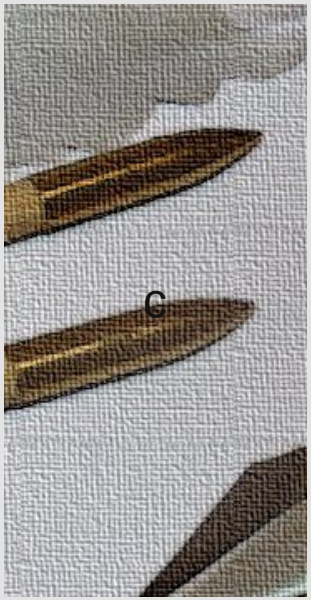
Vector Calculus: Exercises



Exercise 1:

Calculate the angle between the vectors

$$a)\ \vec{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\quad \vec{b} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix};\qquad b)\ \vec{a} = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix},\quad \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$$

Exercise 2:

Determine the parameter t such, that the vectors are mutually orthogonal

$$a)\ \vec{a} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix},\qquad \vec{b} = \begin{pmatrix} 6 \\ t \\ 0 \end{pmatrix}$$

$$b)\ \vec{a} = \begin{pmatrix} -3 \\ t^2 \\ 4\,t \end{pmatrix},\qquad \vec{b} = \begin{pmatrix} 8 \\ 4 \\ t \end{pmatrix}$$

$$a)\quad \vec{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$$

$$\cos\varphi = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{a_x\,b_x + a_y\,b_y}{\sqrt{a_x^2+a_y^2}\cdot\sqrt{b_x^2+b_y^2}} = \frac{0\cdot 1 + 1\cdot\sqrt{3}}{\sqrt{0^2+1^2}\cdot\sqrt{1^2+3}} = \frac{\sqrt{3}}{1\cdot 2} = \frac{\sqrt{3}}{2}$$

$$\varphi = 30^\circ$$

$$b)\quad \vec{a} = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$$

$$\cos\varphi = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{a_x\,b_x + a_y\,b_y + a_z\,b_z}{\sqrt{a_x^2+a_y^2+a_z^2}\cdot\sqrt{b_x^2+b_y^2+b_z^2}} = \frac{5-5}{|\vec{a}|\cdot|\vec{b}|} = 0$$

$$\varphi = 90^\circ$$

$$a)\quad \vec{a} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 6 \\ t \\ 0 \end{pmatrix}, \qquad \vec{a} \perp \vec{b} \quad \Leftrightarrow \quad \varphi = 90^\circ \quad \Rightarrow \quad \cos\varphi = 0$$

$$\cos\varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \frac{a_x\, b_x + a_y\, b_y + a_z\, b_z}{|\vec{a}| \cdot |\vec{b}|} = \frac{12 + 3\,t}{|\vec{a}| \cdot |\vec{b}|} = 0 \quad \Rightarrow$$

$$12 + 3\,t = 0 \quad \Rightarrow \quad t = -4$$

$$b)\quad \vec{a} = \begin{pmatrix} -3 \\ t^2 \\ 4\,t \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 8 \\ 4 \\ t \end{pmatrix}$$

$$\cos\varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \frac{a_x\, b_x + a_y\, b_y + a_z\, b_z}{|\vec{a}| \cdot |\vec{b}|} = \frac{-24 + 4\,t^2 + 4\,t^2}{|\vec{a}| \cdot |\vec{b}|} = 0 \quad \Rightarrow$$

$$-24 + 8\,t^2 = 0 \quad \Leftrightarrow \quad t^2 = 3 \quad \Rightarrow \quad t = \pm\sqrt{3}$$

Exercise 3:

Which of the following expressions are meaningful and which are not?

$a)\ (\vec{a}+\vec{b})\cdot\vec{c}\,,$ $b)\ (\vec{a}\cdot\vec{b})\cdot\vec{c}$

$c)\ \vec{a}+\vec{b}\cdot\vec{c}\,,$ $d)\ |\vec{a}|\,(\vec{b}\cdot\vec{c})$

$e)\ (|\vec{a}|\,\vec{b})\cdot\vec{c}\,,$ $f)\ \vec{a}\,\vec{b}+|\vec{a}|\,\vec{c}$

Exercise 4:

Calculate the cross product $\vec{a}\times\vec{b}$

$$a)\ \vec{a}=\begin{pmatrix}1\\1\\0\end{pmatrix},\qquad \vec{b}=\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$b)\ \vec{a}=\begin{pmatrix}2\\0\\1\end{pmatrix},\qquad \vec{b}=\begin{pmatrix}3\\1\\0\end{pmatrix}$$

Sokution 3: meaningful: *a), b), d), e);* not meaningful: *c), f)*

Solution 4: $a)\ \vec{a} \times \vec{b} = (0, 0, -2)$

$b)\ \vec{a} \times \vec{b} = (-1, 3, 2)$