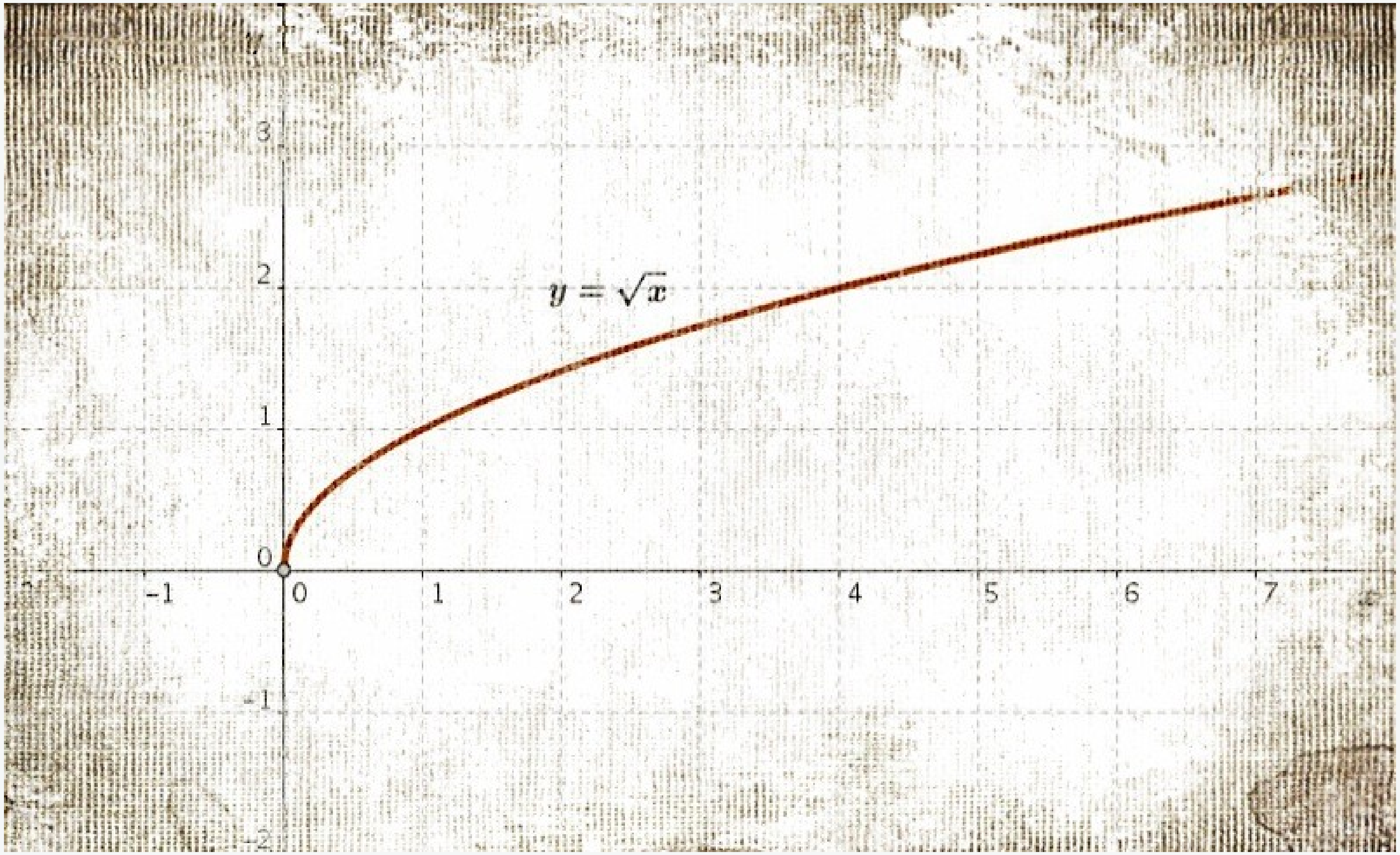

Radical Equations

Definition:

An equation is called a radical equation, if the unknown is contained in the argument of the root function.

Question:

Which of the following equations are radical equations?

$a)\ \sqrt{2x-3}+5-3x=0$

$b)\ \sqrt{21}\,x+x^2-\sqrt{3}\,(x-2)=0$

$c)\ \sqrt{6x-2}=3+x$

$d)\ \sqrt{4+x-\sqrt{2+x}}=7$

$e)\ \sqrt[4]{4+x-\sqrt{2+x^2}}=\sqrt{3}+x$

Radical equations: *Answer*

$$a)\ \sqrt{2x-3}+5-3x=0$$

$$b)\ \sqrt{21}\,x+x^2-\sqrt{3}\,(x-2)=0$$

$$c)\ \sqrt{6x-2}=3+x$$

$$d)\ \sqrt{4+x-\sqrt{2+x}}=7$$

$$e)\ \sqrt[4]{4+x-\sqrt{2+x^2}}=\sqrt{3}+x$$

Equations *a), c), d)* and *e)* are radical equations, the unknown x appears in the argument of a root function. However, equation *b)* is not a radical equation. There are roots, but none has the unknown x in the argument. This equation is a quadratic equation.

Radical equations are solved for the unknown by removing the roots by exponentiation of both sides of the equation. Sometimes it is necessary to raise the equation to powers several times. However, it may happen that the exponentiation results in an equation whose domain is larger than that of the original equation. That is, apparent solutions may show up which do not solve the original equation. Therefore it is very important to check whether the found "solution" actually solves the original equation.

!

- Squaring an equation does not lead to an equivalent equation !
- Check all solutions !

We determine the solutions of the following two radical equations:

$$a)\quad E_1 \quad : \sqrt{x+1} - 2 = 0$$

$$b)\quad E_2 \quad : \sqrt{x+1} + 2 = 0$$

$$E_1 : \sqrt{x+1} - 2 = 0$$

$$x + 1 \geqslant 0 \quad \Rightarrow \quad D(E_1) = [-1, \infty)$$

Solution by squaring the equation:

1. $(\sqrt{x+1} - 2)^2 = 0^2$ One may use the binomial formula $(a+b)^2 = a^2 + 2ab + b^2$

$$(\sqrt{x+1} - 2)^2 = 0^2 \quad \rightarrow \quad x + 1 - 4\sqrt{x+1} + 4 = 0$$

It is obvious that we gained just nothing. The unknown x is still inside the root.

!

Before squaring, the root should be isolated at one side of the equation.

2. $\sqrt{x+1} - 2 = 0 \quad \Leftrightarrow \quad \sqrt{x+1} = 2$

squared:

$\tilde{E}_1 : \; x + 1 = 2^2 \quad \rightarrow \quad x + 1 = 4 \quad \Leftrightarrow \quad x = 3$

$D(\tilde{E}_1) = \mathbb{R}, \qquad D(E_1) \neq D(\tilde{E}_1)$

$S_{\tilde{E}_1} = \{ 3 \}$

Now we have to check, whether the solution of the transformed equation also solves the original equation.

The check:

$x = 3 \quad : \sqrt{3+1} - 2 = 2 - 2 = 0$

Indeed, $x = 3$ is a solution of the original equation.

$$E_2 \quad : \sqrt{x+1} + 2 = 0 \quad \Leftrightarrow \quad \sqrt{x+1} = -2$$

$$\tilde{E}_2 : \quad x + 1 = (-2)^2 \quad \Rightarrow \quad x + 1 = 4, \quad x = 3$$

$$D(E_2) = [-1, \infty), \quad D(\tilde{E}_2) = \mathbb{R}, \quad D(E_2) \neq D(\tilde{E}_2)$$

$$S_{\tilde{E}_2} = \{ 3 \}$$

The check

$$x = 3 \quad : \sqrt{3+1} + 2 = 2 + 2 = 0$$

shows, that $x = 3$ is not a solution of the original equation

Squaring led to a “no solution”. Equation E_2 has no solution.

$$S_{E_2} = \{ \emptyset \}$$

This result could have been seen immediately without solving the equation. The sum of two positive numbers can never be zero.

Solution 1: Graphical demonstration

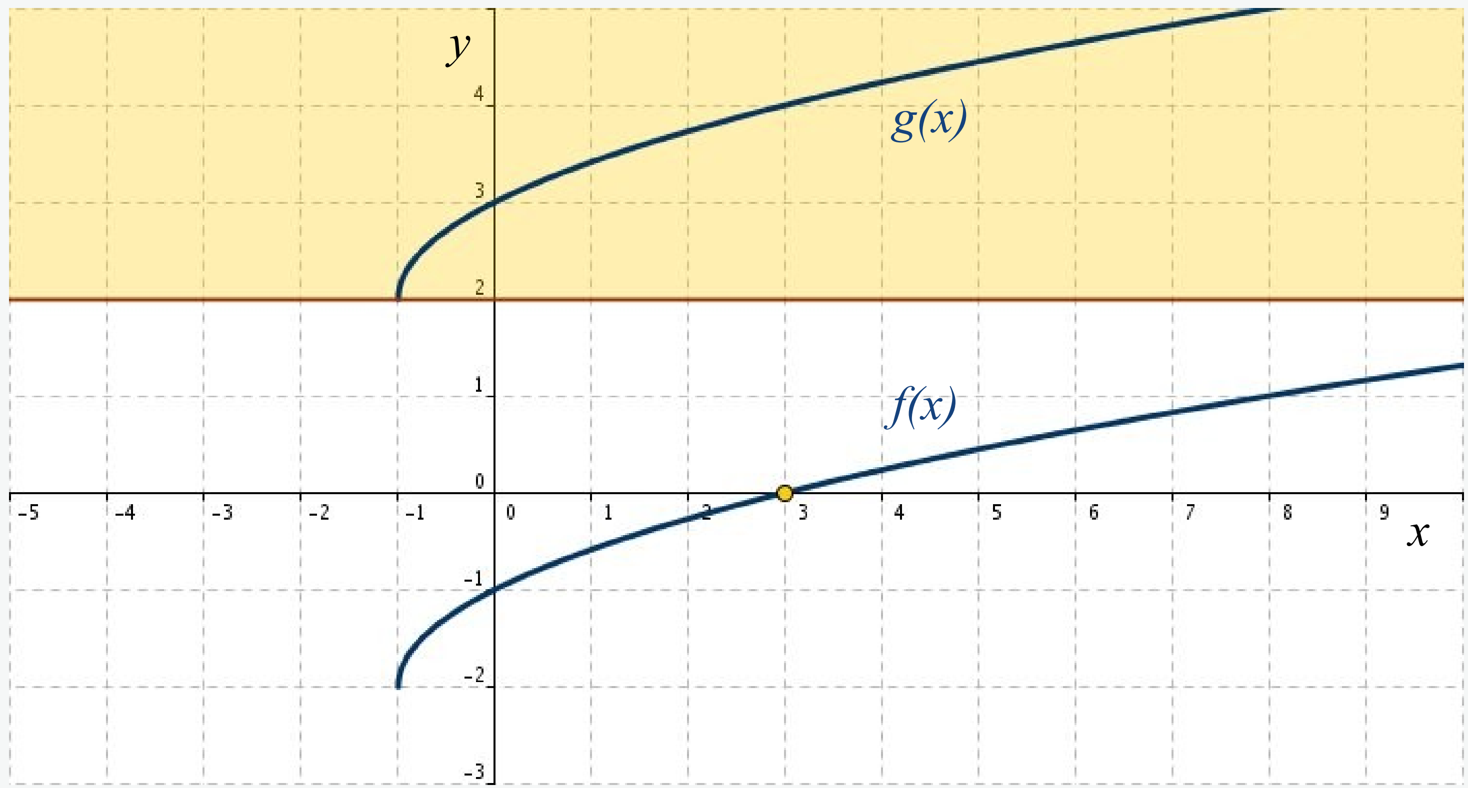

Fig. 1: Root functions f (x) and g(x)

$$f(x) = \sqrt{x+1} - 2, \qquad g(x) = \sqrt{x+1} + 2$$

The equations $f(x) = 0$ and $g(x) = 0$ determine the x coordinates of the intersections with the x-axis. The range of the function $g(x)$, $R(g) \geq 2$, is positive only. There is no intersection of $g(x)$ with the x-axis.

Solve the following two equations:

Exercise 3: $1 + \sqrt{x + 5} = x$

Exercise 4: $\sqrt{x - 1} + 7 = x$

1. $E: \quad 1 + \sqrt{x+5} = x$

2. Domain of the equation: $\quad x + 5 \geqslant 0 \quad \Rightarrow \quad D(E) = [-5, \infty)$

3. Isolation of the root: $\quad \sqrt{x+5} = x - 1$

4. Squaring the equation: $\quad (\sqrt{x+5})^2 = (x-1)^2$

   $\tilde{E}: \quad (\sqrt{x+5})^2 = (x-1)^2 \quad \Rightarrow \quad x + 5 = x^2 - 2x + 1$

   Be careful ! E and $\tilde{E}$ are not equivalent!

   $\Rightarrow \quad x^2 - 3x - 4 = 0, \qquad D(\tilde{E}) = \mathbb{R}$

   $x_{1,2} = \frac{3}{2} \pm \sqrt{\frac{9}{4} + \frac{16}{4}}, \qquad x_1 = 4, \quad x_2 = -1$

5. The set of solutions of the transformed equation: $\quad S(\tilde{E}) = \{-1, 4\}$

6. Check: $x_1 = 4 \qquad 1 + \sqrt{4+5} = 4 \qquad 4 = 4$

   $x_2 = -1 \qquad 1 + \sqrt{-1+5} = -1 \qquad 3 \neq -1$

   $\Rightarrow \quad S(E) \neq S(\tilde{E})$

7. Solution of equation $E: \quad S(E) = \{4\}$

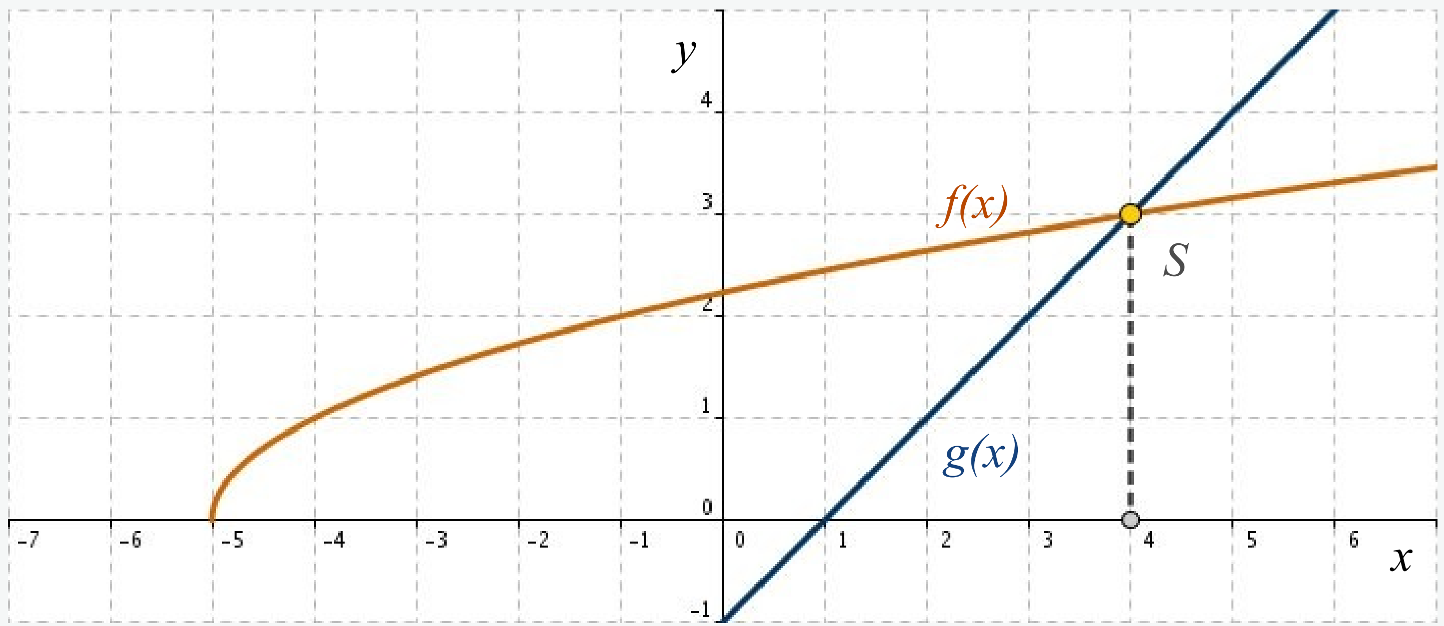

Fig. S-3a: Functions f (x) and g (x)

$$E: \quad 1 + \sqrt{x+5} = x \quad \Leftrightarrow \quad \sqrt{x+5} = x - 1$$

$$E: \quad f(x) = g(x), \quad f(x) = \sqrt{x+5}, \quad g(x) = x - 1$$

S (4, 3) is the intercept of the functions *f (x)* and *g (x)*.

The abscissa of the intercept S, i.e. $x = 4$, is the solution of equation E.

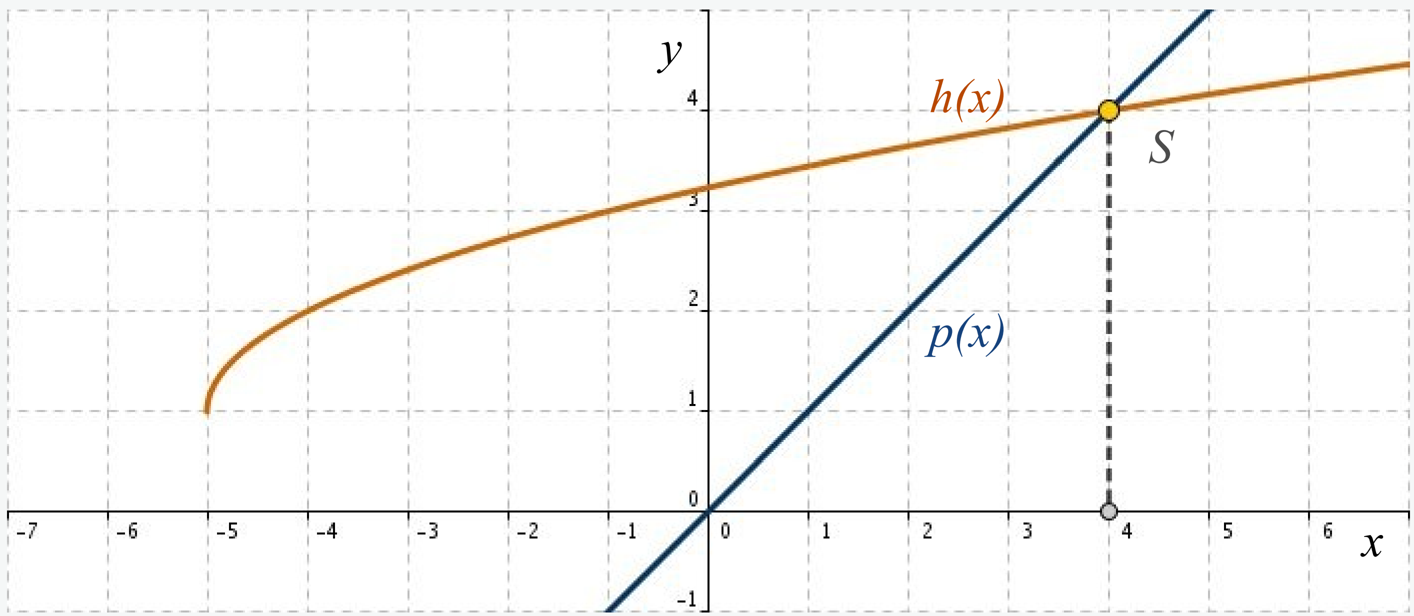

Fig. S-3b: Functions h (x) and p (x)

$$E: \quad 1 + \sqrt{x + 5} = x \qquad \Leftrightarrow \qquad h(x) = p(x)$$

$$h(x) = \sqrt{x + 5} + 1, \qquad p(x) = x$$

S (4, 4) is intercept of the functions *h (x)* and *p (x)*.

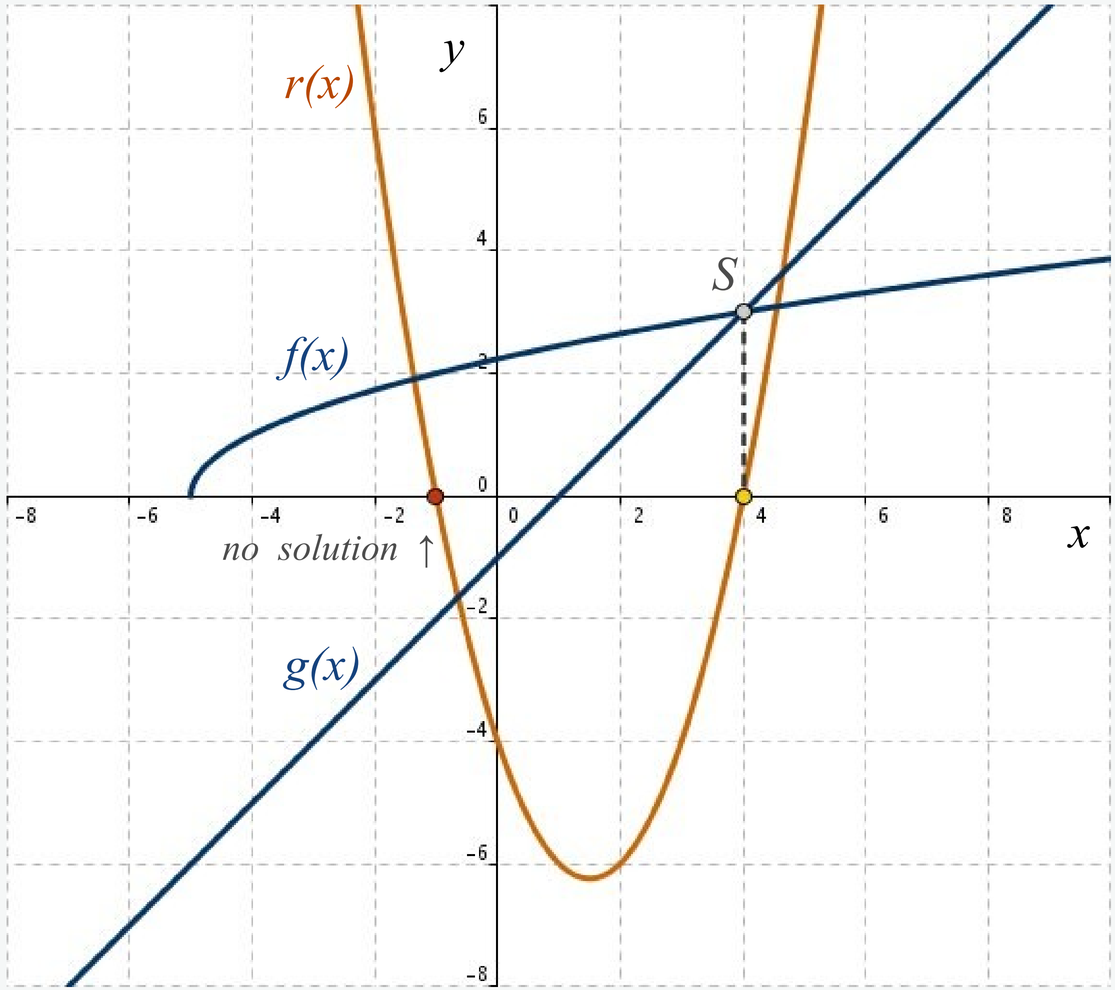

Fig. S-3c: Functions f (x), g (x) and r (x)

$$f(x) = \sqrt{x + 5}\,, \qquad g(x) = x - 1, \qquad r(x) = x^2 - 3x - 4$$

1. $E: \quad \sqrt{x-1} + 7 = x$

2. Domain of the equation: $\quad x - 1 \geqslant 0 \quad \Rightarrow \quad D(E) = [1, \infty)$

3. Isolation of the root: $\quad \sqrt{x-1} = x - 7$

4. Squaring the equation: $\quad (\sqrt{x-1})^2 = (x-7)^2$

   $\tilde{E}: \quad x - 1 = x^2 - 14x + 49 \quad \Leftrightarrow \quad x^2 - 15x + 50 = 0$

   E and $\tilde{E}$ are not equivalent!

   $D(\tilde{E}) = \mathbb{R}, \qquad D(E) \neq D(\tilde{E})$

   $$x_{1,2} = \frac{15}{2} \pm \sqrt{\left(\frac{15}{2}\right)^2 - 50}, \qquad x_1 = 5, \quad x_2 = 10$$

5. The set of solutions of the transformed equation: $\quad S(\tilde{E}) = \{5, 10\}$

6. Check: $x_1 = 5 \qquad \sqrt{5-1} + 7 = 5 \qquad 9 \neq 5$

   $x_2 = 10 \qquad \sqrt{10-1} + 7 = 10 \qquad 10 = 10$

   $\Rightarrow \quad S(E) \neq S(\tilde{E})$

7. Solution of equation $E: \quad S(E) = \{10\}$

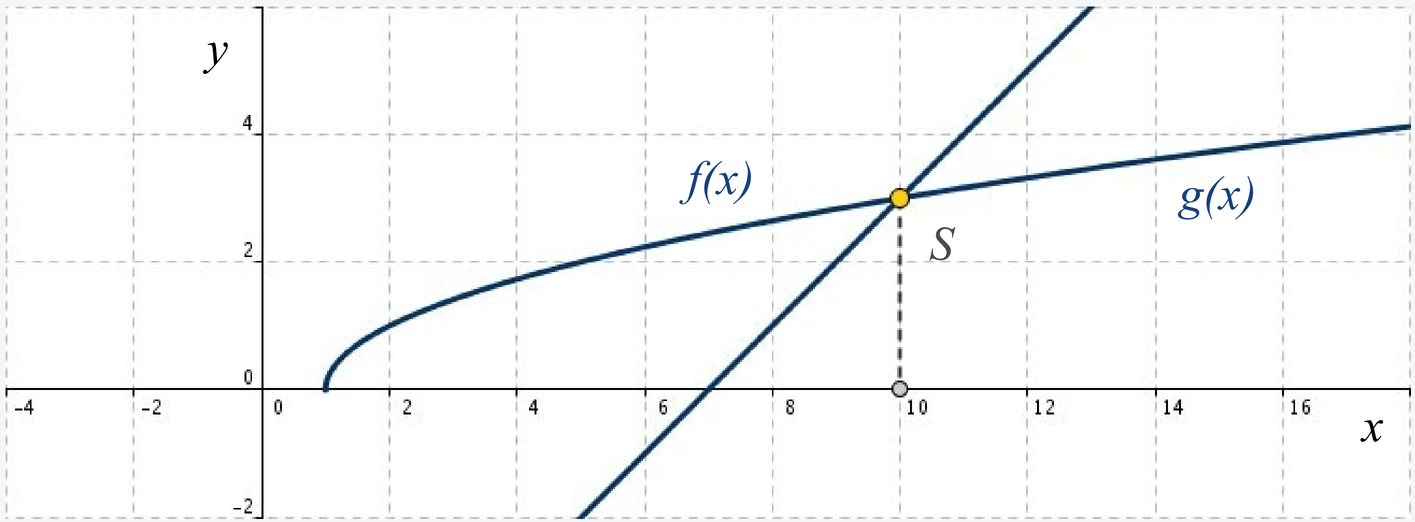

Fig. S-4a: Functions f (x) and g (x)

$$f(x) = \sqrt{x - 1}, \qquad g(x) = x - 7$$

S (10, 3) is the intercept of the functions *f (x)* and *g (x)*.

The abscissa of the intercept *S* , i.e. $x = 10$, is the solution of equation *E*.

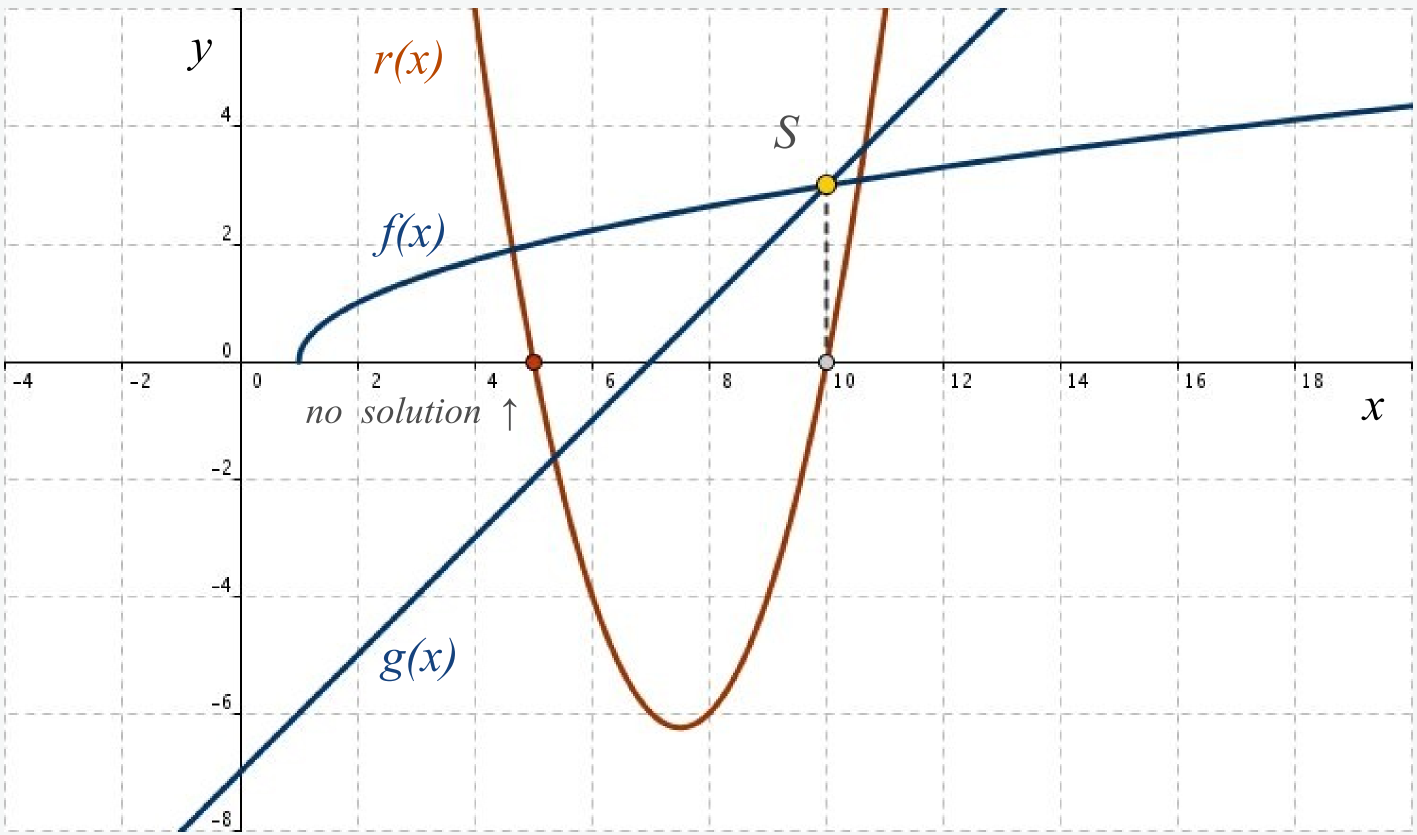

Fig. S-4b: Functions f (x), g (x) and r (x)

$$f(x) = \sqrt{x - 1}, \qquad g(x) = x - 7, \qquad r(x) = x^2 - 15x + 50$$