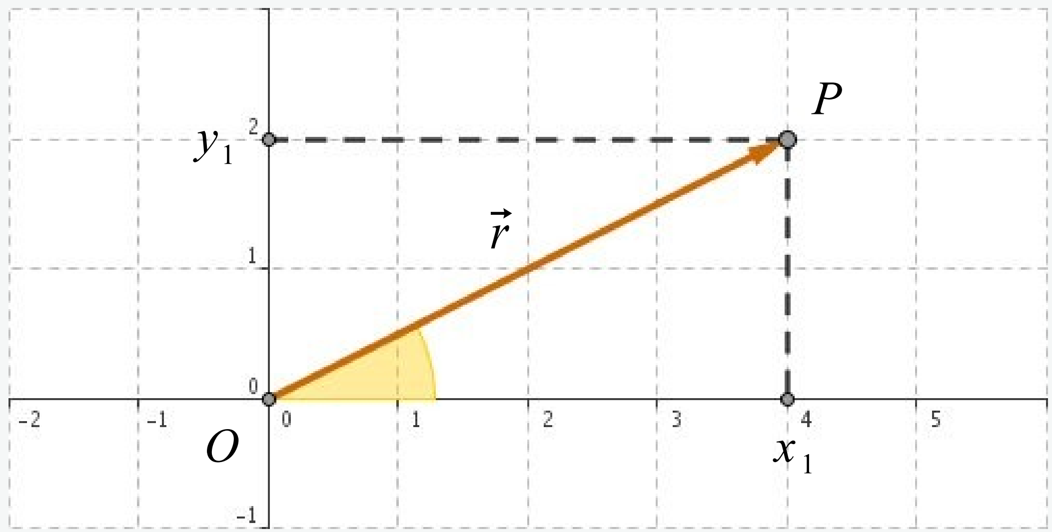
Vector calculus on planes



$$\vec{r} = \overrightarrow{OP} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\cos\varphi = \frac{x_1}{|\vec{r}|}, \quad \sin\varphi = \frac{y_1}{|\vec{r}|} \quad \Rightarrow \quad x_1 = |\vec{r}|\cos\varphi, \quad y_1 = |\vec{r}|\sin\varphi$$

The dependence of a two-dimensional vector on its length and its angle with respect to the x-axis can be represented as

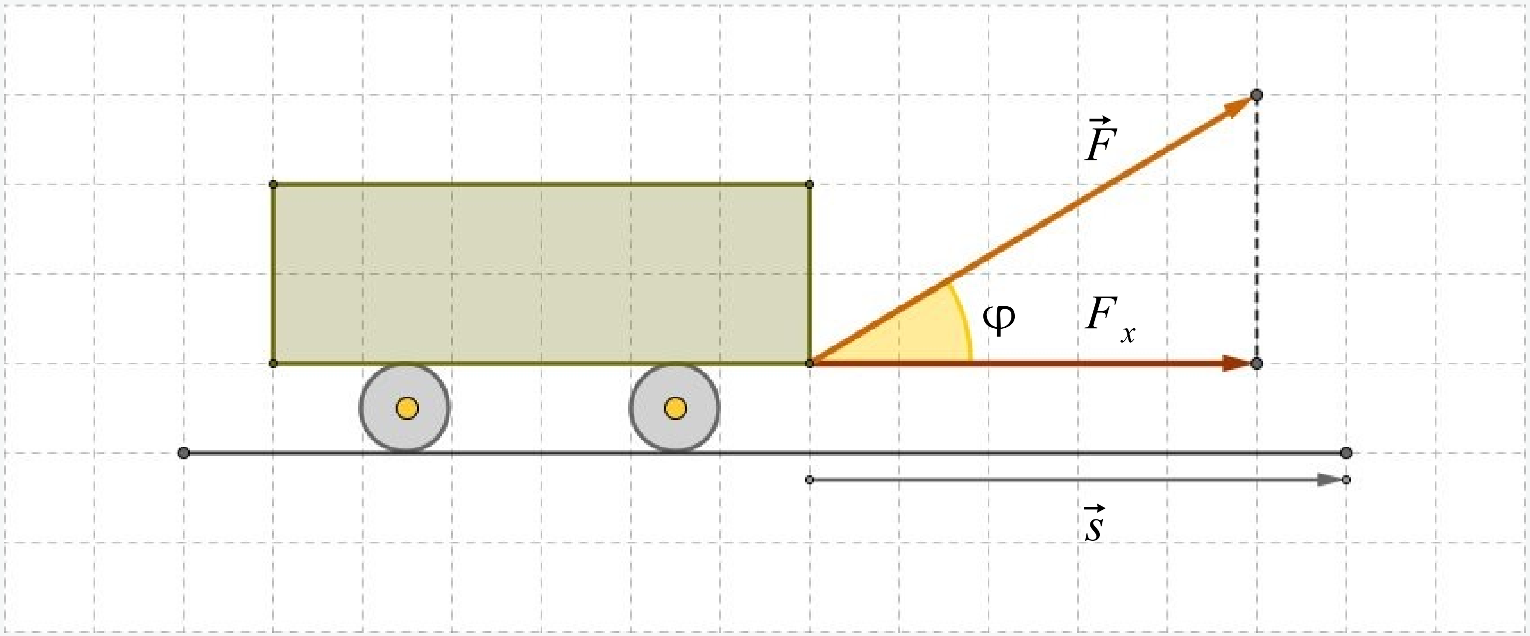
$$\vec{r} = |\vec{r}| \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

Scalar product (or dot product)

There are situations where only one direction of a vector is of interest.

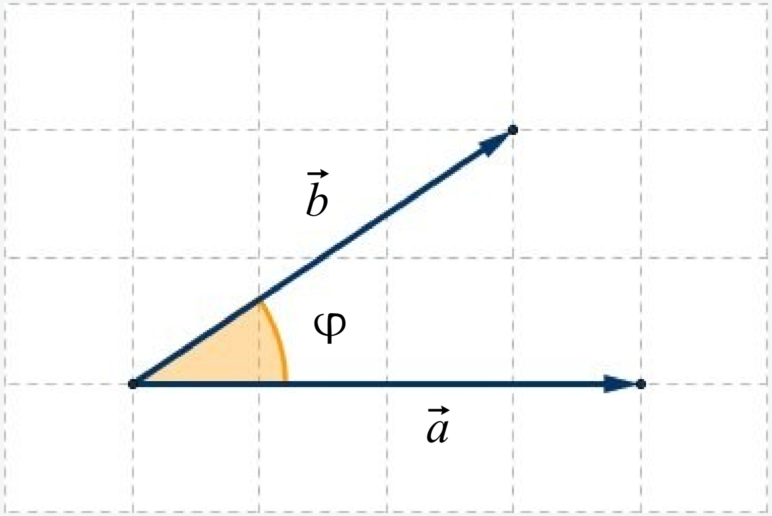
When moving a trolley, the force $\vec{F}$ may be sloping upwards - for anatomic reasons - but only the horizontal component $\vec{F}_x$ of the force is relevant, and the vertical component has no effect. The magnitude of the horizontal component is given by

$$|\vec{F}_x| = |\vec{F}| \cos\varphi$$



The effective work moving the trolley horizontally with path vector $\vec{s}$ is

$$W = |\vec{F}_x| \cdot |\vec{s}| = |\vec{F}| \cdot |\vec{s}| \cos\varphi$$



$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi = a\ b \cdot \cos\varphi$$

$$(0^{\circ} \leqslant \varphi \leqslant 180^{\circ})$$

$$\vec{a} \cdot \vec{b} = (a_x ,\ a_y) \cdot \begin{pmatrix} b_x \\ b_y \end{pmatrix} = a_x\ b_x + a_y\ b_y$$

The scalar product is an operation on the set of vectors. It assigns a real number to two vectors, that is, the result of the operation is not a vector again. This is different with the sum of two vectors.

Rules for scalar products:

Commutative law: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

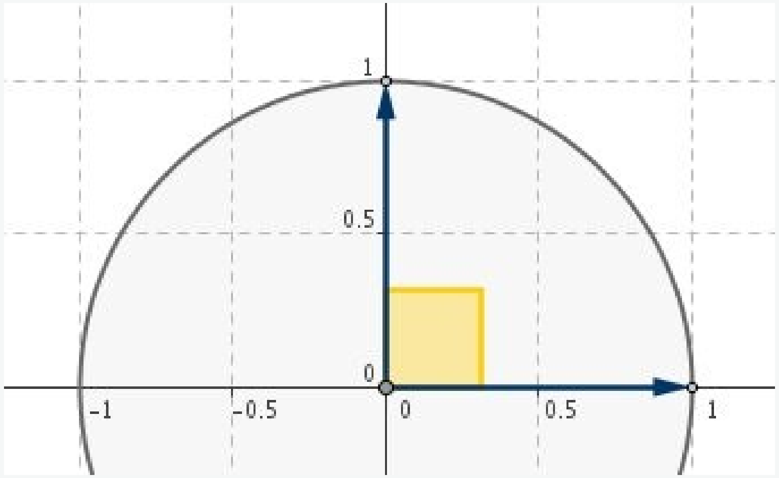
Distributive law: $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi = a_x b_x + a_y b_y$$

The angle between two vectors

$$\cos\varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \frac{a_x b_x + a_y b_y}{\sqrt{a_x^2 + a_y^2} \cdot \sqrt{b_x^2 + b_y^2}}$$

$$\varphi = \arccos\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} \right)$$

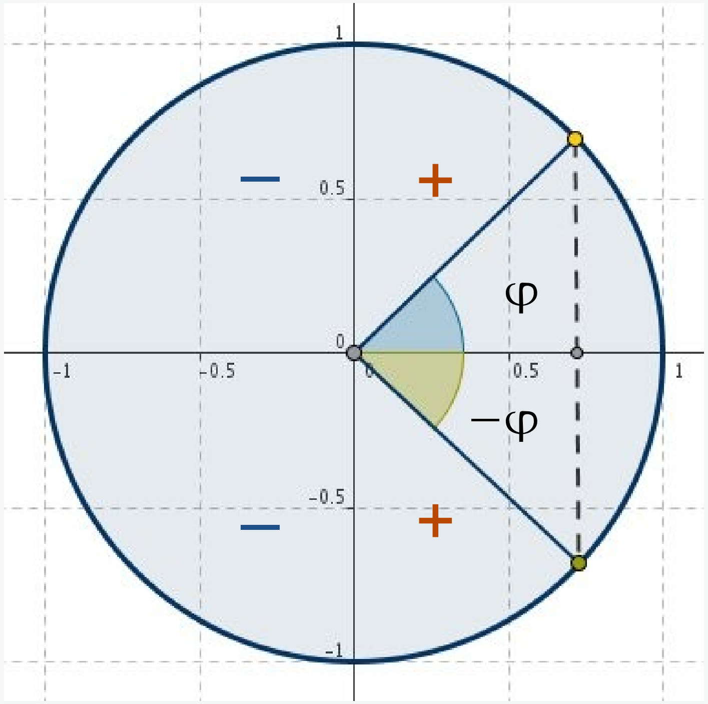


$$\vec{a} \cdot \vec{b} = 0 \quad \Leftrightarrow \quad \vec{a} \perp \vec{b}$$

$$\vec{a} \cdot \vec{a} = |\vec{a}| \cdot |\vec{a}| \cdot \cos 0^\circ = |\vec{a}|^2 = a^2 \quad \Rightarrow$$

$$|\vec{a}| = a = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_x^2 + a_y^2}$$

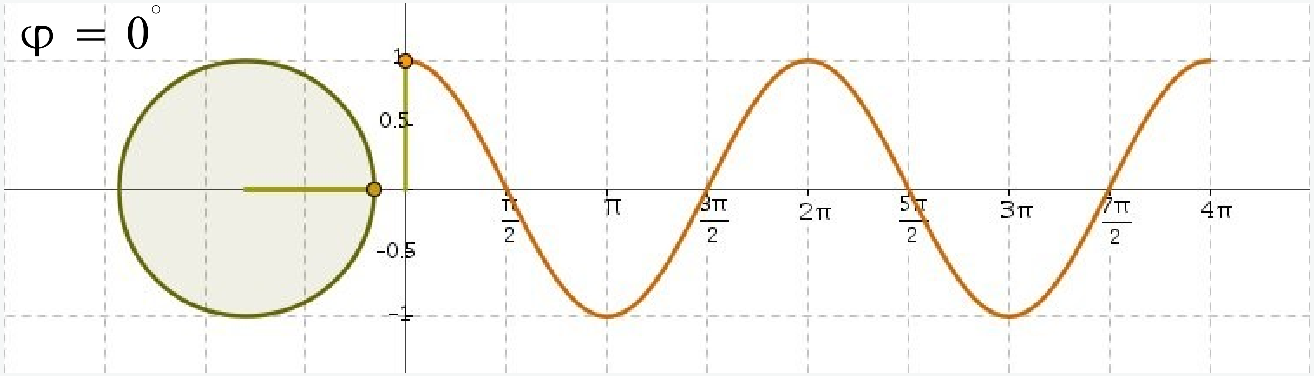
The term $\cos\varphi$ in the definition of the scalar product can be positive zero or negative in dependence on the angle. Therefore the same is true for the scalar product of two vectors.

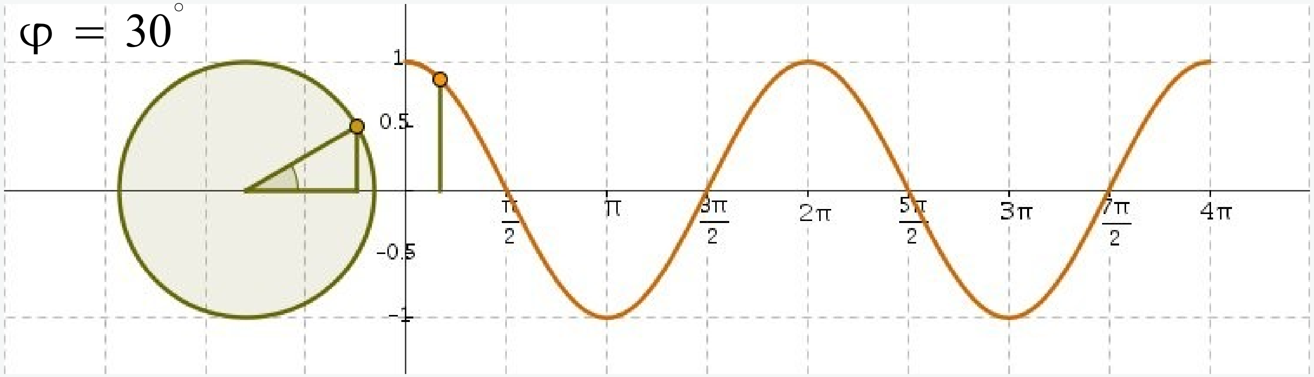


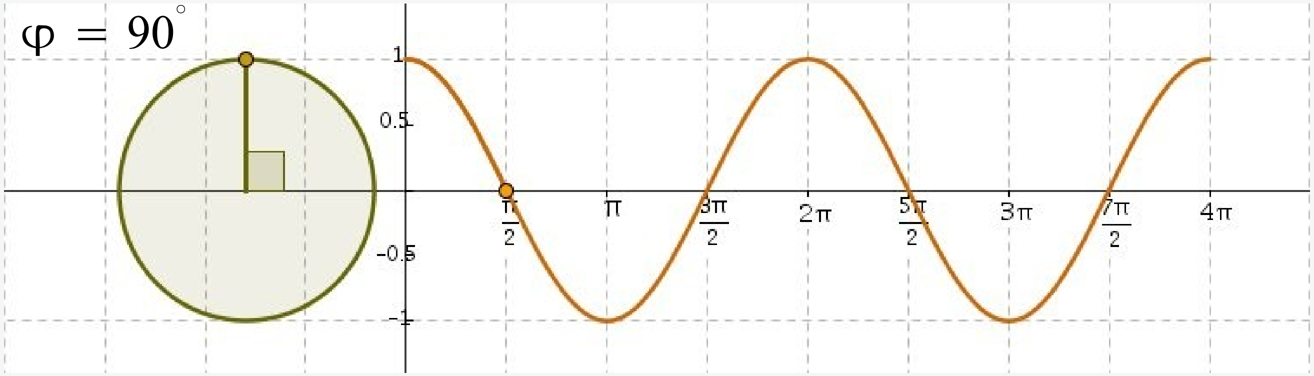
$$\cos(-\varphi) = \cos\varphi$$

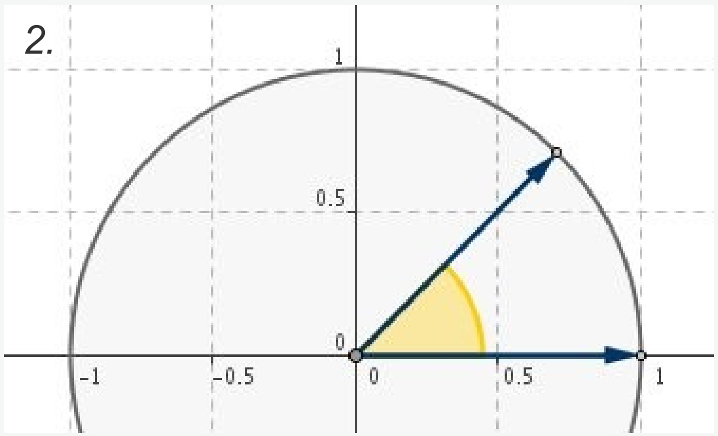
$$\cos 0^\circ = 1, \quad \cos 30^\circ = \frac{\sqrt{3}}{2}, \quad \cos 45^\circ = \frac{\sqrt{2}}{2}, \quad \cos 60^\circ = \frac{1}{2}, \quad \cos 90^\circ = 0$$

Mathematical background: Cosine









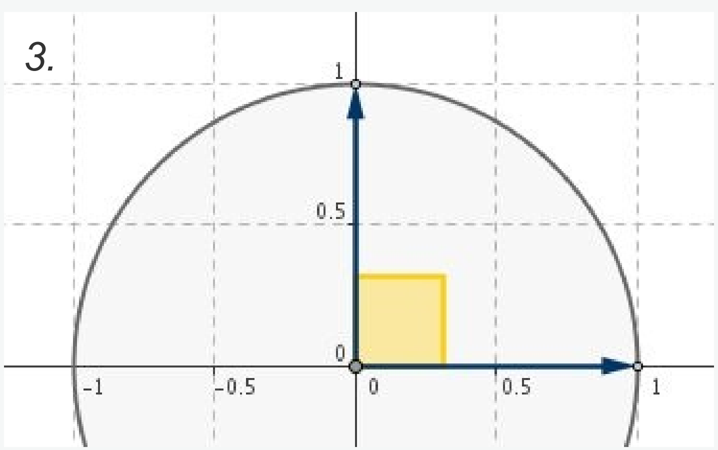
$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi$$

1. $\varphi = 0^\circ$

   $$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| > 0$$

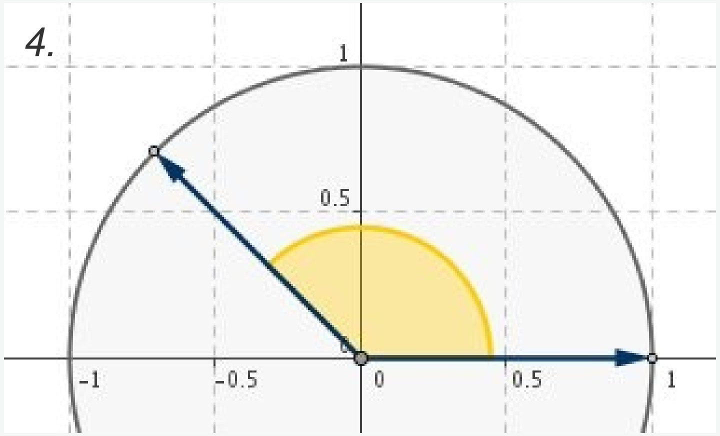
2. $0^\circ \leqslant \varphi \leqslant 90^\circ$

   $$\vec{a} \cdot \vec{b} > 0$$



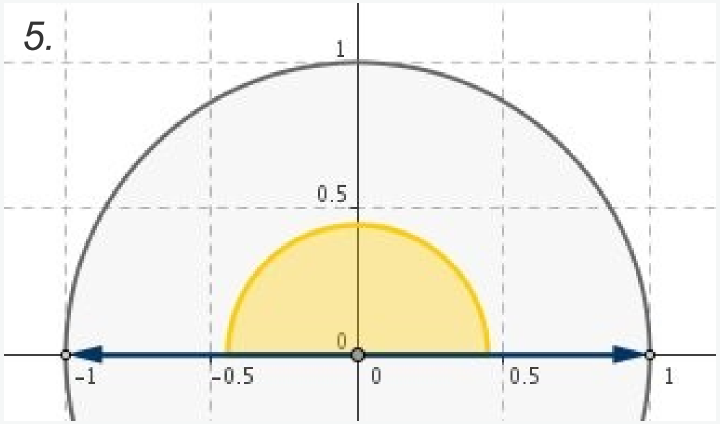
3. $\varphi = 90^\circ$

   $$\vec{a} \cdot \vec{b} = 0$$



4. $90^\circ \leqslant \varphi \leqslant 180^\circ$

$\vec{a} \cdot \vec{b} < 0$



5. $\varphi = 180^\circ$

$\vec{a} \cdot \vec{b} = -|\vec{a}| \cdot |\vec{b}| < 0$

Scalar product: Exercise 1, 2

Exercise 1: Calculate the scalar product of two vectors $\vec{a}$ and $\vec{b}$:

$$a)\ \ \vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}; \qquad b)\ \ \vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Exercise 2: Determine the angle between vector $\vec{a}$ and $\vec{b}$:

$$a)\ \ \vec{a} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \qquad b)\ \ \vec{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Solution 1:

a) $\vec{a} \cdot \vec{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \end{pmatrix} = 3 \cdot (-1) + 2 \cdot 5 = -3 + 10 = 7$

b) $\vec{a} \cdot \vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 1 \cdot (-1) + 1 \cdot 1 = 0 \quad \Rightarrow \quad \vec{a} \perp \vec{b}$

Solution 2:

a) $\cos\varphi = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \dfrac{\sqrt{3} \cdot 0 + 1 \cdot 1}{\sqrt{3+1} \cdot \sqrt{0+1}} = \dfrac{1}{2 \cdot 1} = \dfrac{1}{2} \quad \Rightarrow \quad \varphi = 60^\circ$

b) $\cos\varphi = = \dfrac{2 \cdot (-1) + 3 \cdot 2}{\sqrt{4+9} \cdot \sqrt{1+4}} = \dfrac{-2+6}{\sqrt{13} \cdot \sqrt{5}} = \dfrac{4}{\sqrt{65}} \sim 0.496 \quad \Rightarrow \quad \varphi = 60.26^\circ$