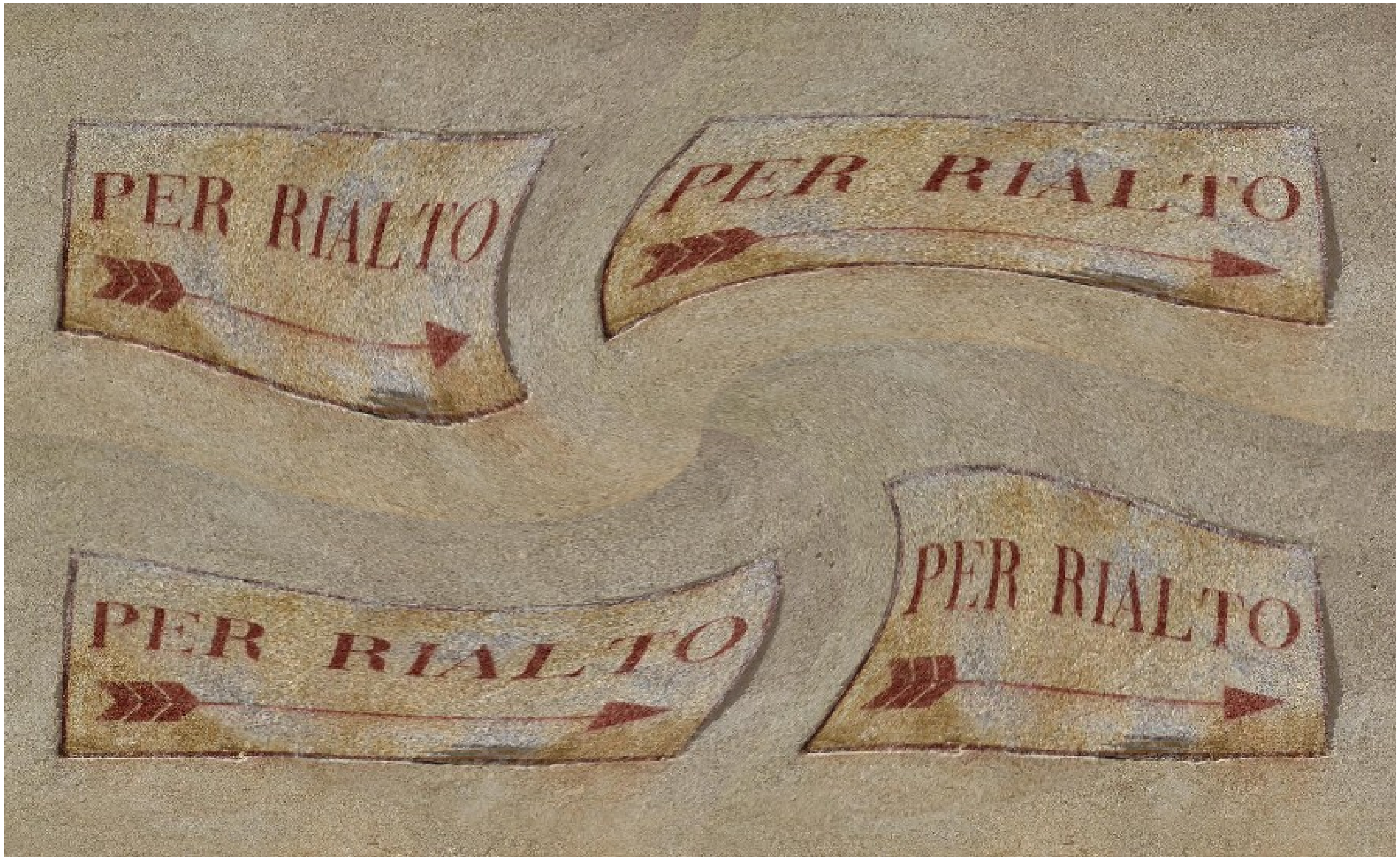
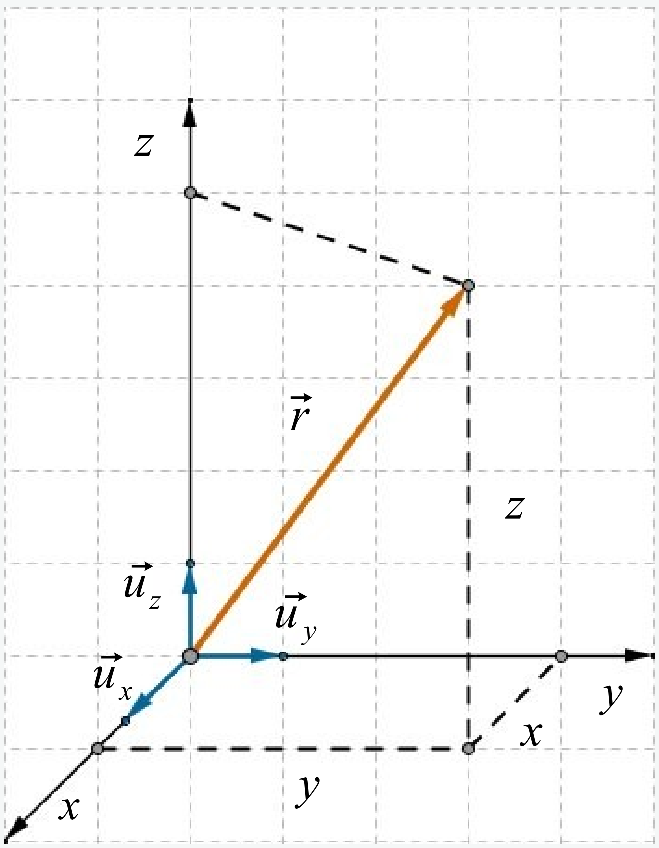


Vector calculus in 3-dimensional Space



representation by components:

$$\vec{r} = x\,\vec{u}_x + y\,\vec{u}_y + z\,\vec{u}_z = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

magnitude:

$$|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

zero vector:

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

unit vector:

$$\vec{u}_r = \frac{\vec{r}}{|\vec{r}|}$$

Exercise 1: Given are three vectors:

$$\vec{a} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

Perform the following additions:

a. $\vec{a} + \vec{b}$, *b.* $\vec{a} + \vec{c}$, *c.* $\vec{b} + \vec{c}$, *d.* $\vec{a} + \vec{b} + \vec{c}$

e. $\vec{a} - \vec{b} + \vec{c}$, *f.* $-\vec{a} + \vec{b} - \vec{c}$

Exercise 2: Determine the missing coordinates such, that the two vectors are collinear

$$\vec{a} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ 3 \\ b_3 \end{pmatrix}$$

Exercis 3: Examine whether the three points *A*, *B* and *C* are situated on a common straight line.

$$A(0,\ 1,\ -1), \quad B(-2,\ 1,\ -2), \quad C(6,\ 1,\ 2)$$

Solution 1:

$$a.\ \begin{pmatrix} 6 \\ 6 \\ -1 \end{pmatrix}, \quad b.\ \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \quad c.\ \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}, \quad d.\ \begin{pmatrix} 4 \\ 7 \\ 1 \end{pmatrix}, \quad e.\ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad f.\ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Solution 2: $\vec{a} = \alpha\, \vec{b}\,, \quad b_1 = 6\,, \quad b_3 = 15$

Solution 3: Condition: $\overrightarrow{AB} = \alpha \cdot \overrightarrow{AC}\,, \quad \alpha = -\dfrac{1}{3}$

The three points *A, B* and *C* are on a common straight line.

Scalar product

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi = a\ b \cdot \cos\varphi \qquad (0^\circ \leqslant \varphi \leqslant 180^\circ)$$

$$\vec{a} \cdot \vec{b} = (a_x,\ a_y,\ a_z) \cdot \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = a_x\ b_x + a_y\ b_y + a_z\ b_z$$

The angle between the vectors:

$$\cos\varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \frac{a_x\ b_x + a_y\ b_y + a_z\ b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2} \cdot \sqrt{b_x^2 + b_y^2 + b_z^2}}$$

$$\varphi = \arccos\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} \right)$$

$$\vec{a} \cdot \vec{b} = 0 \quad \Leftrightarrow \quad \vec{a} \perp \vec{b}$$

$$\vec{a} \cdot \vec{a} = |\vec{a}| \cdot |\vec{a}| \cdot \cos 0^\circ = |\vec{a}|^2 = a^2 \quad \Rightarrow$$

$$|\vec{a}| = a = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

Exercise 1: Calculate the scalar product of the vectors given below:

$$a)\quad \vec{a} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \qquad b)\quad \vec{a} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$c)\quad \vec{a} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 5 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 3 \\ 4 \\ 1 \\ -1 \end{pmatrix} \qquad d)\quad \vec{a} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 5 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 6 \\ -1 \end{pmatrix}$$

Exercise 2: Determine the missing coordinates such, that the scalar product takes the value k

$$a)\quad \vec{a} = \begin{pmatrix} 0 \\ a_2 \\ 3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ 1 \\ -5 \end{pmatrix}, \quad k = -10$$

$$b)\quad \vec{a} = \begin{pmatrix} 2 \\ 4 \\ a_3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \quad k = 2$$

Exercise 3: Determine the length of the vector

$$\vec{v} = (1,\ 2,\ 5,\ 3,\ 4,\ -3,\ 0)$$

Exercise 4: Show, that the 3 vectors form a rectangular triangle

$$\vec{a} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} -1 \\ 6 \\ 1 \end{pmatrix}$$

Exercise 5: Determine the length of the vectors given below and transform them to unit vectors

$$\vec{a} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 4 \\ -5 \\ -2 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{d} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Exercise 6: Determine the parameter λ such, that the vector $\boldsymbol{a}$ gets length 3.

$$a)\quad \vec{a} = \begin{pmatrix} \lambda \\ 2 \\ 1 \end{pmatrix}, \qquad b)\quad \vec{a} = \overrightarrow{AB}, \quad A = (\lambda,\ 0,\ 1), \quad B = (1,\ 2,\ 2)$$

Scalar product, length: Solutions

Solution 1: $a)\ -8, \quad b)\ 0, \quad c)\ 6, \quad d)\ 0$

Solution 2: $a)\quad 0 \cdot 5\, b_1 + a_2 - 15 = -10, \quad a_2 = 5, \quad b_1 \in \mathbb{R}$

$b)\quad 4 + 4 + 2\, a_3 = 2, \quad a_3 = -3$

Solution 3: $|\,\vec{v}\,| = \sqrt{1^2 + 2^2 + 5^2 + 3^2 + 4^2 + (-3)^2 + 0^2} = \sqrt{64} = 8$

Solution 4: $\vec{c} = \vec{a} + \vec{b}, \quad \vec{a} \perp \vec{b}$

Solution 5:

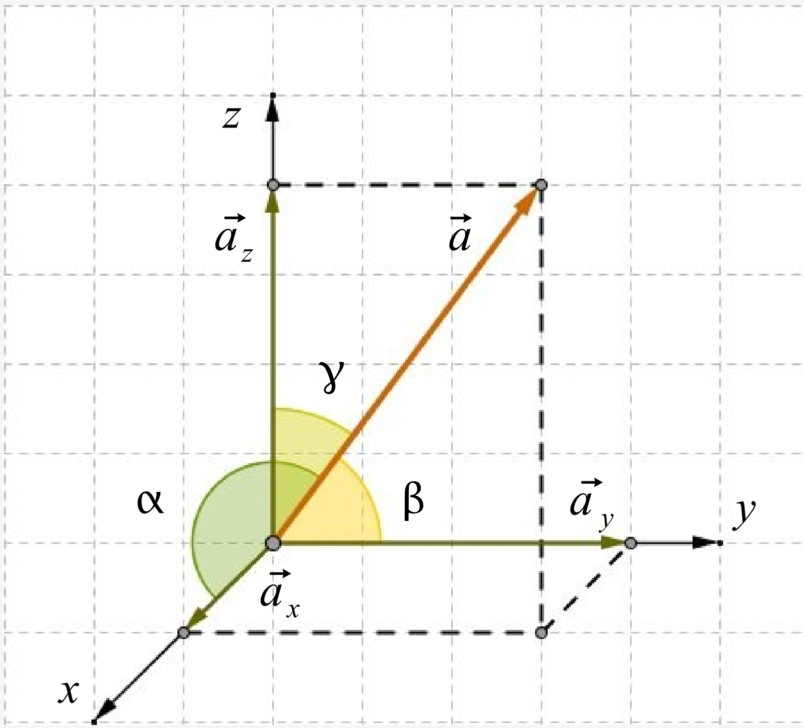
$$|\,\vec{a}\,| = \sqrt{38}, \quad |\,\vec{b}\,| = \sqrt{45} = 3\sqrt{5}, \quad |\,\vec{c}\,| = \sqrt{6}, \quad |\,\vec{d}\,| = \sqrt{2}$$

$$\vec{u}_a = \frac{1}{\sqrt{38}}\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}, \quad \vec{u}_b = \frac{1}{\sqrt{45}}\begin{pmatrix} 4 \\ -5 \\ -2 \end{pmatrix}, \quad \vec{u}_c = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_d = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Solution 6:

$a)\quad |\,\vec{a}\,| = \sqrt{\lambda^2 + 5} = 3, \quad \lambda^2 + 5 = 9, \quad \lambda^2 = 4, \quad \lambda_{1,2} = \pm 2$

$b)\quad |\,\vec{a}\,| = \sqrt{(1-\lambda)^2 + 2^2 + 1} = 3, \quad (1-\lambda)^2 = 4, \quad \lambda_1 = 3, \quad \lambda_2 = -1$



A vector is uniquely defined by its length and direction. The direction can be defined by the angles between the vector and the three basis vectors.

$$\cos\alpha = \frac{\vec{a}\cdot\vec{e}_x}{|\vec{a}|\cdot|\vec{e}_x|} = \frac{a_x}{|\vec{a}|\cdot 1} = \frac{a_x}{|\vec{a}|}$$

α – is here the angle between the vector and the x-axis.

Direction cosines:

$$\cos\alpha = \frac{a_x}{|\vec{a}|}, \qquad \cos\beta = \frac{a_y}{|\vec{a}|}, \qquad \cos\gamma = \frac{a_z}{|\vec{a}|}$$

The direction angless are not independent due to the relation of the direction cosines:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Exercise 7: Determine the angle between the vectors ***a*** and ***b***. How large is the angle between vector ***a*** and the *x*-axis?

$$a)\quad \vec{a} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 4 \\ -5 \\ -2 \end{pmatrix}, \qquad b)\quad \vec{a} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Exercise 8: Determine the length, the unit vector and the angles with respect to the basis vectors

$$\vec{v} = 2\,\vec{u}_x - \vec{u}_y - 2\,\vec{u}_z$$

Exercise 9: Determine the direction angles of the following vectors

$$\vec{v}_1 = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -3 \\ 5 \\ -8 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 11 \\ -2 \\ 10 \end{pmatrix}$$

Solution 7:

$$a)\quad \vec{a} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 4 \\ -5 \\ -2 \end{pmatrix}, \quad |\,\vec{a}\,| = \sqrt{38}\,, \qquad |\,\vec{b}\,| = \sqrt{45} = 3\sqrt{5}$$

$$\cos\varphi = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{a_x\,b_x + a_y\,b_y + a_z\,b_z}{\sqrt{a_x^2+a_y^2+a_z^2}\cdot\sqrt{b_x^2+b_y^2+b_{zy}^2}}$$

$$\cos\varphi = \frac{13}{\sqrt{38}\cdot\sqrt{45}} \simeq 0.314\,, \qquad \varphi = 71.68^\circ$$

$$\cos\alpha = \frac{\vec{a}\cdot\vec{e}_x}{|\vec{a}|\cdot|\vec{e}_x|} = \frac{a_x}{|\vec{a}|} = \frac{2}{\sqrt{38}} \simeq 0.324\,, \qquad \alpha = 71.07^\circ$$

$$b)\quad \vec{a} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \qquad |\,\vec{a}\,| = \sqrt{6}\,, \qquad |\,\vec{b}\,| = \sqrt{2}$$

$$\cos\varphi = 0\,, \qquad \varphi = 90^\circ$$

$$\cos\alpha = \frac{2}{\sqrt{6}} \simeq 0.816\,, \qquad \alpha = 35.26^\circ$$

Solution 8:

$$\vec{v} = 2\vec{u}_x - \vec{u}_y - 2\vec{u}_z = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \qquad |\vec{v}| = \sqrt{2^2 + 1^2 + 2^2} = 3$$

$$\vec{u}_v = \frac{\vec{v}}{|\vec{v}|} = \frac{2}{3}\vec{u}_x - \frac{1}{3}\vec{u}_y - \frac{2}{3}\vec{u}_z$$

$$\cos\alpha = \frac{v_x}{|\vec{v}|} = \frac{2}{3}, \qquad \cos\beta = \frac{v_y}{|\vec{v}|} = -\frac{1}{3}, \qquad \cos\gamma = \frac{v_z}{|\vec{v}|} = -\frac{2}{3}$$

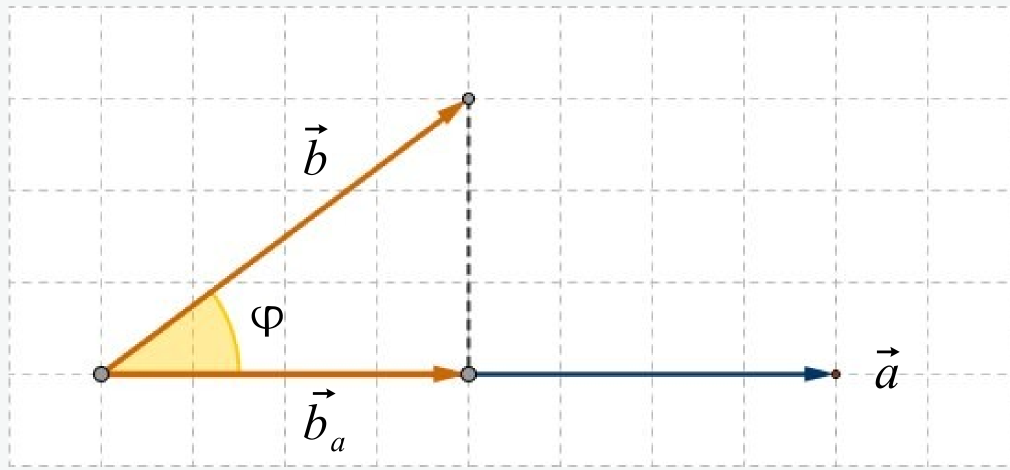
$$\alpha = 48.11^\circ, \qquad \beta = 109.28^\circ, \qquad \gamma = 131.49^\circ$$

Solution 9:

$$\vec{v}_1: \quad \alpha = 39{,}51^\circ, \qquad \beta = 81{,}12^\circ, \qquad \gamma = 51{,}89^\circ$$

$$\vec{v}_2: \quad \alpha = 107{,}64^\circ, \qquad \beta = 59{,}66^\circ, \qquad \gamma = 143{,}91^\circ$$

$$\vec{v}_3: \quad \alpha = 42{,}83^\circ, \qquad \beta = 97{,}66^\circ, \qquad \gamma = 48{,}19^\circ$$



$$|\vec{b}_a| = |\vec{b}| \cdot \cos\varphi = |\vec{b}| \cdot \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

$$\vec{b}_a = |\vec{b}_a|\, \vec{e}_a = |\vec{b}_a| \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \frac{\vec{a}}{|\vec{a}|} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a}$$

The projection of the vector ***b*** onto the vector ***a*** results in the vector

$$\boxed{\vec{b}_a = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a}}$$

It is called the vector component of ***b*** in the direction of ***a***.

Determine the projection of vector ***b*** onto vector ***a*** :

$$a)\quad \vec{a} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 4 \\ -1 \\ 7 \end{pmatrix}, \qquad b)\quad \vec{a} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 10 \\ 4 \\ -2 \end{pmatrix}$$

$$\boxed{\vec{b}_a = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a}}$$

Solution 10 a):

$$\vec{a}\cdot\vec{b} = (3,\ 0,\ 4)\cdot\begin{pmatrix}4\\-1\\7\end{pmatrix} = 12 + 28 = 40$$

$$|\vec{a}|^2 = 3^2 + 0^2 + 4^2 = 25$$

$$\vec{b}_a = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a} = \frac{40}{25}\begin{pmatrix}3\\0\\4\end{pmatrix} = \frac{8}{5}\begin{pmatrix}3\\0\\4\end{pmatrix} = \begin{pmatrix}4.8\\0\\6.4\end{pmatrix}$$

Solution 10 b):

$$\vec{a}\cdot\vec{b} = (2,\ -2,\ 1)\begin{pmatrix}10\\4\\-2\end{pmatrix} = 10$$

$$|\vec{a}|^2 = 2^2 + (-2)^2 + 1^2 = 9$$

$$\vec{b}_a = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a} = \frac{10}{9}\begin{pmatrix}2\\-2\\1\end{pmatrix} = \begin{pmatrix}\frac{20}{9}\\-\frac{20}{9}\\\frac{10}{9}\end{pmatrix}$$