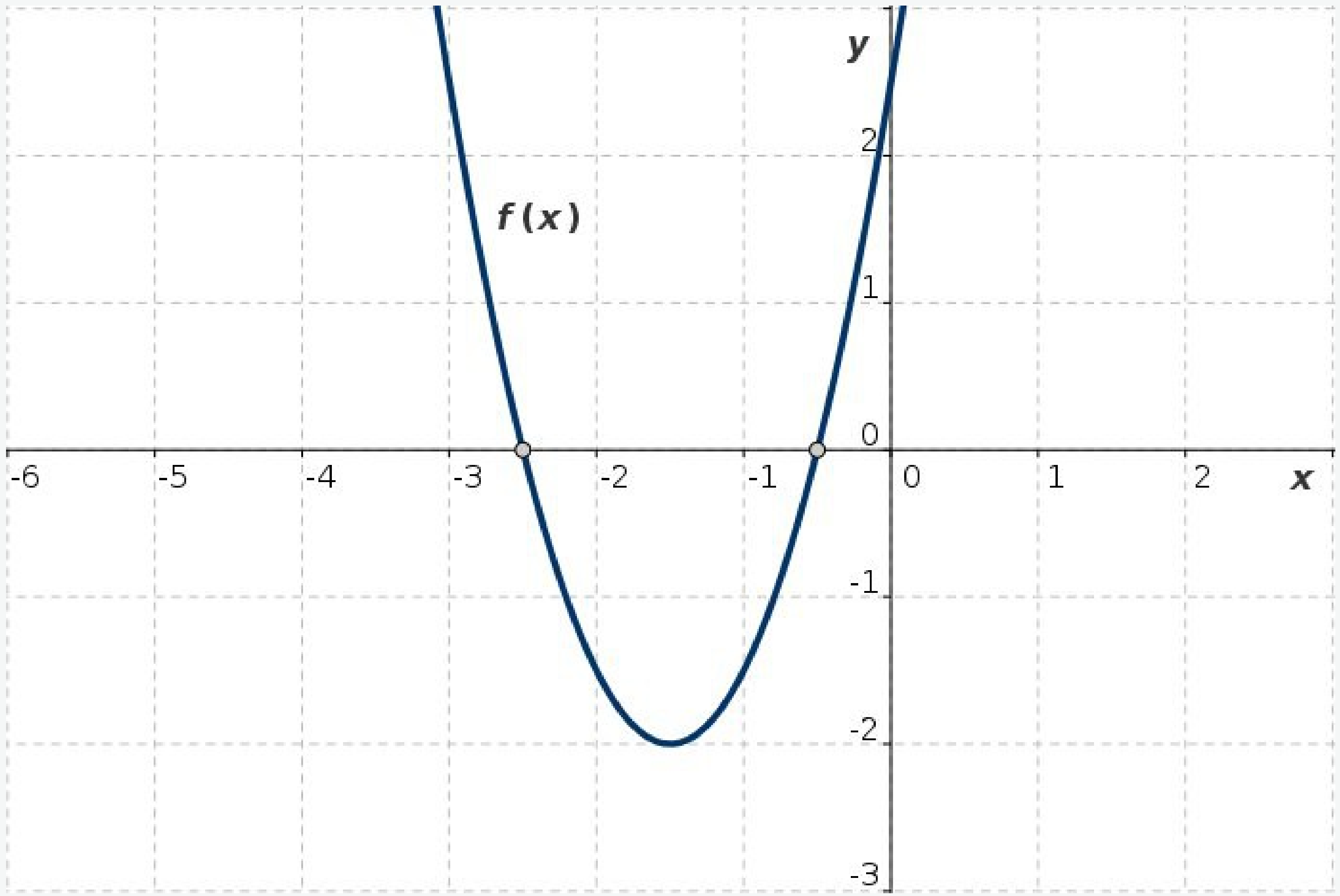

Fig. L4: Function $y = f(x)$

$$f(x) = 2x^2 + 6x + \frac{5}{2}, \qquad x^2 + 3x + \frac{5}{4} = 0, \qquad x_1 = -\frac{5}{2}, \qquad x_2 = -\frac{1}{2}$$

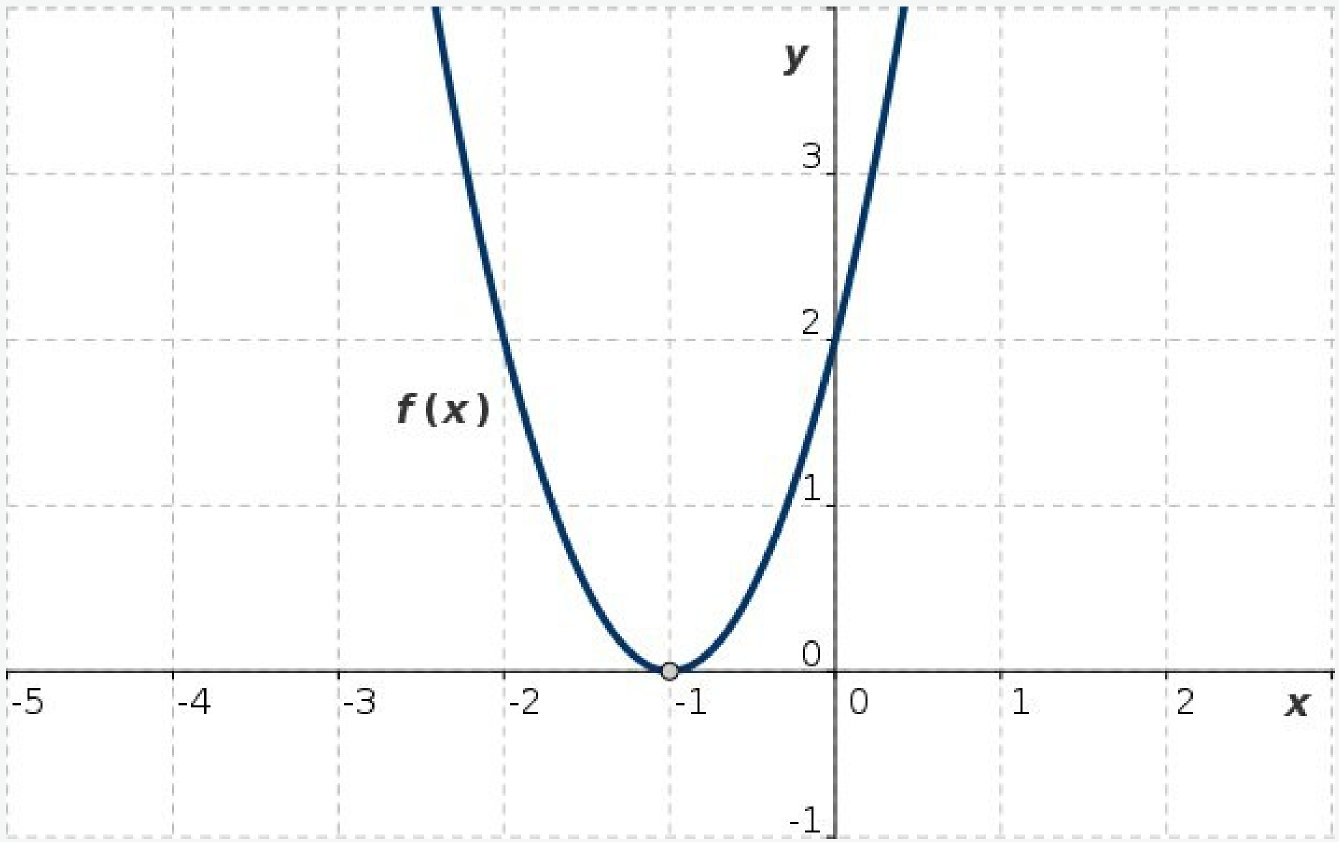

Fig. L5: Function $y = f(x)$

$$f(x) = 2x^2 + 4x + 2, \qquad x^2 + 2x + 1 = 0, \qquad x_1 = x_2 = -1$$

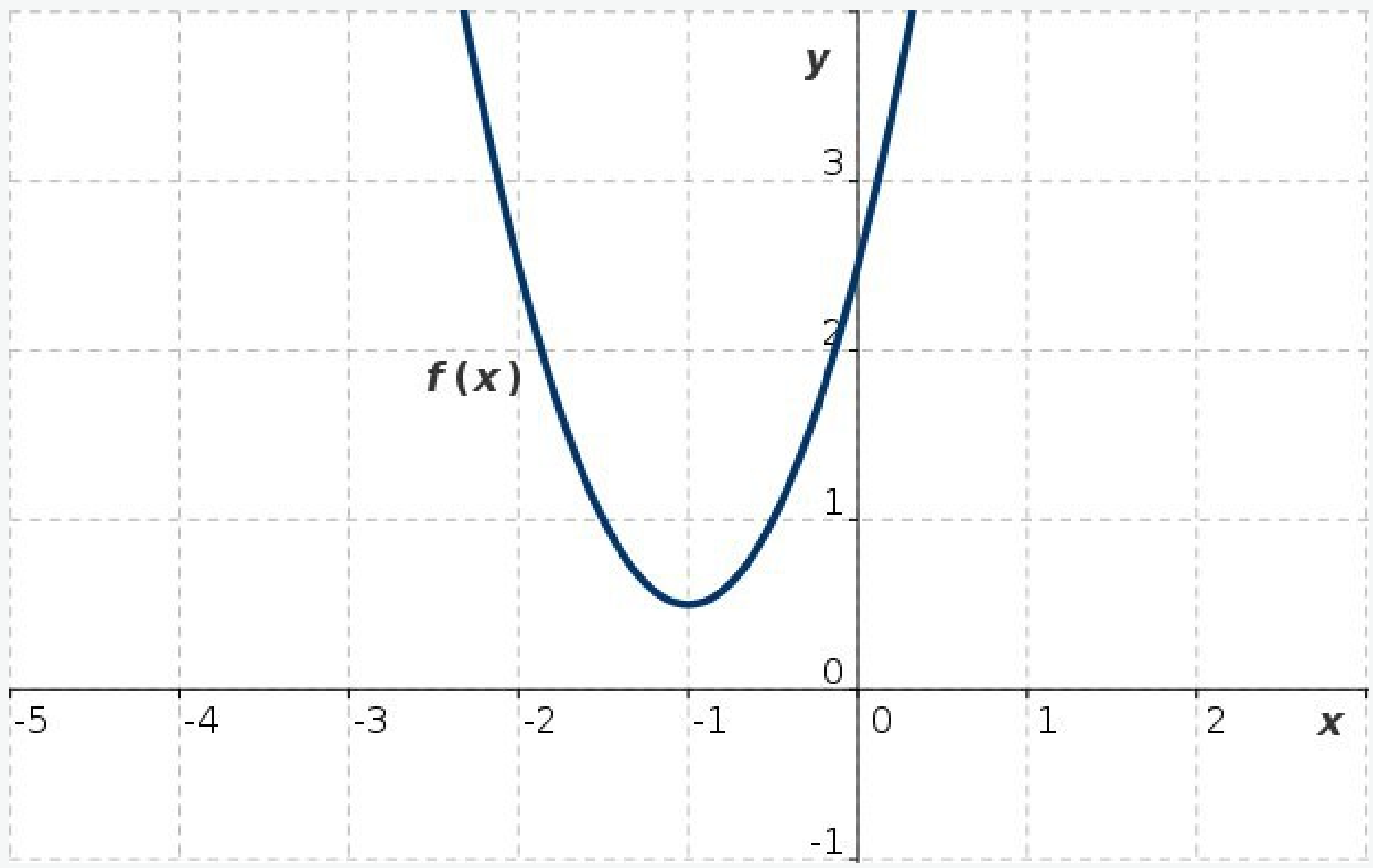

Fig. L6: Function $y = f(x)$

$$f(x) = 2x^2 + 4x + \frac{5}{2}, \qquad x^2 + 2x + \frac{5}{4} = 0, \qquad x_{1,2} = -1 \pm \sqrt{-\frac{1}{4}}$$

$D < 0$: no real solution