Vertex, Intersections, Completing the square

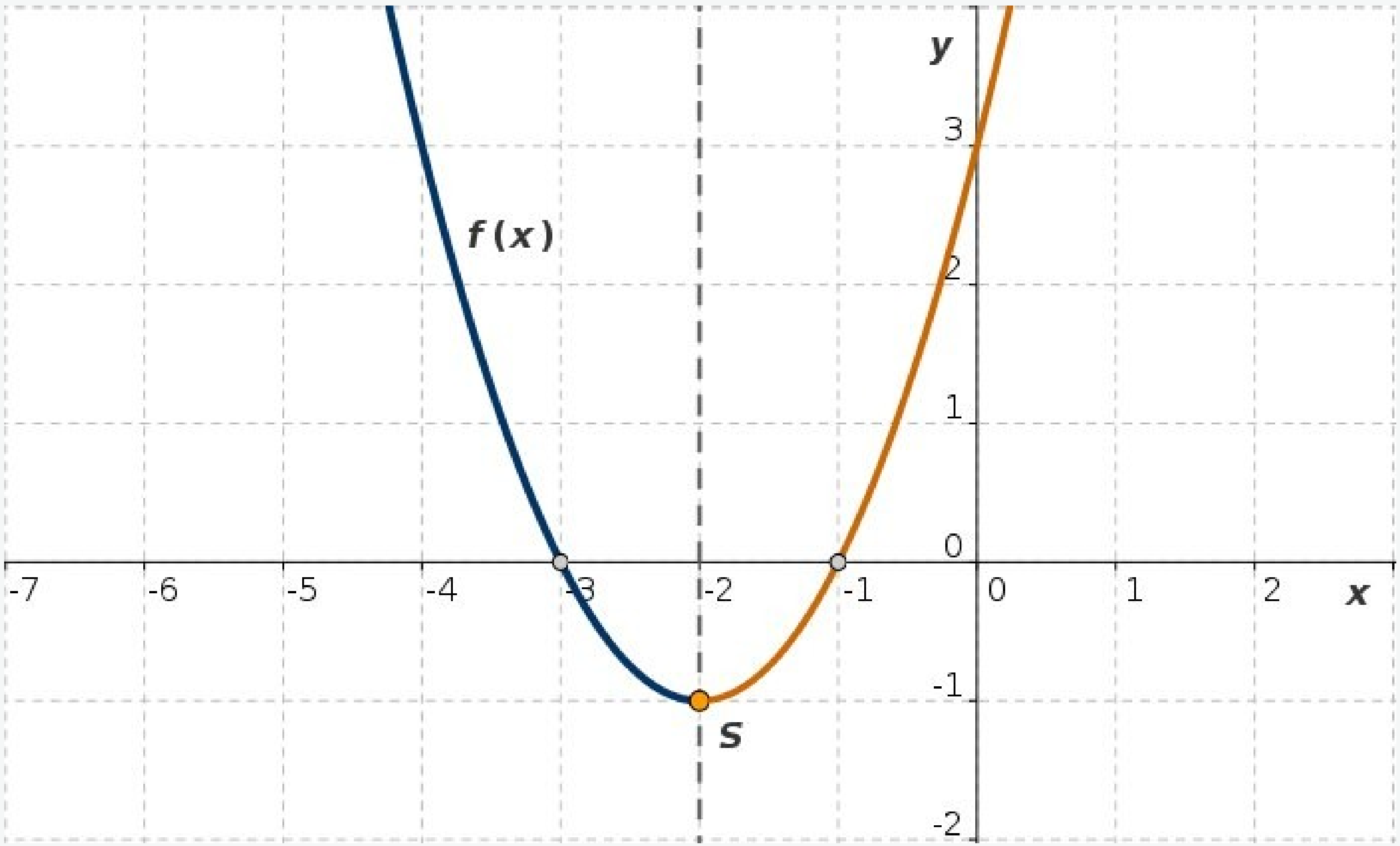

Fig. 6-1: Representation of the function $y = f(x)$

$$f(x) = a x^2 + b x + c = x^2 + 4x + 3 \quad (a = 1)$$

$a > 0$: the parabola opens up. The vertex $S = (-2, -1)$ is the lowest point of the parabola. The line $x = -2$ is an axis of symmetry.

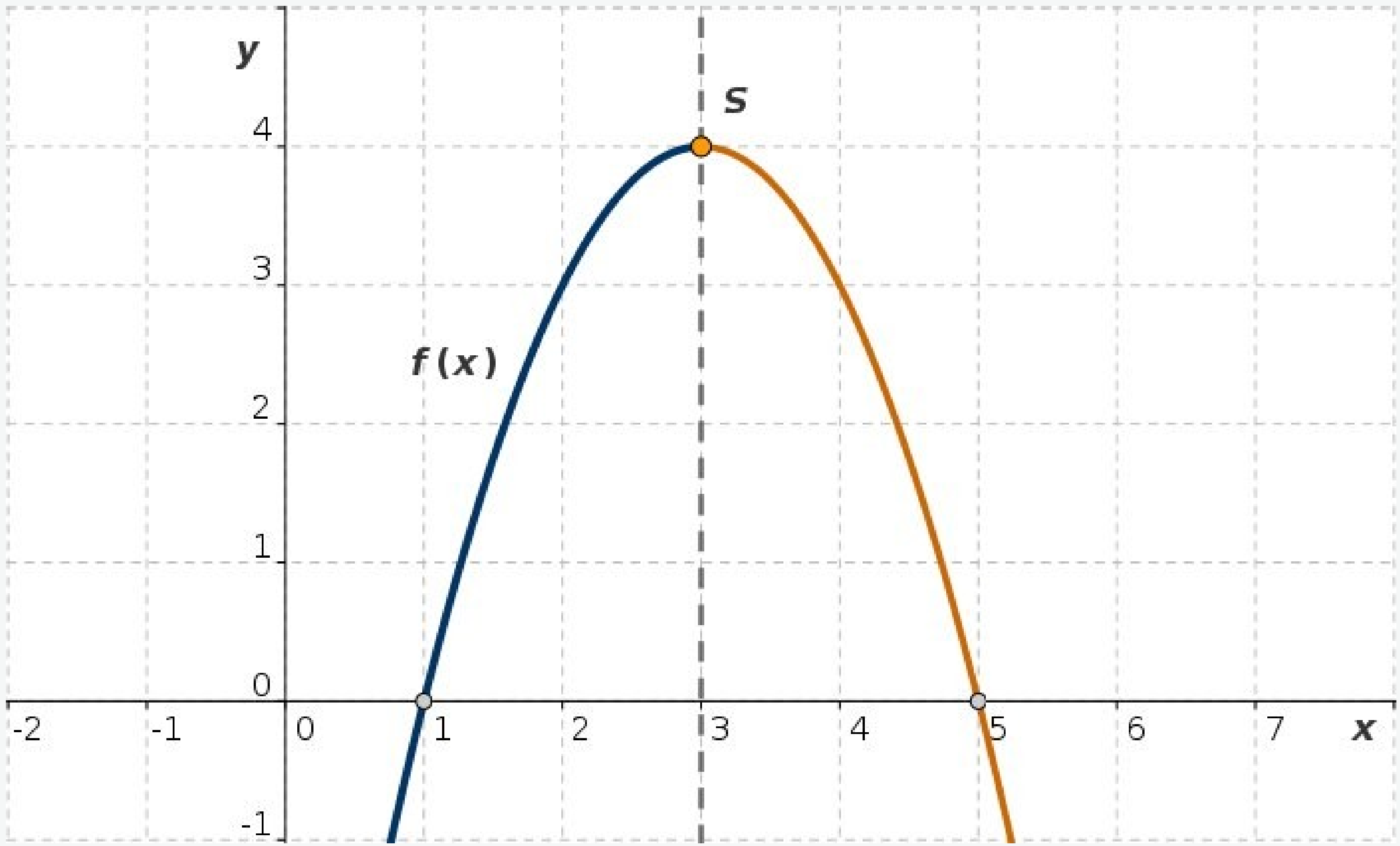

Fig. 6-2: Representation of the function y = f (x)

$$f(x) = a x^2 + b x + c = -x^2 + 6x - 5 \quad (a = -1)$$

$a < 0$: the parabola opens down. The vertex $S = (3, 4)$ is the highest point of the parabola. The line $x = 3$ is an axis of symmetry.

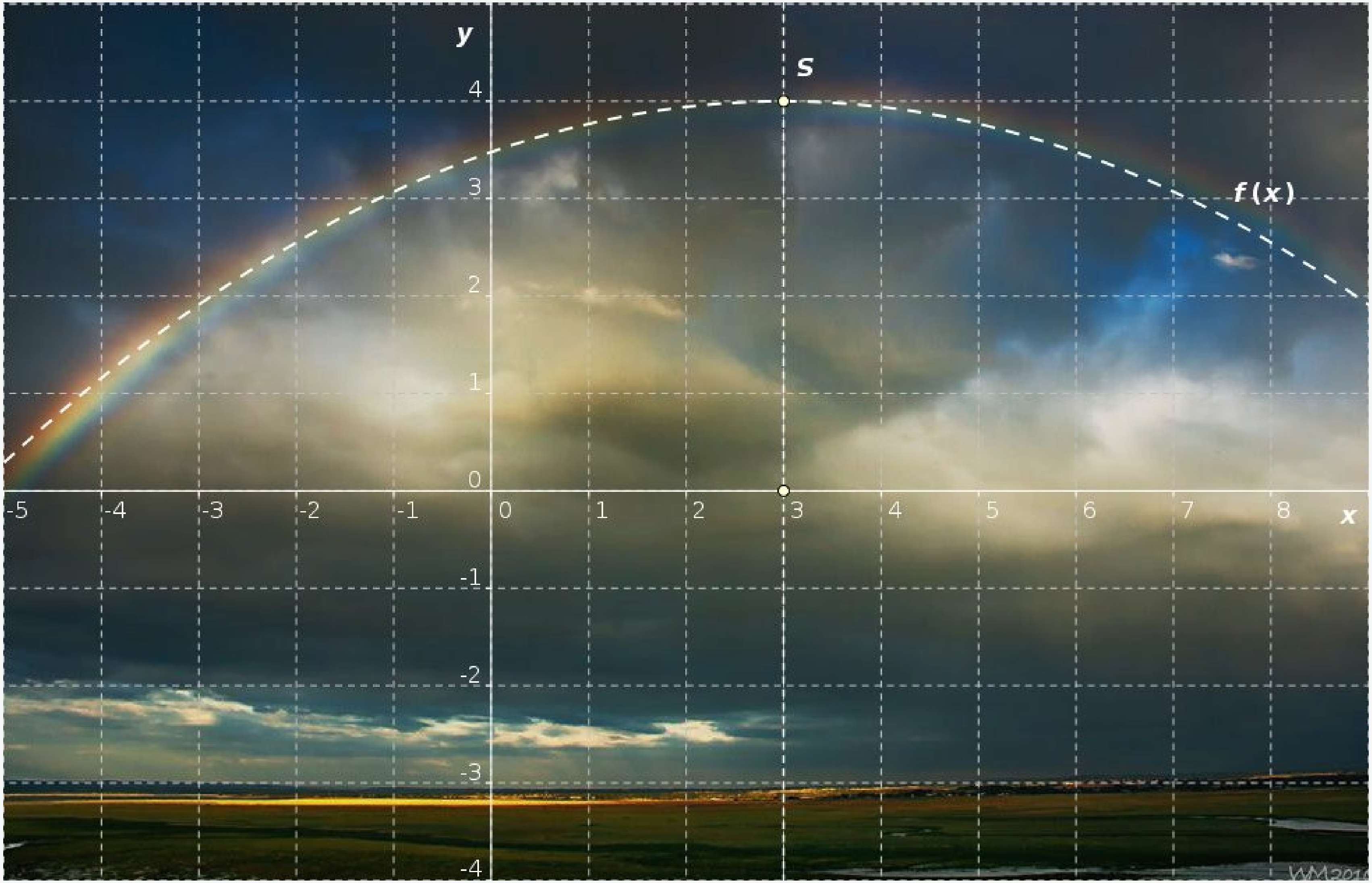

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Fig. 6-3: A representation of the function $y = f(x)$, which opens down

$$f(x) = -0.06\,(x - 3)^2 + 4$$

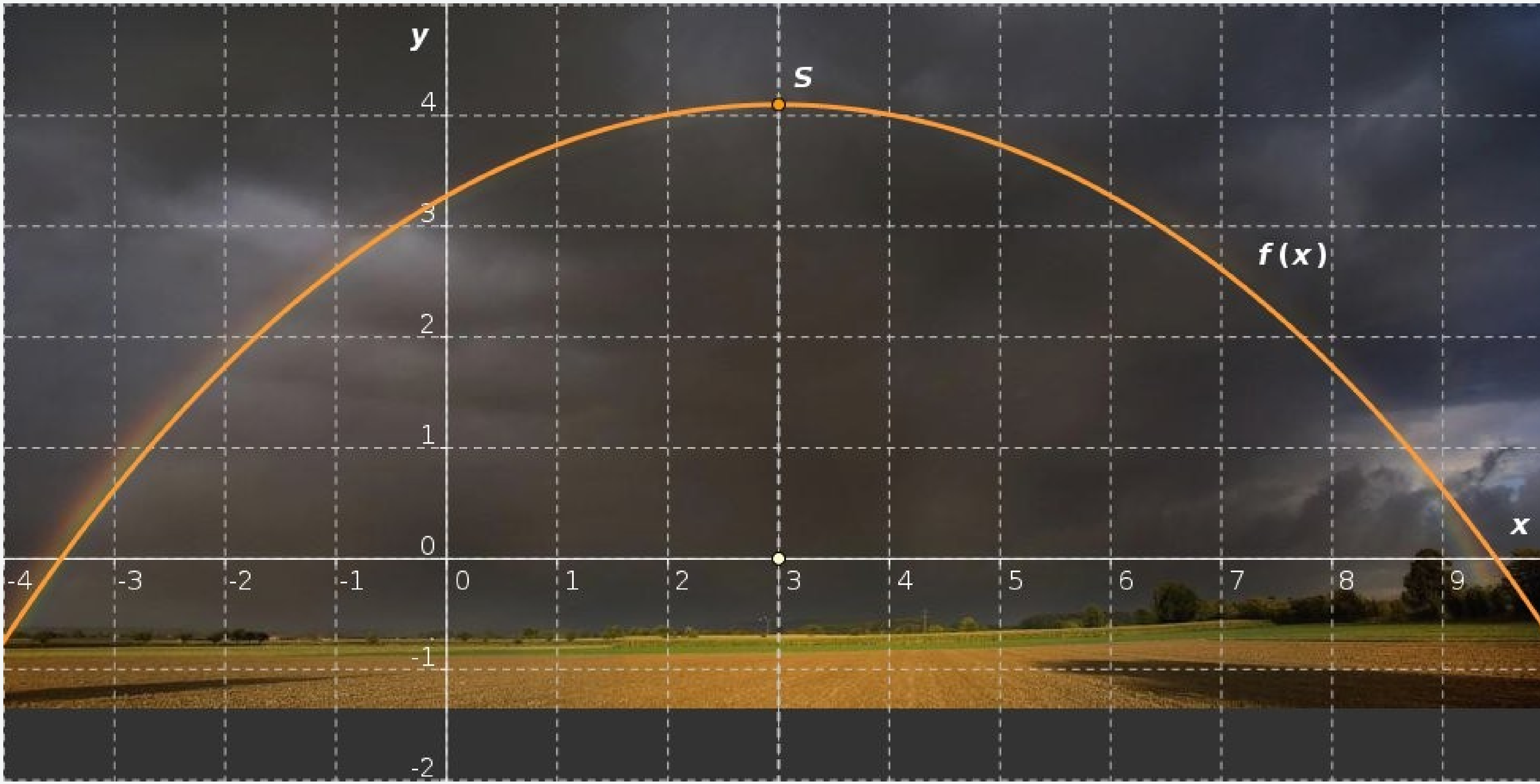

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Fig. 6-4: A representation of the function $y = f(x)$, which opens down

$$f(x) = -0.09\,(x - 3)^2 + 4.1$$

We show that a quadratic function

$$y = a x^2 + b x + c$$

can be transformed to $y = a(x - m)^2 + n$

$$a x^2 + b x + c = a \left(x^2 + \frac{b}{a} x + \frac{c}{a} \right) =$$

$$= a \left[x^2 + 2 x \frac{b}{2a} + \left(\frac{b}{2a} \right)^2 - \left(\frac{b}{2a} \right)^2 + \frac{c}{a} \right] =$$

$$= a \left[\left(x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a^2} \right] =$$

$$= a \left(x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a}$$

$$= a (x - m)^2 + n$$

$$m = -\frac{b}{2a}, \qquad n = \frac{4ac - b^2}{4a}$$

We have shown, that a quadratic function

$$y = a x^2 + b x + c$$

can be transformed to

$$y = a (x - m)^2 + n$$

From this equation we can read off

- the vertex $S (m, n)$,
- the dilation by scale factor a compared to the simplest parabola $y = x^2$
- the direction of opening (sign of a).

Exercise 5:

Transform the following quadratic functions

a) $y = x^2 + 4x - 1$

b) $y = x^2 - 4x + 7$

c) $y = -2x^2 - 4x + 2$

to $y = a(x - m)^2 + n$

Exercise 6:

What are the lengths of the sides of the rectangle with largest area among all rectangles which have a perimeter of 24 ?

Exercise 7:

Determine the minimal value of the function

$$y = \sqrt{x^2 + x + 1}$$

$$y = a x^2 + b x + c = a (x - m)^2 + n$$

$$m = -\frac{b}{2a}, \qquad n = \frac{4ac - b^2}{4a}$$

a) $y = x^2 + 4x - 1, \quad a = 1, \quad b = 4, \quad c = -1 \quad \Rightarrow$

$$m = -\frac{b}{2a} = -2, \qquad n = \frac{4ac - b^2}{4a} = \frac{-4 - 16}{4} = -5$$

$$y = x^2 + 4x - 1 = (x + 2)^2 - 5$$

b) $y = x^2 - 4x + 7, \quad a = 1, \quad b = -4, \quad c = 7 \quad \Rightarrow$

$$m = -\frac{b}{2a} = 2, \qquad n = \frac{4ac - b^2}{4a} = 3$$

$$y = x^2 - 4x + 7 = (x - 2)^2 + 3$$

c) $y = -2x^2 - 4x + 2 = -2(x + 1)^2 + 4$

Another way to transform the quadratic function $y = a\,x^2 + b\,x + c$ to the equation $y = a\,(x - m)^2 + n$ is the method of completing the square.

By means of the binomial theorem

$$(x + r)^2 = x^2 + 2\,r\,x + r^2$$

it is possible to express the equation by a quadratic binomial and an absolute term only.

$$a)\;\; x^2 + 4\,x = x^2 + 2\cdot 2\cdot x \overbrace{=}^{(r=2)} x^2 + 2\cdot 2\cdot x + 2^2 - 2^2 = (x + 2)^2 - 4$$

$$y = x^2 + 4\,x - 1 = (x + 2)^2 - 4 - 1 = (x + 2)^2 - 5$$

$$b)\;\; x^2 - 4\,x = x^2 - 2\cdot 2\cdot x \overbrace{=}^{(r=-2)} x^2 - 2\cdot 2\cdot x + 2^2 - 2^2 = (x - 2)^2 - 4$$

$$y = x^2 - 4\,x + 7 = (x - 2)^2 - 4 + 7 = (x - 2)^2 + 3$$

$$c)\;\; y = -2\,x^2 - 4\,x + 2 = -2\,(x^2 + 2\,x) + 2 = -2\,(x^2 + 2\,x + 1 - 1) + 2 =$$

$$= -2\,((x + 1)^2 - 1) + 2 = -2\,(x + 1)^2 + 4$$

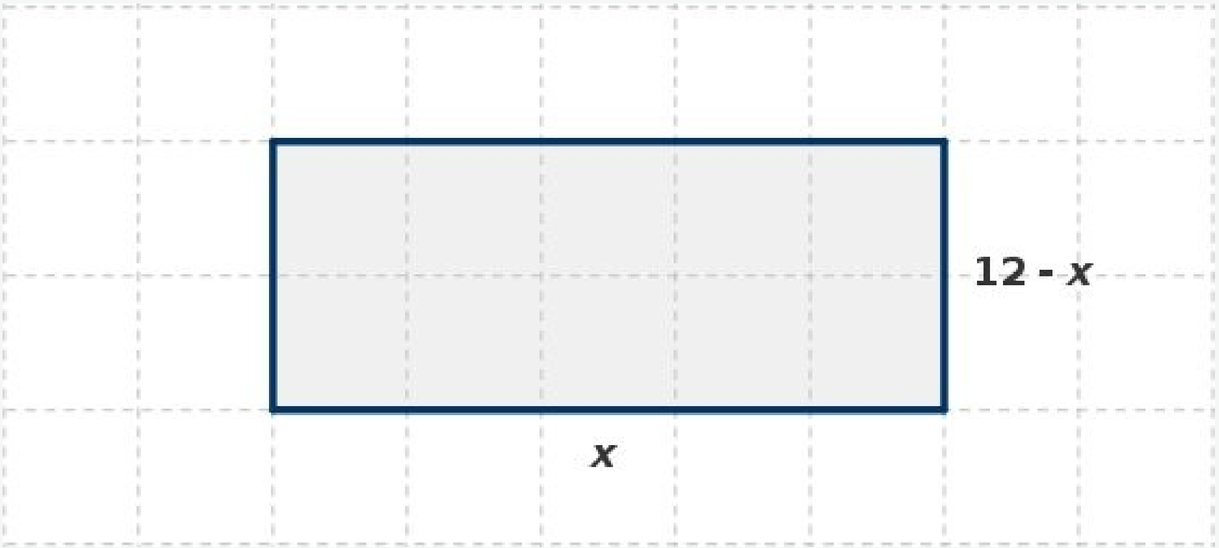

Fig. L6-1: Illustration to exercise 6

$$U = 2x + 2y \quad \Rightarrow \quad y = 12 - x$$

Area of the rectangle:

$$S = x(12 - x) = 12x - x^2 = 36 - (x - 6)^2$$

The area reaches a maximal value of 36 at $x = 6$.
The rectangle is a square in this case.

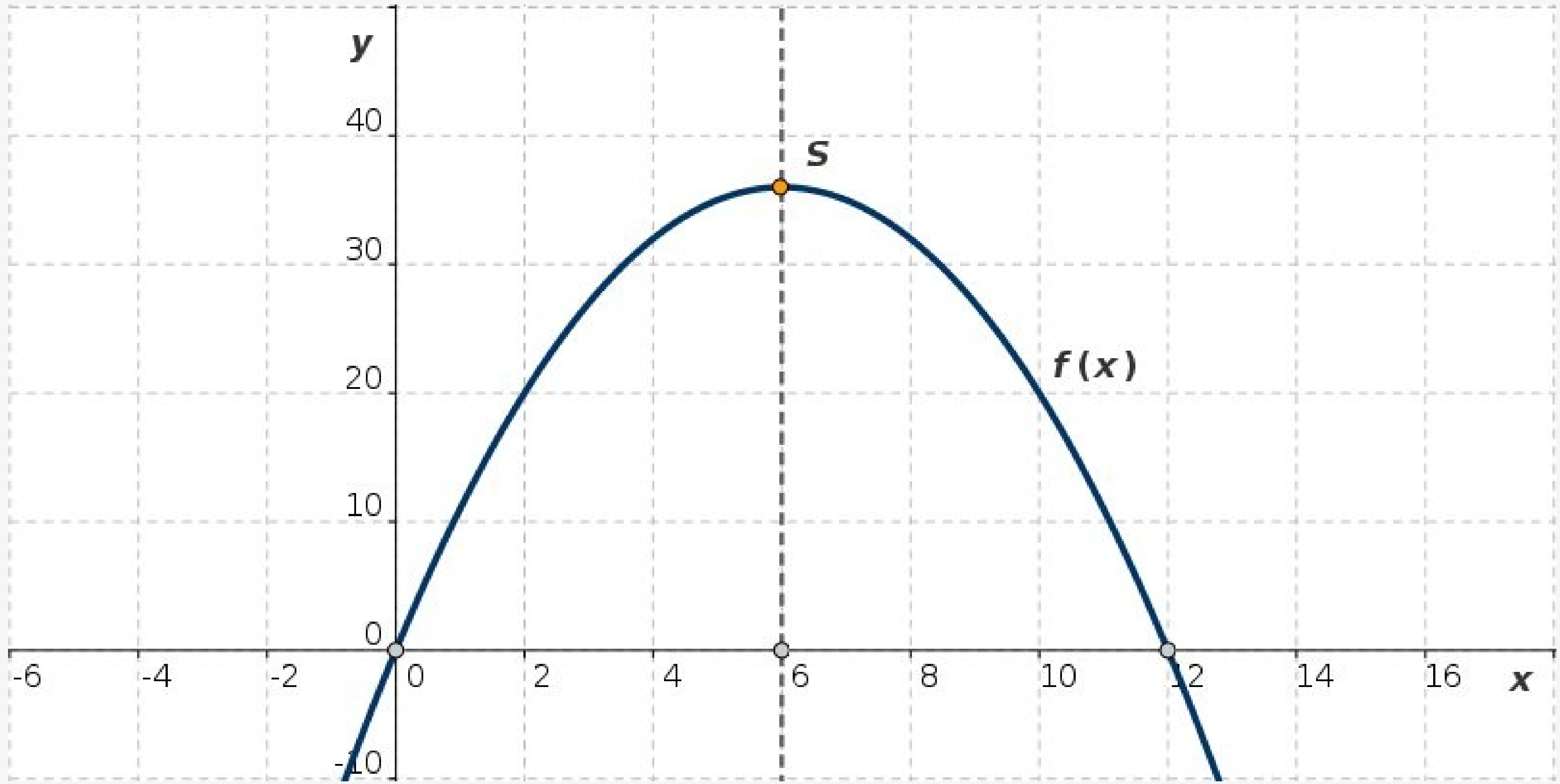

Fig. L6-2: Function $f(x) = 12x - x^2$

To find the minimal value of the function

$$y = \sqrt{x^2 + x + 1} \equiv \sqrt{f(x)} \ ,$$

we first determine the minimal value of the function

$$f(x) = x^2 + x + 1$$

This function is a parabola which opens up. The lowest point is given by the vertex *S (m, n)*

$$f(x) = x^2 + x + 1 = x^2 + 2 \cdot \frac{1}{2} \cdot x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1 =$$

$$= \left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$S = \left(-\frac{1}{2}, \ \frac{3}{4}\right)$$

$$y_{min} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} \simeq 0.866$$

Quadratic functions: Solution 7

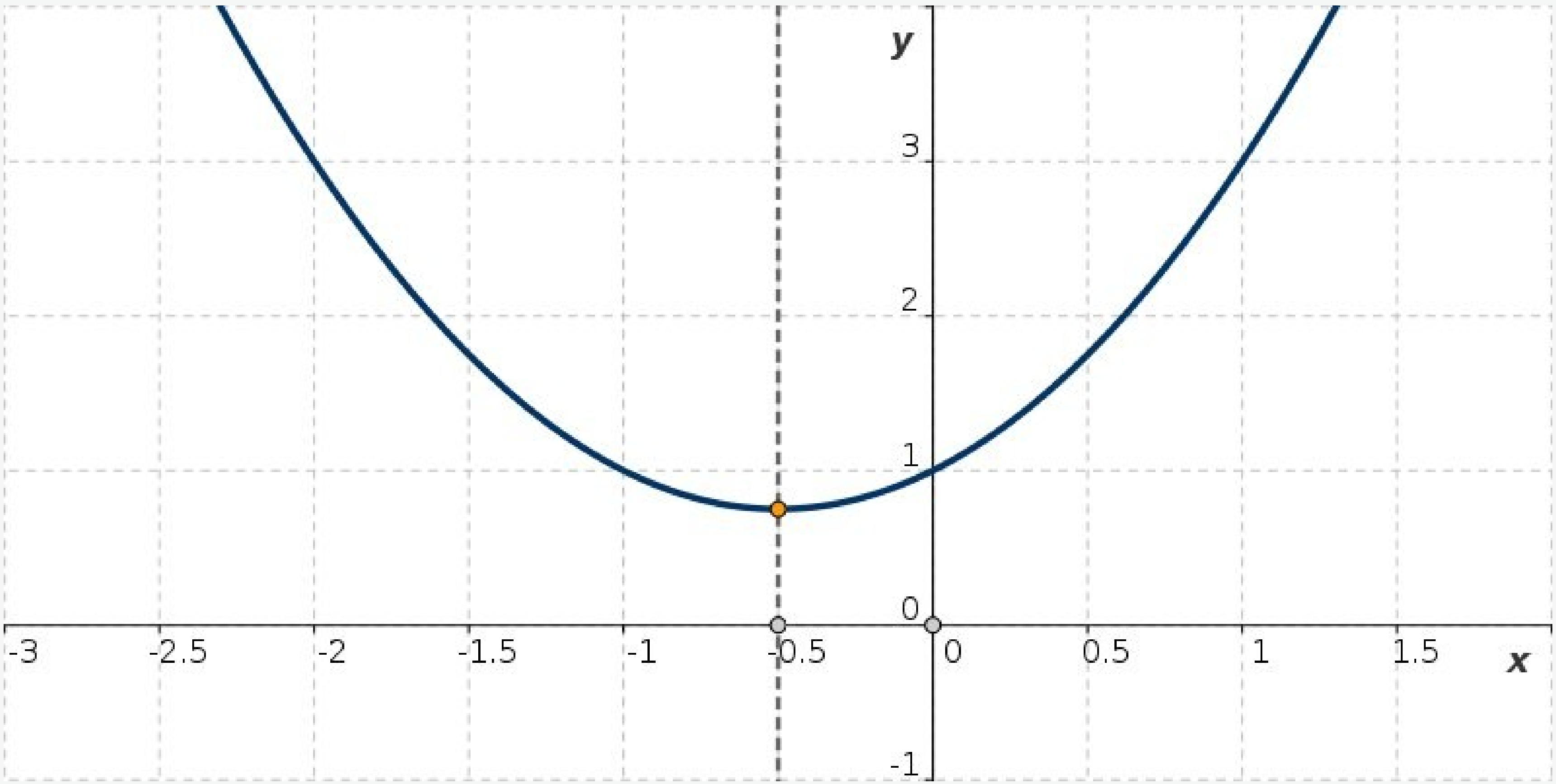

Fig. L7: Function $f(x) = x^2 + x + 1$

Roots of the function $y = x^2 + px + q$

The graph of the function $f(x) = x^2 + px + q$ may have two intersections with the x-axis (roots or zeros) or none or just one, depending on the position of the parabola.

Due to

$$x^2 + px + q = \left(x + \frac{p}{2}\right)^2 + \left(-\left(\frac{p}{2}\right)^2 + q\right),$$

the vertex of the parabola $f(x) = x^2 + px + q$ has the coordinates

$$S\left(-\frac{p}{2},\ -\left(\frac{p}{2}\right)^2 + q\right) = S\left(-\frac{p}{2},\ -D\right)$$

The term $D = \left(\frac{p}{2}\right)^2 - q$

is the discriminant of the quadratic function

$$f(x) = x^2 + px + q$$

Roots of the function $y = x^2 + px + q$

The roots of the equation $f(x) = x^2 + px + q$ are obtained by solving the quadratic equation

$$x^2 + px + q = 0$$

p-q-formula:

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q} = -\frac{p}{2} \pm \sqrt{D}$$

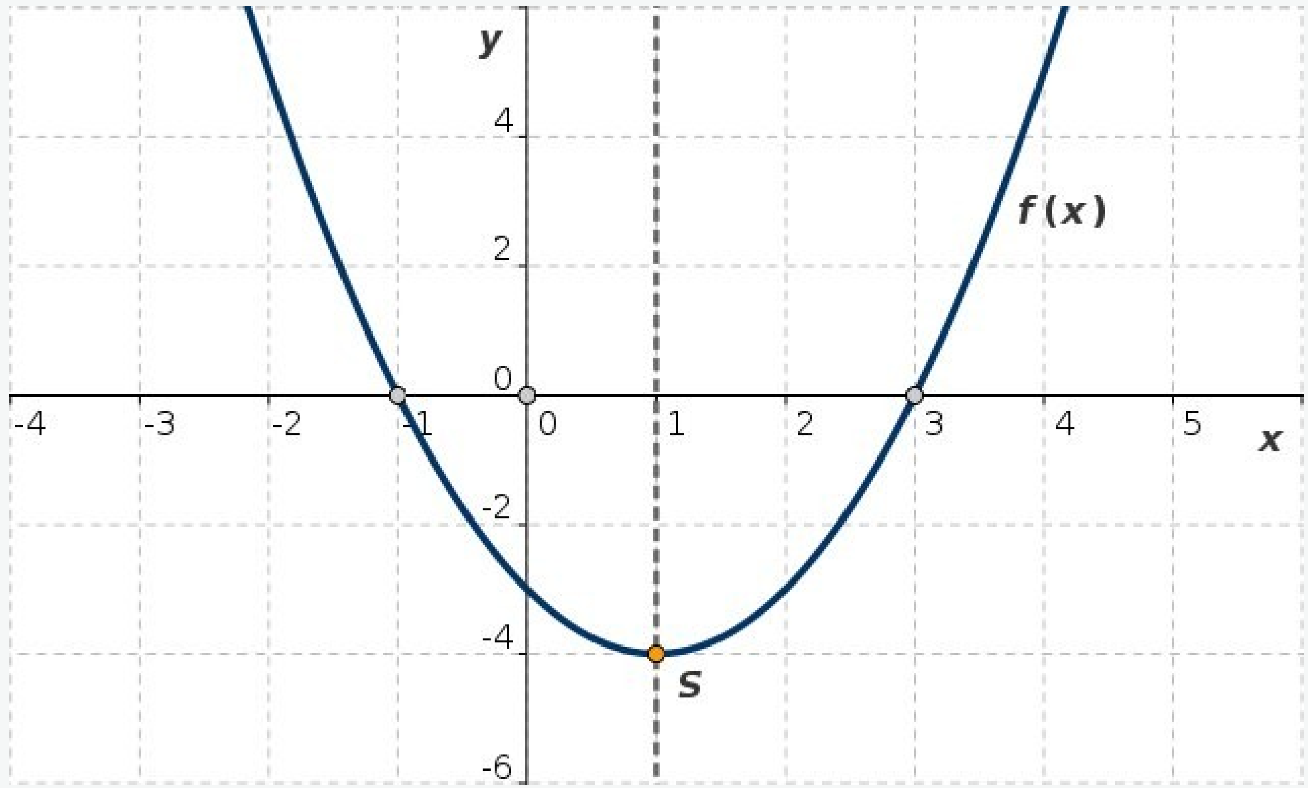

Fig. 7-1: Function $f(x) = x^2 - 2x - 3$

case 1: $D = 4\ (> 0)$, two zeros

function: $f(x) = x^2 - 2x - 3, \quad S(1, -4)$

zeros: $x_1 = -1, \quad x_2 = 3$

Roots of the function $y = x^2 + px + q$

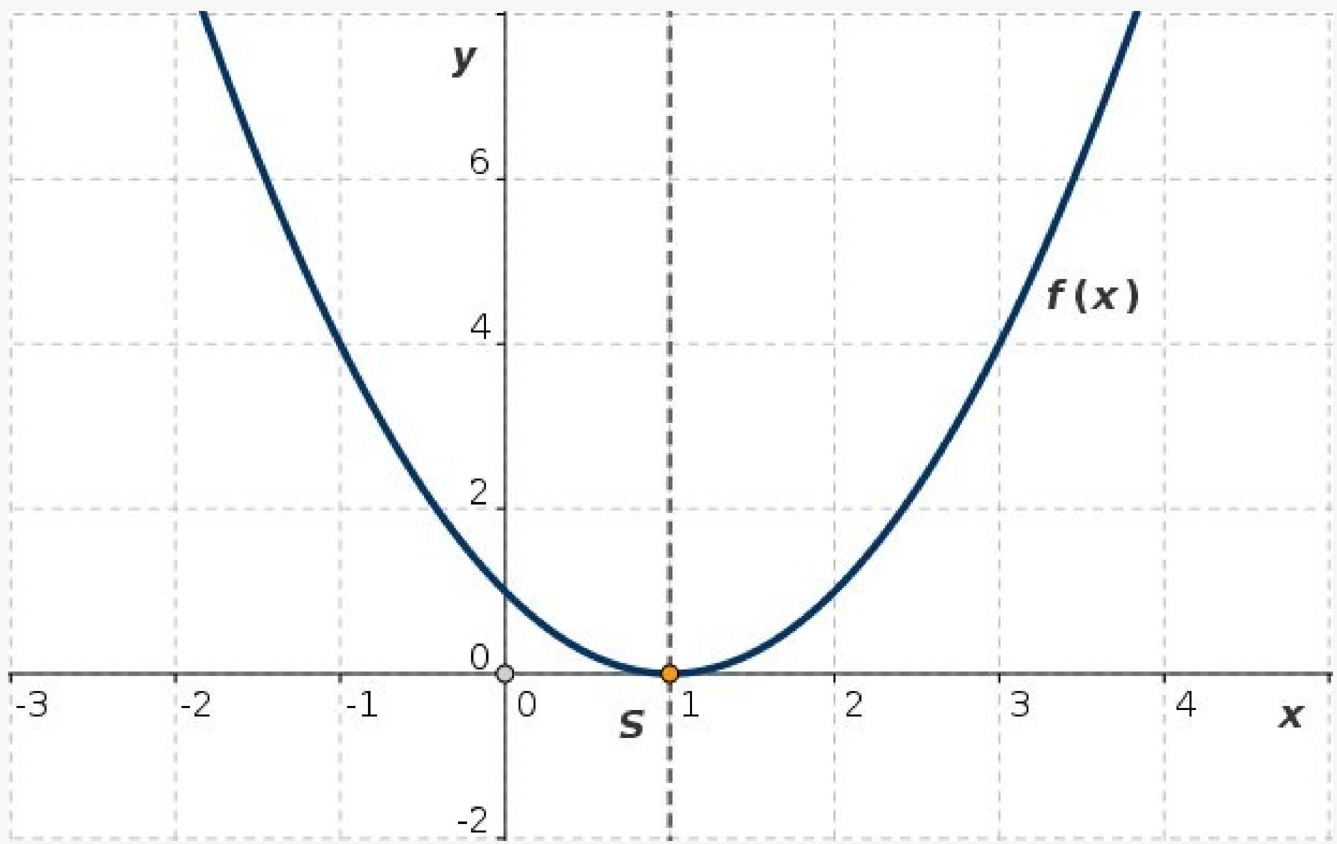

Fig. 7-2: Function $f(x) = x^2 - 2x + 1$

case 2: $D = 0$, one (double) zero

function: $f(x) = x^2 - 2x + 1, \quad S(1, 0)$

zeros: $x_{1,2} = 1$

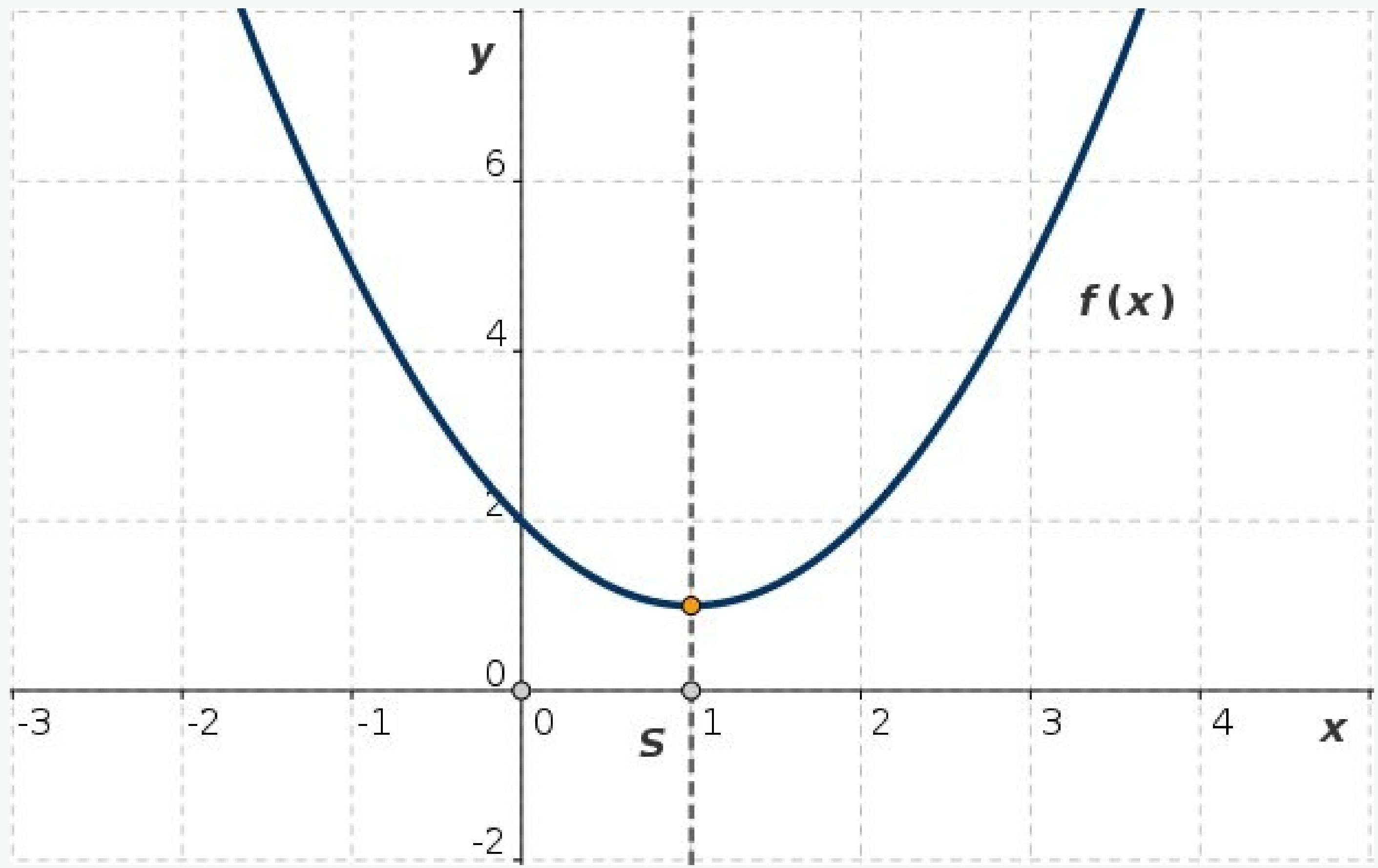

Fig. 7-3: Function $f(x) = x^2 - 2x + 2$

case 3: $D = -1$ (< 0), no zero

function: $f(x) = x^2 - 2x + 2, \quad S(1, 1)$

zeros: none