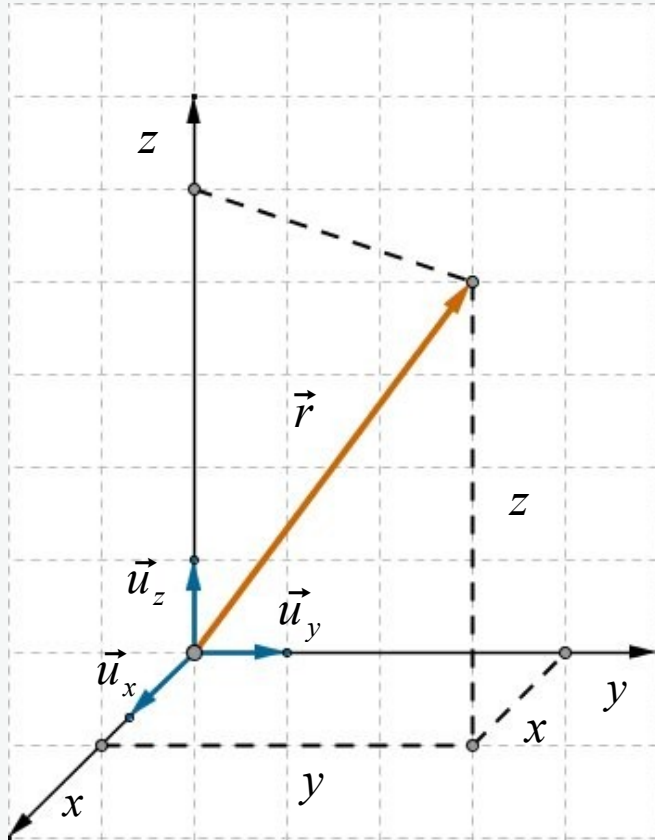


Vector calculus in 3-dimensional Space

Basic Properties



representation by components:

$$\vec{r} = x \vec{u}_x + y \vec{u}_y + z \vec{u}_z = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

magnitude:

$$|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

zero vector:

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

unit vector:

$$\vec{u}_r = \frac{\vec{r}}{|\vec{r}|}$$

Exercises 1-3

Exercise 1: Given are three vectors:

$$\vec{a} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

Perform the following additions:

a. $\vec{a} + \vec{b}$, b. $\vec{a} + \vec{c}$, c. $\vec{b} + \vec{c}$, d. $\vec{a} + \vec{b} + \vec{c}$
e. $\vec{a} - \vec{b} + \vec{c}$, f. $-\vec{a} + \vec{b} - \vec{c}$

Exercise 2: Determine the missing coordinates such, that the two vectors are collinear

$$\vec{a} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ 3 \\ b_3 \end{pmatrix}$$

Exercise 3: Examine whether the three points A , B and C are situated on a common straight line.

$$A(0, 1, -1), \quad B(-2, 1, -2), \quad C(6, 1, 2)$$

Solutions 1-3

Solution 1:

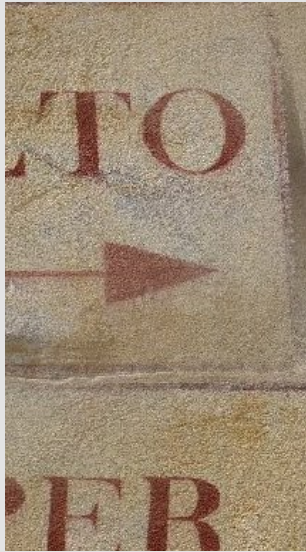
$$a. \begin{pmatrix} 6 \\ 6 \\ -1 \end{pmatrix}, \quad b. \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \quad c. \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}, \quad d. \begin{pmatrix} 4 \\ 7 \\ 1 \end{pmatrix}, \quad e. \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad f. \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Solution 2: $\vec{a} = \alpha \vec{b}, \quad b_1 = 6, \quad b_3 = 15$

Solution 3: Condition: $\vec{AB} = \alpha \cdot \vec{AC}, \quad \alpha = -\frac{1}{3}$

The three points A , B and C are on a common straight line.

Scalar product



$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos \varphi = a b \cdot \cos \varphi \quad (0^\circ \leq \varphi \leq 180^\circ)$$

$$\vec{a} \cdot \vec{b} = (a_x, a_y, a_z) \cdot \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = a_x b_x + a_y b_y + a_z b_z$$

The angle between the vectors:

$$\cos \varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2} \cdot \sqrt{b_x^2 + b_y^2 + b_z^2}}$$

$$\varphi = \arccos \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} \right)$$

$$\vec{a} \cdot \vec{b} = 0 \quad \Leftrightarrow \quad \vec{a} \perp \vec{b}$$

$$\vec{a} \cdot \vec{a} = |\vec{a}| \cdot |\vec{a}| \cdot \cos 0^\circ = |\vec{a}|^2 = a^2 \quad \Rightarrow$$

$$|\vec{a}| = a = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

Scalar product: Exercises 1, 2

Exercise 1: Calculate the scalar product of the vectors given below:

$$a) \quad \vec{a} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \qquad b) \quad \vec{a} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$c) \quad \vec{a} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 5 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 3 \\ 4 \\ 1 \\ -1 \end{pmatrix} \qquad d) \quad \vec{a} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 5 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 6 \\ -1 \end{pmatrix}$$

Exercise 2: Determine the missing coordinates such, that the scalar product takes the value k

$$a) \quad \vec{a} = \begin{pmatrix} 0 \\ a_2 \\ 3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ 1 \\ -5 \end{pmatrix}, \quad k = -10$$

$$b) \quad \vec{a} = \begin{pmatrix} 2 \\ 4 \\ a_3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \quad k = 2$$

Scalar product, length: Exercises 3-6

Exercise 3: Determine the length of the vector

$$\vec{v} = (1, 2, 5, 3, 4, -3, 0)$$

Exercise 4: Show, that the 3 vectors form a rectangular triangle

$$\vec{a} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} -1 \\ 6 \\ 1 \end{pmatrix}$$

Exercise 5: Determine the length of the vectors given below and transform them to unit vectors

$$\vec{a} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 4 \\ -5 \\ -2 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{d} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Exercise 6: Determine the parameter λ such, that the vector \mathbf{a} gets length 3.

$$a) \quad \vec{a} = \begin{pmatrix} \lambda \\ 2 \\ 1 \end{pmatrix}, \quad b) \quad \vec{a} = \overrightarrow{AB}, \quad A = (\lambda, 0, 1), \quad B = (1, 2, 2)$$

Scalar product, length: Solutions

Solution 1: $a) -8, \quad b) 0, \quad c) 6, \quad d) 0$

Solution 2: $a) \quad 0 \cdot 5 b_1 + a_2 - 15 = -10, \quad a_2 = 5, \quad b_1 \in \mathbb{R}$

$b) \quad 4 + 4 + 2 a_3 = 2, \quad a_3 = -3$

Solution 3: $|\vec{v}| = \sqrt{1^2 + 2^2 + 5^2 + 3^2 + 4^2 + (-3)^2 + 0^2} = \sqrt{64} = 8$

Solution 4: $\vec{c} = \vec{a} + \vec{b}, \quad \vec{a} \perp \vec{b}$

Solution 5:

$$|\vec{a}| = \sqrt{38}, \quad |\vec{b}| = \sqrt{45} = 3\sqrt{5}, \quad |\vec{c}| = \sqrt{6}, \quad |\vec{d}| = \sqrt{2}$$

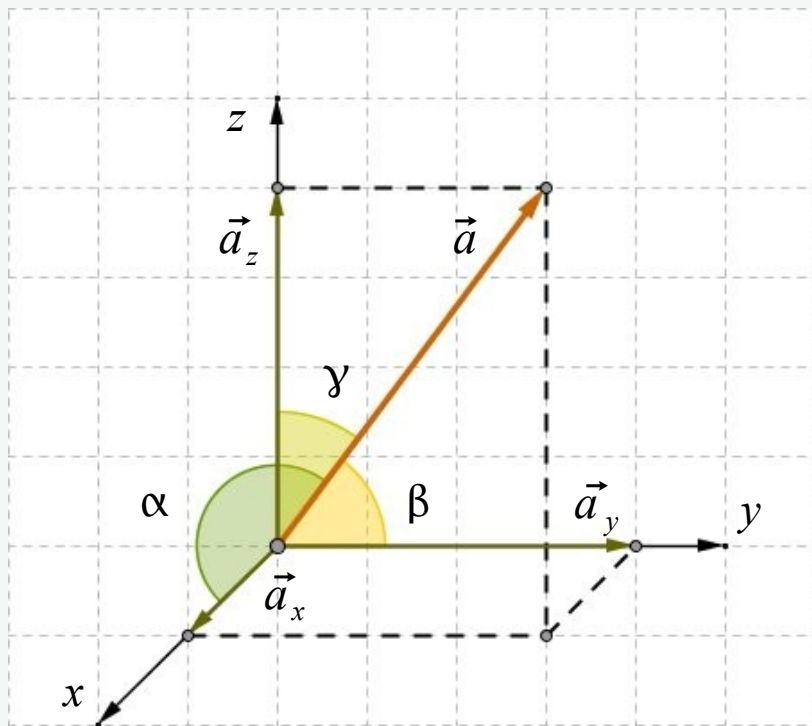
$$\vec{u}_a = \frac{1}{\sqrt{38}} \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}, \quad \vec{u}_b = \frac{1}{\sqrt{45}} \begin{pmatrix} 4 \\ -5 \\ -2 \end{pmatrix}, \quad \vec{u}_c = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_d = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Solution 6:

$a) \quad |\vec{a}| = \sqrt{\lambda^2 + 5} = 3, \quad \lambda^2 + 5 = 9, \quad \lambda^2 = 4, \quad \lambda_{1,2} = \pm 2$

$b) \quad |\vec{a}| = \sqrt{(1-\lambda)^2 + 2^2 + 1} = 3, \quad (1-\lambda)^2 = 4, \quad \lambda_1 = 3, \quad \lambda_2 = -1$

Direction angles of a vector



A vector is uniquely defined by its length and direction. The direction can be defined by the angles between the vector and the three basis vectors.

$$\cos \alpha = \frac{\vec{a} \cdot \vec{u}_x}{|\vec{a}| \cdot |\vec{u}_x|} = \frac{a_x}{|\vec{a}| \cdot 1} = \frac{a_x}{|\vec{a}|}$$

α – is here the angle between the vector and the x -axis.

Direction cosines:

$$\cos \alpha = \frac{a_x}{|\vec{a}|}, \quad \cos \beta = \frac{a_y}{|\vec{a}|}, \quad \cos \gamma = \frac{a_z}{|\vec{a}|}$$

The direction angles are not independent due to the relation of the direction cosines:

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

Exercise 7: Determine the angle between the vectors \mathbf{a} and \mathbf{b} . How large is the angle between vector \mathbf{a} and the x -axis?

$$a) \vec{a} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 4 \\ -5 \\ -2 \end{pmatrix}, \quad b) \vec{a} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Exercise 8: Determine the length, the unit vector and the angles with respect to the basis vectors

$$\vec{v} = 2\vec{u}_x - \vec{u}_y - 2\vec{u}_z$$

Exercise 9: Determine the direction angles of the following vectors

$$\vec{v}_1 = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -3 \\ 5 \\ -8 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 11 \\ -2 \\ 10 \end{pmatrix}$$

Direction angles of a vector: Solution 7

Solution 7: a) $\vec{a} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 4 \\ -5 \\ -2 \end{pmatrix}$, $|\vec{a}| = \sqrt{38}$, $|\vec{b}| = \sqrt{45} = 3\sqrt{5}$

$$\cos \varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2} \cdot \sqrt{b_x^2 + b_y^2 + b_z^2}}$$

$$\cos \varphi = \frac{13}{\sqrt{38} \cdot \sqrt{45}} \simeq 0.314, \quad \varphi = 71.68^\circ$$

$$\cos \alpha = \frac{\vec{a} \cdot \vec{u}_x}{|\vec{a}| \cdot |\vec{e}_x|} = \frac{a_x}{|\vec{a}|} = \frac{2}{\sqrt{38}} \simeq 0.324, \quad \alpha = 71.07^\circ$$

b) $\vec{a} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $|\vec{a}| = \sqrt{6}$, $|\vec{b}| = \sqrt{2}$

$$\cos \varphi = 0, \quad \varphi = 90^\circ$$

$$\cos \alpha = \frac{2}{\sqrt{6}} \simeq 0.816, \quad \alpha = 35.26^\circ$$

Direction angles of a vector: Solutions 8, 9

Solution 8:

$$\vec{v} = 2\vec{u}_x - \vec{u}_y - 2\vec{u}_z = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \quad |\vec{v}| = \sqrt{2^2 + 1^2 + 2^2} = 3$$

$$\vec{u}_v = \frac{\vec{v}}{|\vec{v}|} = \frac{2}{3}\vec{u}_x - \frac{1}{3}\vec{u}_y - \frac{2}{3}\vec{u}_z$$

$$\cos \alpha = \frac{v_x}{|\vec{v}|} = \frac{2}{3}, \quad \cos \beta = \frac{v_y}{|\vec{v}|} = -\frac{1}{3}, \quad \cos \gamma = \frac{v_z}{|\vec{v}|} = -\frac{2}{3}$$

$$\alpha = 48.11^\circ, \quad \beta = 109.28^\circ, \quad \gamma = 131.49^\circ$$

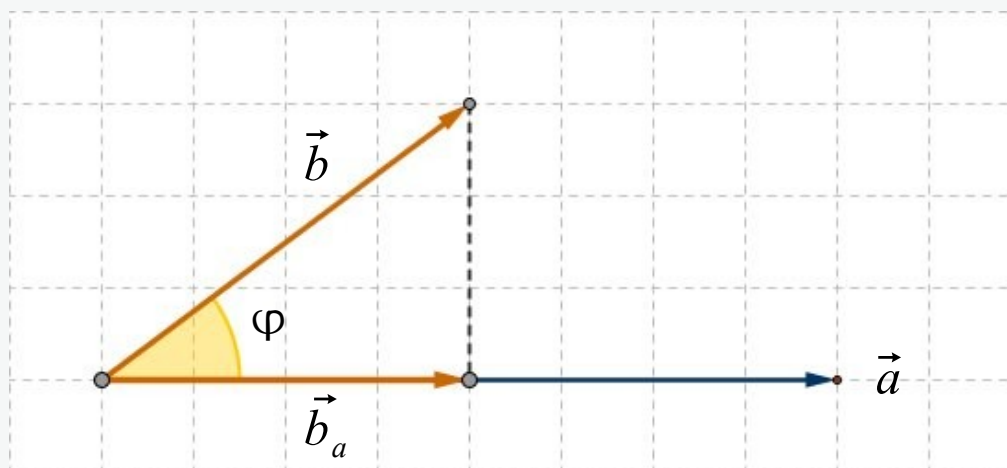
Solution 9:

$$\vec{v}_1: \quad \alpha = 39,51^\circ, \quad \beta = 81,12^\circ, \quad \gamma = 51,89^\circ$$

$$\vec{v}_2: \quad \alpha = 107,64^\circ, \quad \beta = 59,66^\circ, \quad \gamma = 143,91^\circ$$

$$\vec{v}_3: \quad \alpha = 42,83^\circ, \quad \beta = 97,66^\circ, \quad \gamma = 48,19^\circ$$

Projection of a vector onto another one



$$|\vec{b}_a| = |\vec{b}| \cdot \cos \varphi = |\vec{b}| \cdot \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

$$\vec{b}_a = |\vec{b}_a| \vec{e}_a = |\vec{b}_a| \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \frac{\vec{a}}{|\vec{a}|} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a}$$

The projection of the vector \mathbf{b} onto the vector \mathbf{a} results in the vector

$$\vec{b}_a = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a}$$

It is called the vector component of \mathbf{b} in the direction of \mathbf{a} .

Projection of a vector onto another one: Exercise 10

Determine the projection of vector \mathbf{b} onto vector \mathbf{a} :

$$a) \quad \vec{a} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 4 \\ -1 \\ 7 \end{pmatrix}, \quad b) \quad \vec{a} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 10 \\ 4 \\ -2 \end{pmatrix}$$

$$\vec{b}_a = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a}$$

Projection of a vector onto another one: Solution 10



Solution 10 a):

$$\vec{a} \cdot \vec{b} = (3, 0, 4) \cdot \begin{pmatrix} 4 \\ -1 \\ 7 \end{pmatrix} = 12 + 28 = 40$$

$$|\vec{a}|^2 = 3^2 + 0^2 + 4^2 = 25$$

$$\vec{b}_a = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a} = \frac{40}{25} \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = \frac{8}{5} \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 4.8 \\ 0 \\ 6.4 \end{pmatrix}$$

Solution 10 b):

$$\vec{a} \cdot \vec{b} = (2, -2, 1) \cdot \begin{pmatrix} 10 \\ 4 \\ -2 \end{pmatrix} = 10$$

$$|\vec{a}|^2 = 2^2 + (-2)^2 + 1^2 = 9$$

$$\vec{b}_a = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a} = \frac{10}{9} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{20}{9} \\ -\frac{20}{9} \\ \frac{10}{9} \end{pmatrix}$$