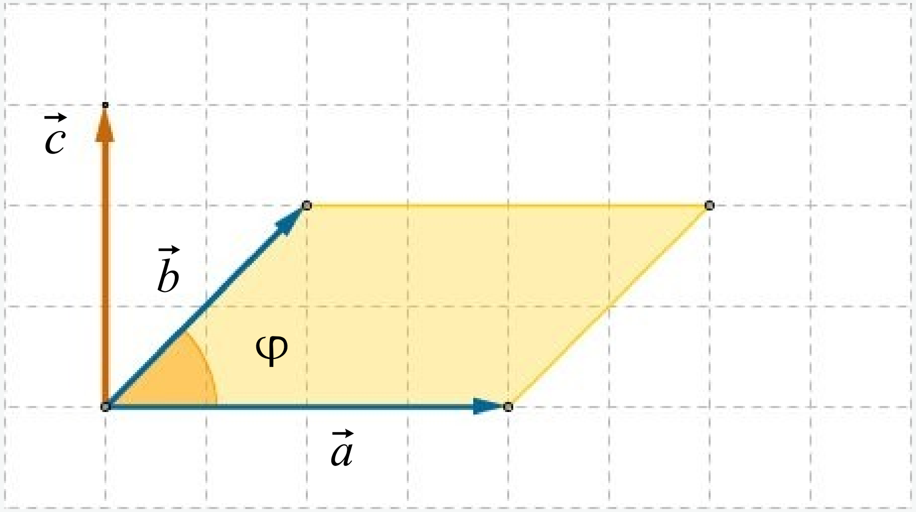
Cross Product

Cross product (also called vector product) :

$$\vec{c} = \vec{a} \times \vec{b}$$

1. $|\vec{c}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin\varphi \qquad (0^\circ \leqslant \varphi \leqslant 180^\circ)$
2. $\vec{c} \perp \vec{a}, \quad \vec{c} \perp \vec{b} \qquad \vec{c} \cdot \vec{a} = \vec{c} \cdot \vec{b} = 0$ (scalar product)

$\vec{c}$ is a vector normal to the plane containing $\vec{a}$ and $\vec{b}$



The cross product is defined for 3-dimensional vectors only !

Cross Product

Formal representation by a determinant of 3 rows

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{u}_x & \vec{u}_y & \vec{u}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} =$$

$$= \vec{u}_x (a_y b_z - a_z b_y) + \vec{u}_y (a_z b_x - a_x b_z) +$$
$$+ \vec{u}_z (a_x b_y - a_y b_x)$$

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}$$

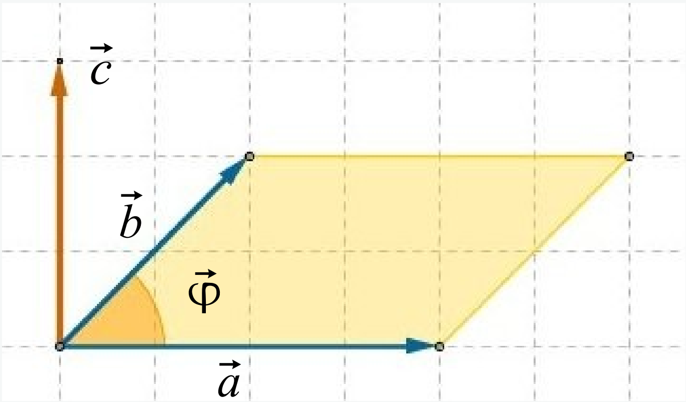
The cross product $\vec{a} \times \vec{b}$ of two vectors $\vec{a}$ and $\vec{b}$ is again a vector. It is characterised by three conditions:

1. $|\vec{c}| = |\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin\varphi$, $\quad \vec{c} = \vec{a} \times \vec{b}$

2. $\vec{c} \perp \vec{a}$, $\quad \vec{c} \perp \vec{b} \quad \Rightarrow \quad \vec{c} \cdot \vec{a} = \vec{c} \cdot \vec{b} = 0$ (scalar product)

3. $(\vec{a}, \vec{b}, \vec{c})$ form a right handed system, in case $\vec{a}$ and $\vec{b}$ are linearly independent.

It follows from this definition, that the magnitude of the vector product corresponds to the area of the parallelogram spanned by the two vectors.



The cross product is

anticommutative	$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
distributive over addition	$(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
compatible with scalar multiplication	$\lambda\,(\vec{a} \times \vec{b}) = (\lambda\,\vec{a}) \times \vec{b} = \vec{a} \times (\lambda\,\vec{b})$

Exercise 1: Determine the cross product of vectors $\vec{a}$ and $\vec{b}$

$$\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$$

Exercise 2: Find all vectors which are perpendicular to the following vectors

$$\vec{a} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

Solution 1: $\vec{a} \times \vec{b} = -14\,\vec{u}_x + 27\,\vec{u}_y - 5\,\vec{u}_z = \begin{pmatrix} -14 \\ 27 \\ -5 \end{pmatrix}$

Solution 2: $\vec{c} = \vec{a} \times \vec{b} = \begin{pmatrix} -1 \\ -10 \\ -7 \end{pmatrix}, \quad \vec{c} \perp \vec{a}, \quad \vec{c} \perp \vec{b}$

$$L = \left\{ \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \quad \vec{c} = \lambda \begin{pmatrix} -1 \\ -10 \\ -7 \end{pmatrix}, \quad \lambda \in \mathbb{R} \right\}$$

Exercise 3: Determine the area of the parallelogram which is spanned by the two vectors: :

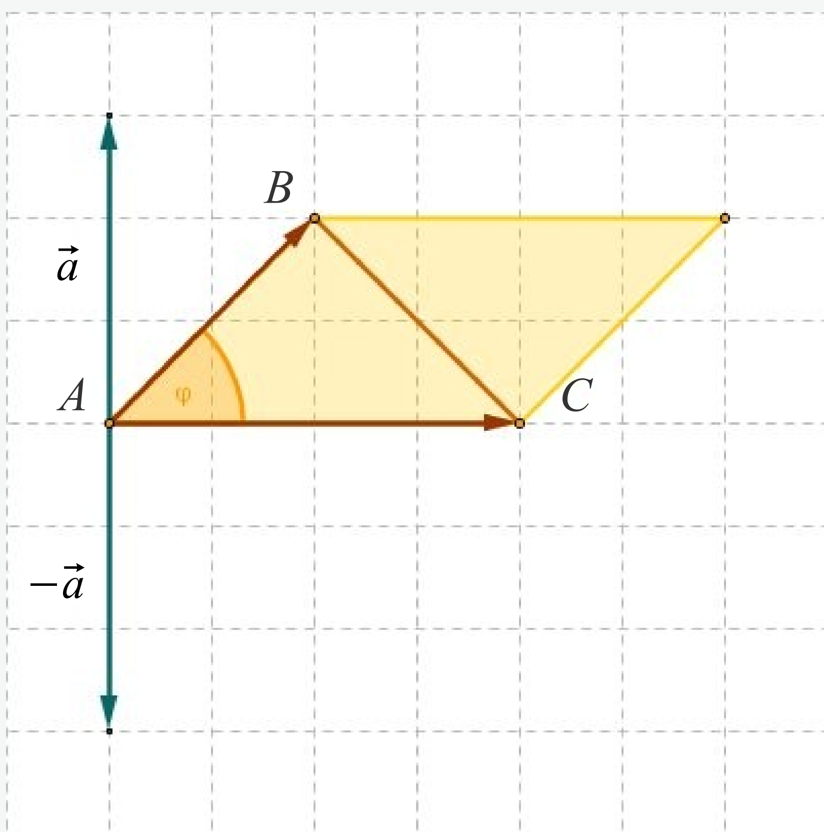
$$\vec{a} = \begin{pmatrix} 4 \\ -10 \\ 5 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -3 \\ -1 \\ -3 \end{pmatrix}$$

Exercise 4: The triangle ABC is given by the points $A\,(1, 1, 0)$, $B\,(3, 2, 2)$ and C $(-3, -1, 2)$. Determine a vector with length 1 which is perpendicular to the triangle (2 solutions).

Vector product: Solutions 3, 4

Solution 3: $|\vec{a} \times \vec{b}| = \sqrt{2390} = \quad = 48.89$

Solution 4: $A = (1,\ 1,\ 0)\,, \quad B = (3,\ 2,\ 2)\,, \quad C = (-3,\ -1,\ 2)$



$$\overrightarrow{AB} = (2,\ 1,\ 2)\,, \quad \overrightarrow{AC} = (-4,\ -2,\ 2)$$

$$\vec{a} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{u}_x & \vec{u}_y & \vec{u}_z \\ 2 & 1 & 2 \\ -4 & -2 & 2 \end{vmatrix} =$$

$$= 6\ \vec{e}_x - 12\ \vec{e}_y$$

$$\vec{a} = (6,\ -12,\ 0) = 6\ (1,\ -2,\ 0)$$

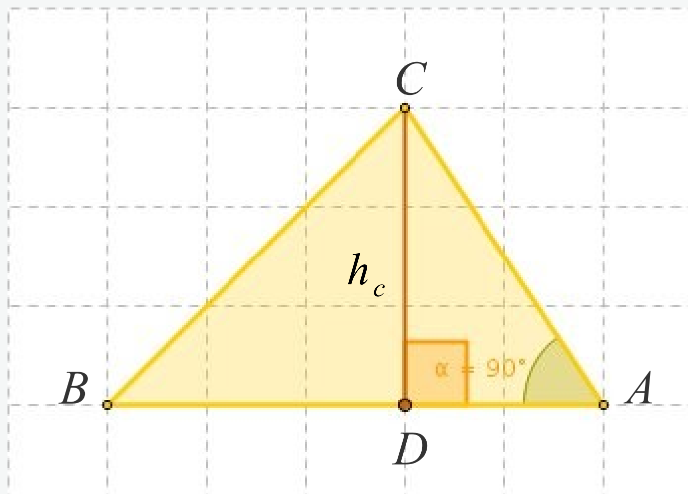
$$|\vec{a}| = \sqrt{6^2 + 12^2} = \sqrt{6^2(1+2^2)} = 6\sqrt{5}$$

$$\vec{u}_a = \frac{\vec{a}}{|a|} = \frac{1}{\sqrt{5}}\ (1,\ -2,\ 0)$$

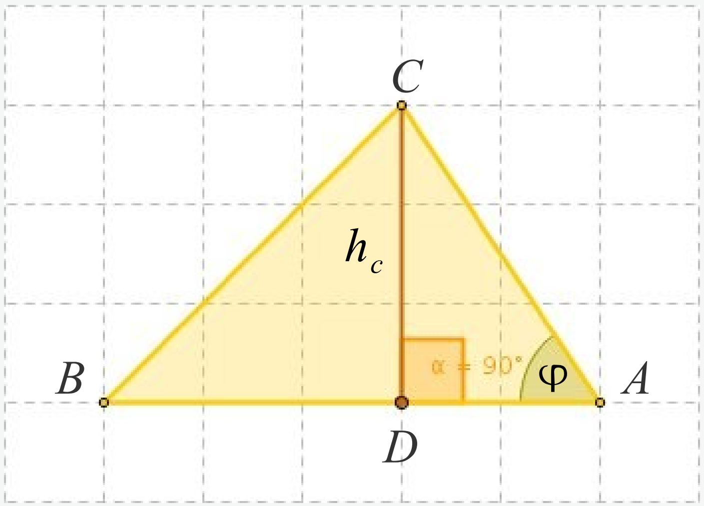
$$-\vec{u}_a = -\frac{\vec{a}}{|a|} = \frac{1}{\sqrt{5}}\ (-1,\ 2,\ 0)$$

Exercise 5: Determine the area of the triangle ABC and the height h_c (perpendicular to the side $[AB]$ through C)

$$A = (2,\ 3,\ -6),\quad B = (6,\ 4,\ 4),\quad C = (3,\ 7,\ 4)$$



$$A = (2,\ 3,\ -6)\,,\quad B = (6,\ 4,\ 4)\,,\quad C = (3,\ 7,\ 4)$$



$$A_{ABC} = \frac{1}{2}\,|\,\vec{AB} \times \vec{AC}\,|$$

$$\vec{AB} = (4,\ 1,\ 10)\,,\quad \vec{AC} = (1,\ 4,\ 10)$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{u}_x & \vec{u}_y & \vec{u}_z \\ 4 & 1 & 10 \\ 1 & 4 & 10 \end{vmatrix} =$$

$$= -30\ \vec{u}_x - 30\ \vec{u}_y + 15\ \vec{u}_z =$$

$$= 15\ (-2,\ -2,\ 1)$$

$$A_{ABC} = \frac{1}{2}\,|\,\vec{AB} \times \vec{AC}\,| = \frac{15}{2}\sqrt{(-2)^2 + (-2)^2 + 1^2} = \frac{45}{2} = 22.5\ \text{AU}$$

$$|\,\vec{AB} \times \vec{AC}\,| = |\,AB\,| \cdot |\,AC\,| \sin\varphi\,,\qquad \sin\varphi = \frac{|\,\vec{AB} \times \vec{AC}\,|}{|\,AB\,| \cdot |\,AC\,|}$$

$$h_c = |\,AC\,| \sin\varphi = |\,AC\,| \frac{|\,\vec{AB} \times \vec{AC}\,|}{|\,AB\,| \cdot |\,AC\,|} = \frac{|\,\vec{AB} \times \vec{AC}\,|}{|\,AB\,|} = \frac{45}{\sqrt{117}} \simeq \frac{45}{10.82} \simeq 4.16\ \text{LU}$$