Trigonometric functions for arbitrary angles

We defined the trigonometric functions of acute angles θ

$$0^\circ < \theta < 90^\circ$$

of right triangles.

Now we define trigonometric functions for arbitrary angles in a unit circle (circle with radius 1) with the center at the point of origin O.

By any point $P\,(x, y)$ on the unit circle, an angle θ is given between the x-axis and the line O - P (Fig. 2-5). The angle θ is measured counter clock wise with respect to the x-axis. This is the mathematically positive sense.

Using the signed coordinates x and y of point P, the trigonometric functions are defined by

$$\sin\theta = y\,, \qquad \cos\theta = x\,, \qquad \tan\theta = \frac{y}{x}\,, \qquad \cot\theta = \frac{x}{y}$$

Trigonometric functions for arbitrary angles

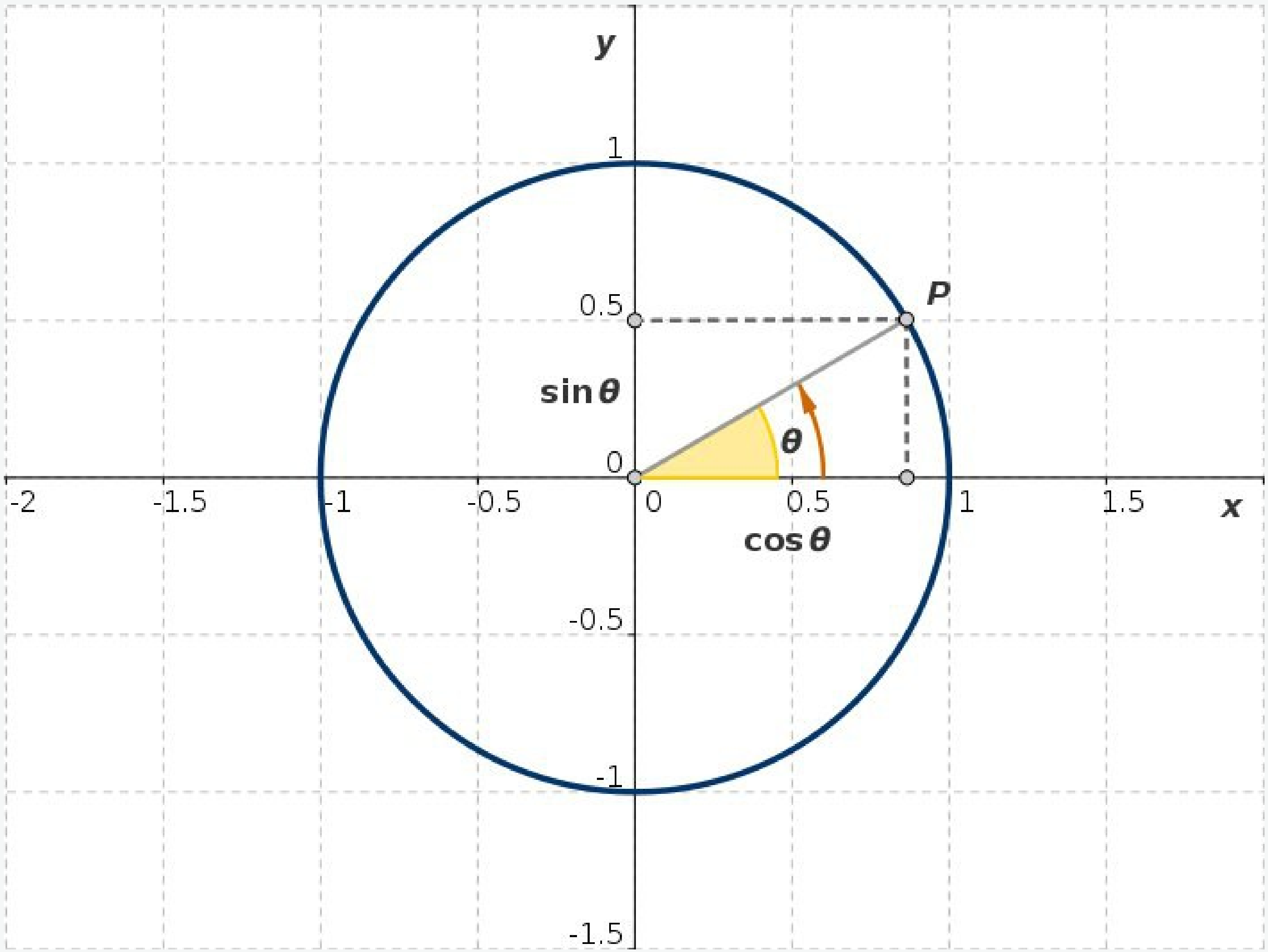


Fig. 2-5: Definition of trigonometric functions for arbitrary angles

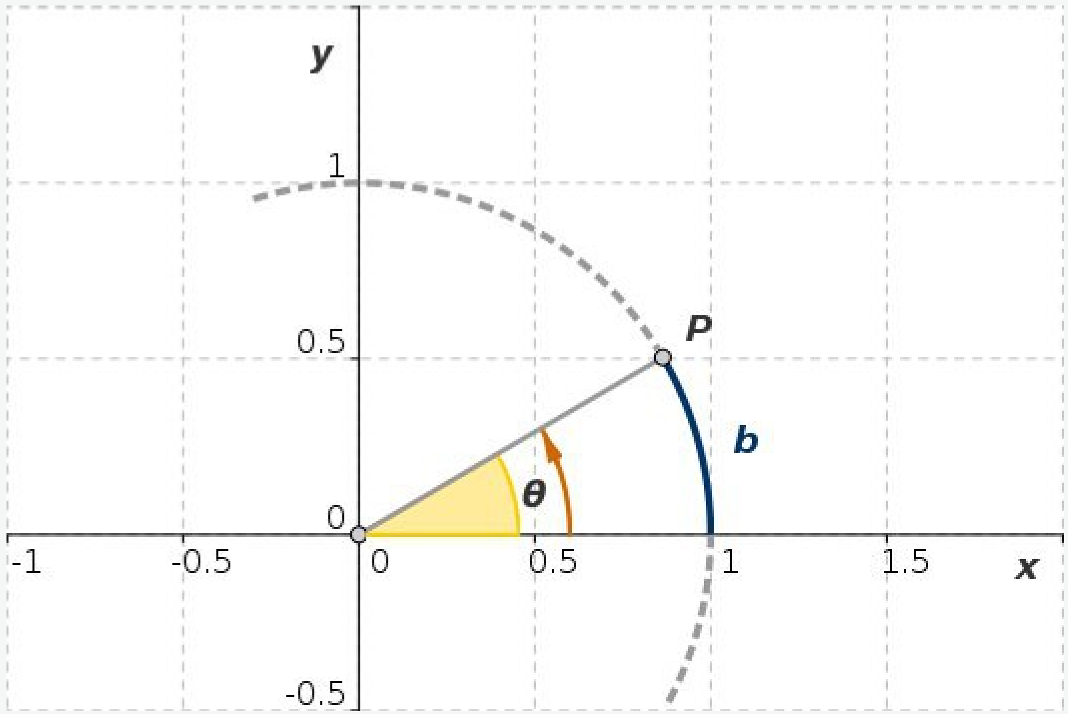


Abb. 2-6: Radian measure b of the angle θ

The length of the path on the unit circle from the x-axis to the point P is the radian measure b of the angle θ, or expressed in other words, b is the angle θ measured in radians.

θ and b are positive (negative), if P moves on the unit circle in mathematically positive (negative) direction of rotation.

By the above equations on the unit circle, the trigonometric functions are defined for any angle θ in degrees or any real number b in radians, if the respective denominators are unequal zero.

The oriented Cartesian coordinate axes divide the plane into four quadrants (see Fig. 2-7).

The points of the first quadrant have both, positive x- and y-coordinates, the points of the second quadrant have negative x- and positive y-coordinates, the points of the third quadrant have negative x- and negative y-coordinates, and the points of the forth quadrant have postive x- and negative y-coordinates.

Cartesian coordinate system: quadrants

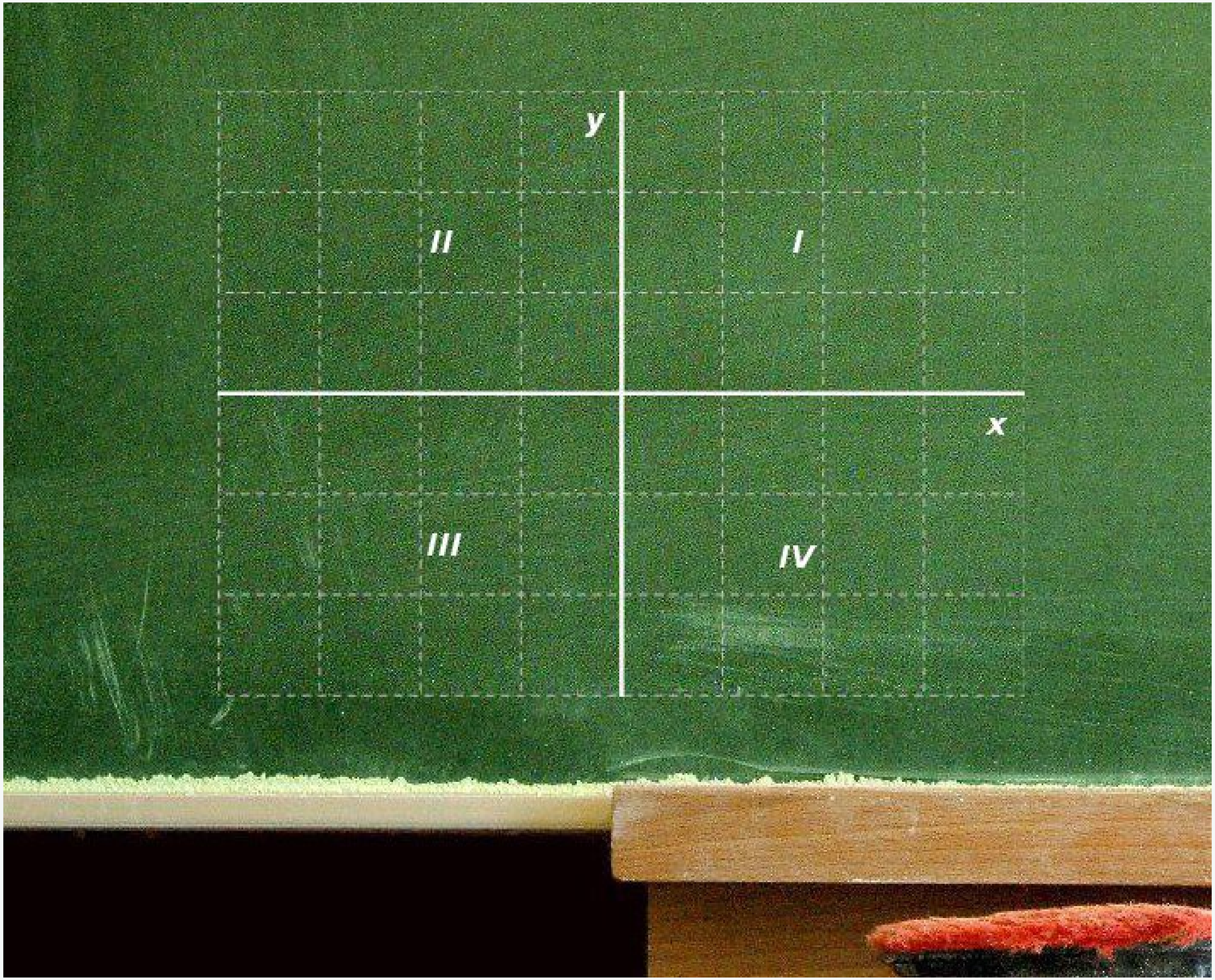
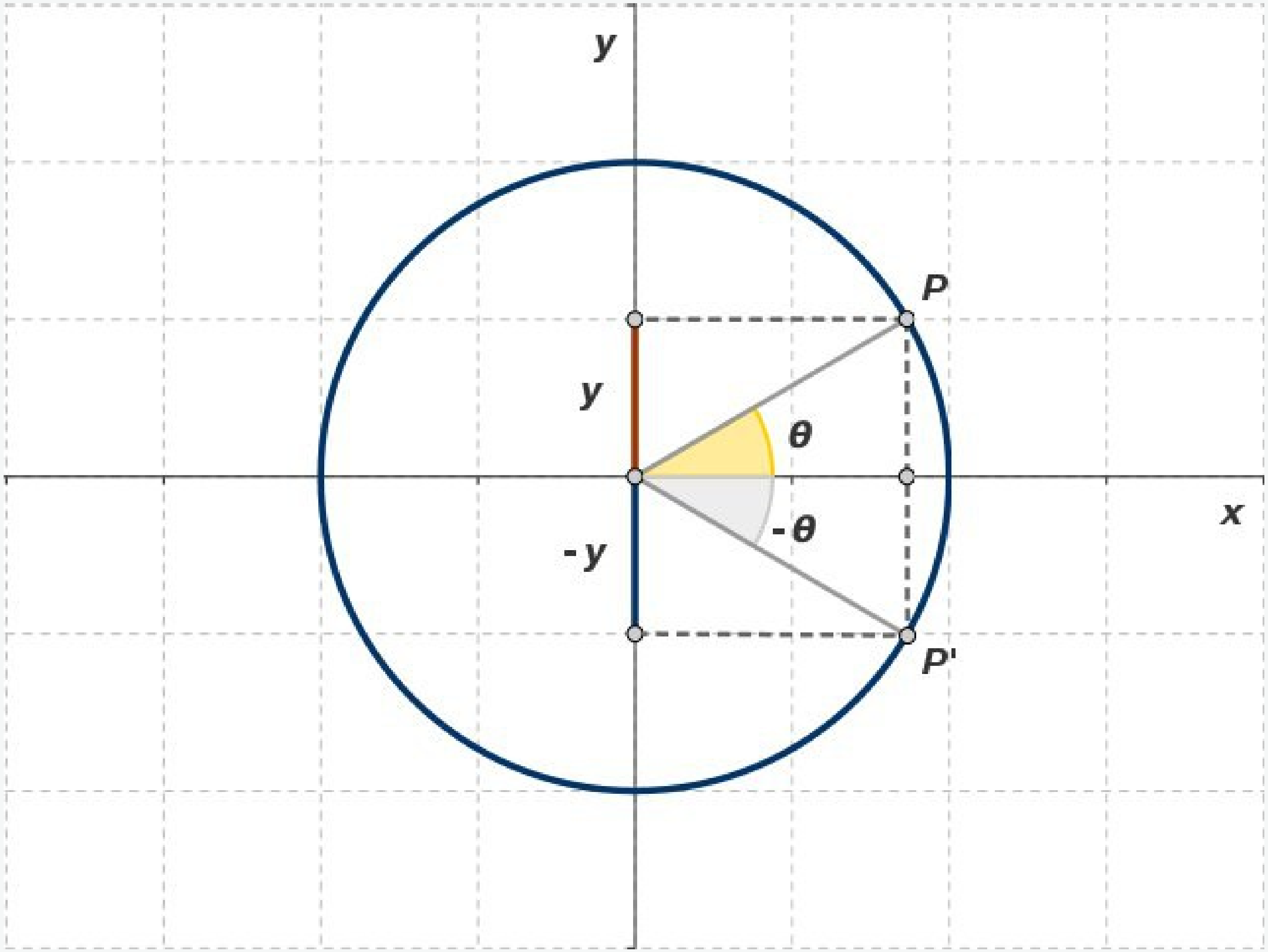


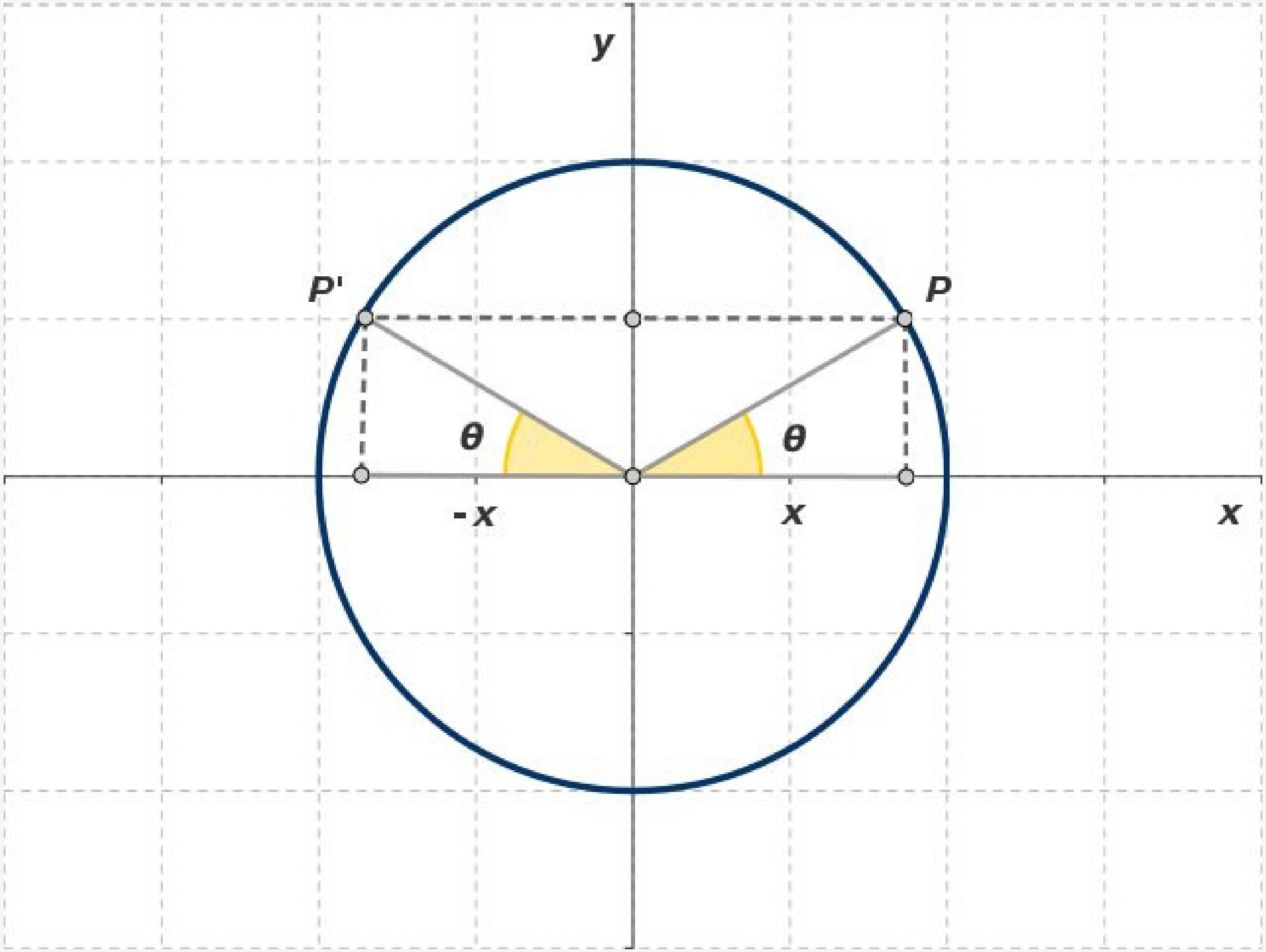
Fig. 2-7: Four quadrants

Quadrant	*sin*	*cos*	*tan*	*cot*
I	+	+	+	+
II	+	−	−	−
III	−	−	+	+
IV	−	+	−	−

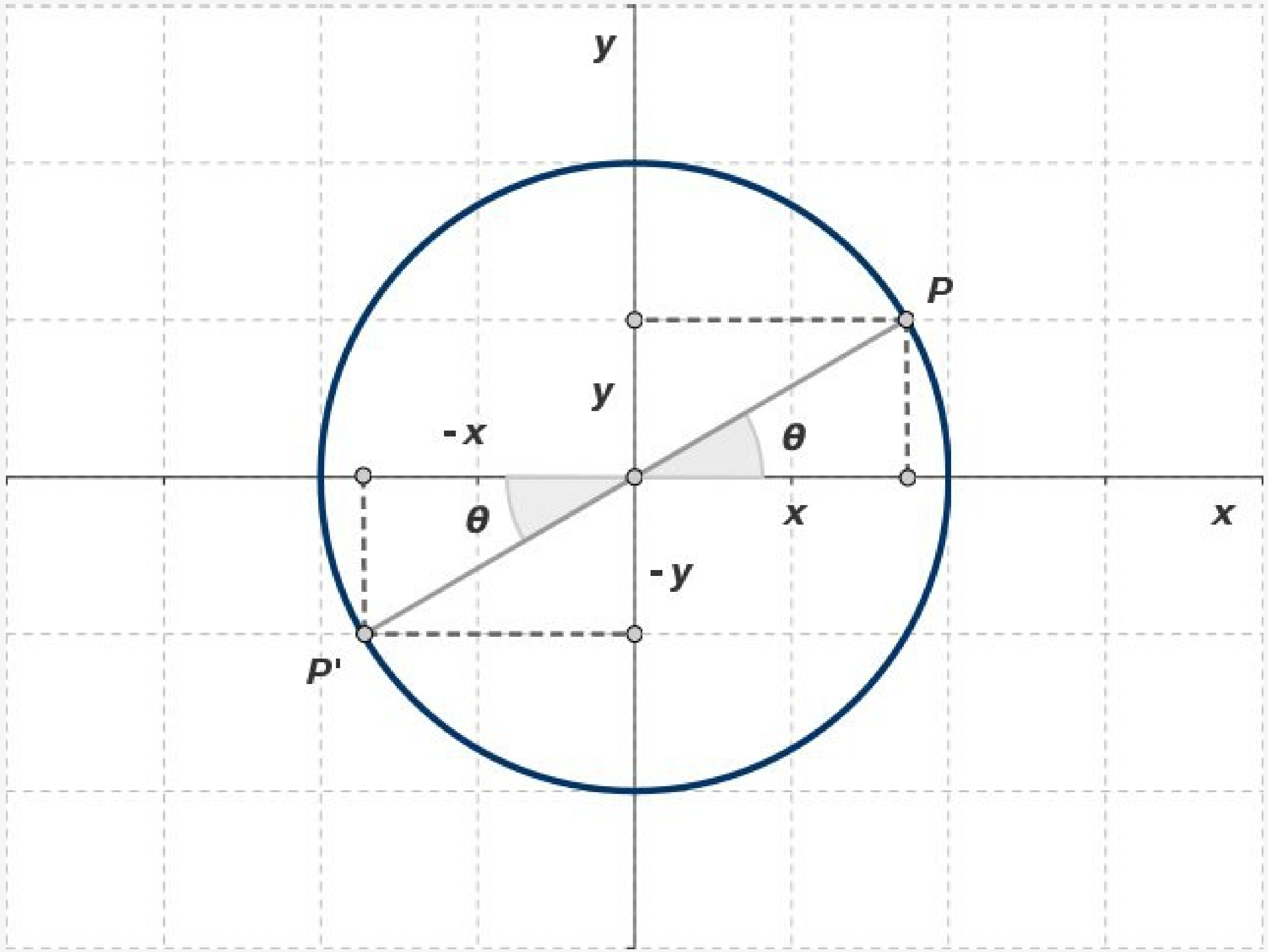
Fig. 2-8: Signs of trigonometric functions in different quadrants



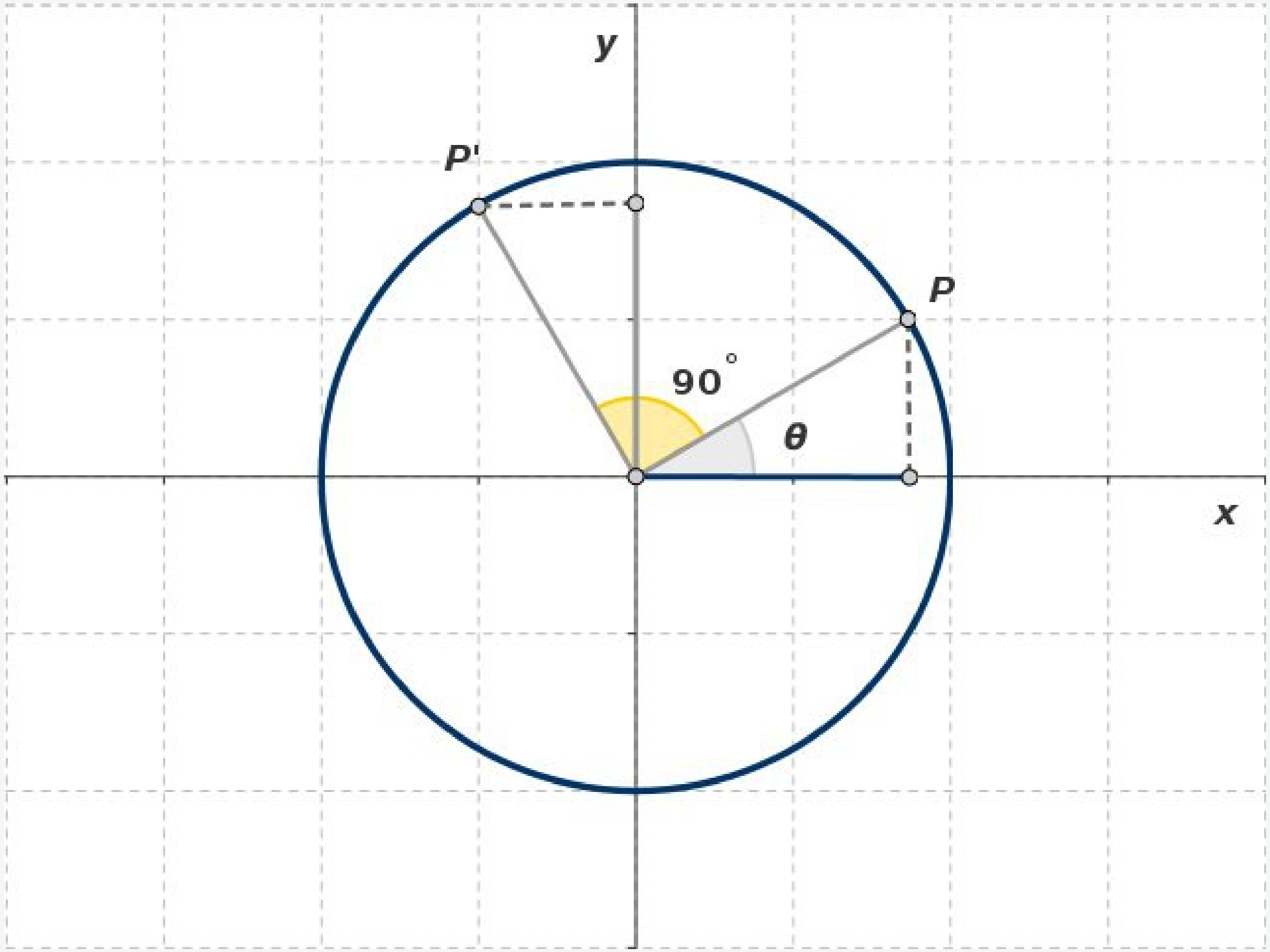
$$\sin(-\theta) = -\sin\theta\,, \qquad \cos(-\theta) = \cos\theta$$



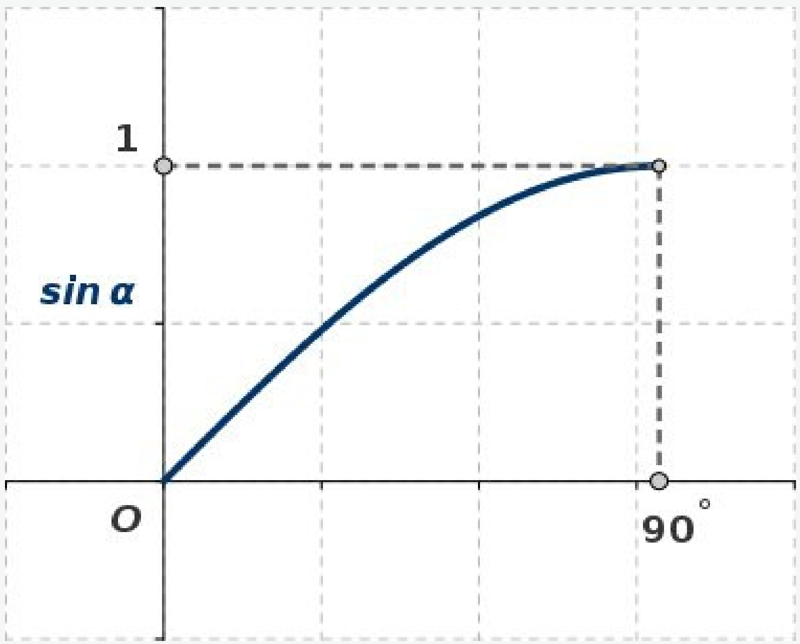
$$\sin(180^\circ - \theta) = \sin\theta\,, \qquad \cos(180^\circ - \theta) = -\cos\theta$$



$$\sin\left(180^{\circ} + \theta\right) = -\sin\theta\,, \qquad \cos\left(180^{\circ} + \theta\right) = -\cos\theta$$



$$\sin(90^\circ + \theta) = \cos\theta\,, \qquad \cos(90^\circ + \theta) = -\sin\theta$$



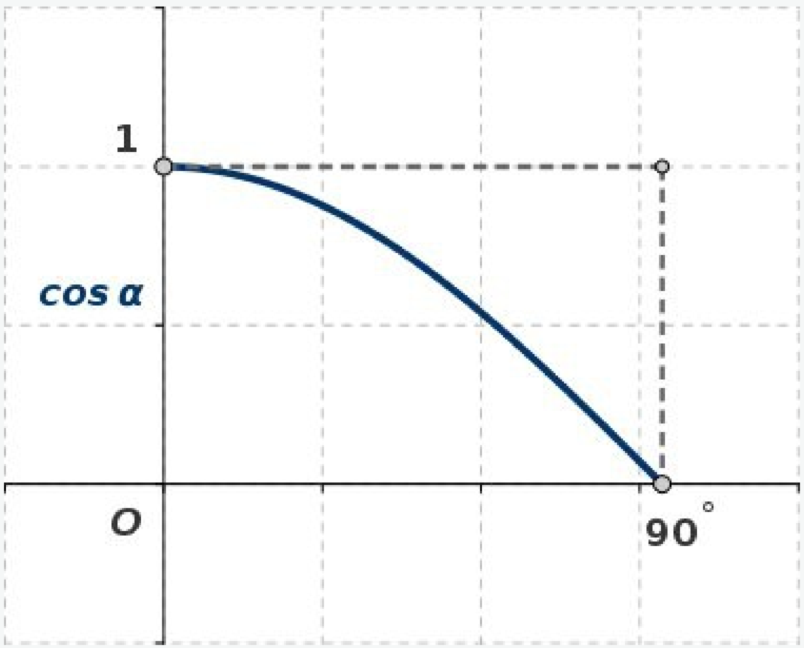


Fig. 3-1: The values of sin α and cos α in a right triangle

Note that the actual size of the triangle has no influence on the values of of *sin α* and *cos α,* they depend on the angle *α* only. This dependence is shown in Fig. *3-1*.

Trigonometric functions

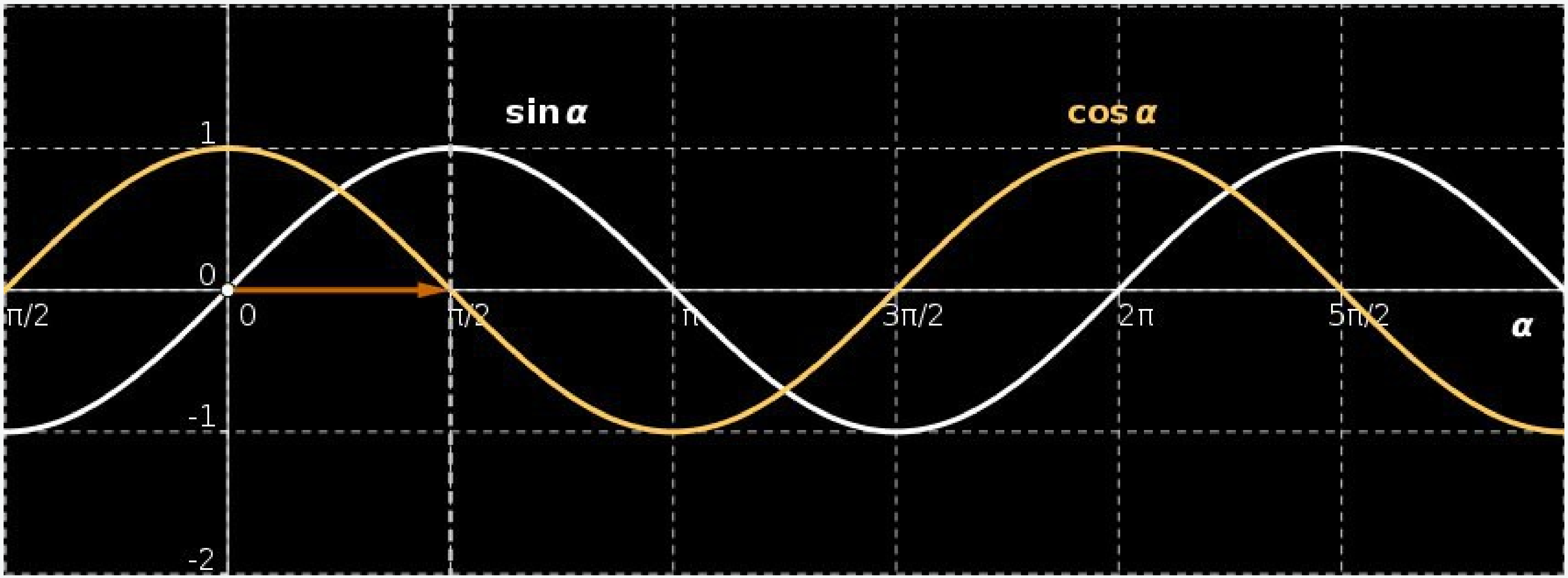


Fig. 3-2: Sine and cosine functions.

When seeing expressions like $\sin\alpha$ or $\cos\alpha$ mathematicians, think rather about functions like shown in Fig. 3-2, than about geometry of triangles. One notices, that the curves have the same shape, besides a shift of $+\pi/2$, respectively $-\pi/2$.

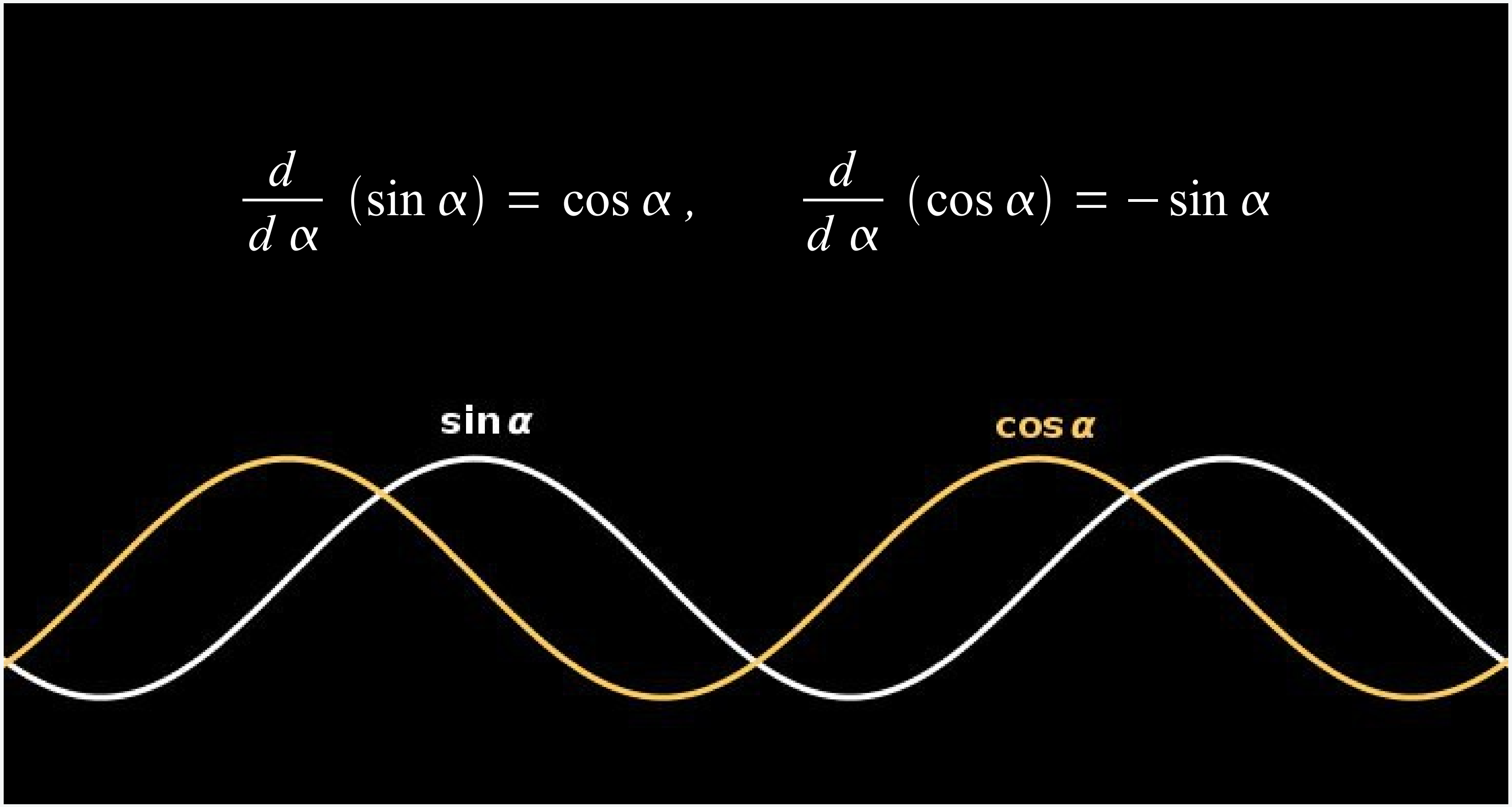


Fig. 3-3: Sine and cosine functions.

There is another relation between these functions which loosely can be expressed as follows: for small shifts in α, the rate of change of $\sin\alpha$ per shift in α, is given by $\cos\alpha$. The same is true the other way round, besides the sign. This is discussed more thoroughly in the chapter Differential Equations.